CONILPOTENCY AND WEAK CATEGORY

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Let $f: X \to Y$ be a map and let $e': Y \to \Omega\Sigma Y$ be the usual embedding. Then we prove the following results.

THEOREM 1. $\operatorname{cat} f = \operatorname{cat}(e'f)$, $\operatorname{w\,cat} f = \operatorname{w\,cat}(e'f)$ *if* Y *is an* H-*space*.

THEOREM 2. $\operatorname{conil} f = \operatorname{w} \Sigma \operatorname{cat}(e'f) \leqq \Sigma \operatorname{w\,cat}(e'f) \leqq \operatorname{w\,cat}(e'f)$, *where* Σ *is the suspension functor. If we take* $X = Y$ *and* $f = 1_X$, *this result yields* $\operatorname{conil} X \leqq \operatorname{w\,cat} e'$, *a result due to Ganea, Hilton, and Peterson* (**4**).

THEOREM 3. *Suppose that* Y *is* $(m - 1)$*-connected and*

$$\dim X \leqq 2m(\operatorname{conil} f + 1) - 2.$$

Then $\operatorname{conil} f = \operatorname{w} \Sigma \operatorname{cat}(e'f) = \Sigma \operatorname{w\,cat}(e'f) = \operatorname{w\,cat}(e'f)$.

THEOREM 4. *Suppose that* Y *is* $(m - 1)$*-connected, where* $m \geqq 2$. *Then if* $\dim Y \leqq m(\operatorname{conil} Y + 2) - 2$, *we have* $\operatorname{cat} Y = \operatorname{conil} Y = \operatorname{w\,cat} e' = \operatorname{w\,cat} Y$.

THEOREM 5. $\operatorname{nil} f \leqq 1$ *if and only if* $fe\nabla: \Sigma\Omega X \vee \Sigma\Omega X \to Y$ *extends to* $\Sigma\Omega X \times \Sigma\Omega X$, *where* $e: \Sigma\Omega X \to X$ *is the projection.*

In this paper we shall work in the category $\mathscr{T}$ of spaces with base point and having the homotopy type of countable CW-complexes. All maps and homotopies shall be with respect to base points, and for simplicity we shall use the same symbol for a map and its homotopy class. Given spaces X, Y we denote the set of homotopy classes of maps from X to Y by $[X, Y]$. We have an isomorphism $\tau: [\Sigma X, Y] \to [X, \Omega Y]$, where Σ and Ω are the suspension and loop functors, respectively. We denote $\tau(1_{\Sigma X})$ by e' and $\tau^{-1}(1_{\Omega X})$ by e.

1. For convenience, we recall some notions from Peterson's theory of structures (**5**). We shall follow the definitions and notation of (**2**). Let $\mathscr{C}$ be a category. By a right structure $\mathscr{R}$ over $\mathscr{C}$ we mean $(R, P, T; d, j)$, where R, P, and T are covariant functors from $\mathscr{C}$ to $\mathscr{T}$, d is a natural transformation from R to P, and j is a natural transformation from T to P. Given an object X of $\mathscr{C}$, we say that X is $\mathscr{R}$-structured if there exists a map $\phi: RX \to TX$ such that $j(X)\phi \simeq d(X)$. We may assume that j is a natural fibration. Given a right structure $\mathscr{R} = (R, P, T; d, j)$ over $\mathscr{C}$, we have a right structure $\Sigma\mathscr{R} = (\Sigma R, \Sigma P, \Sigma T; \Sigma d, \Sigma j)$ over $\mathscr{C}$, where Σ is the suspension functor. Clearly, if $X \in \mathscr{C}$ can be $\mathscr{R}$-structured, it can be $\Sigma\mathscr{R}$-structured. Given a category $\mathscr{C}$, we have a category $\mathscr{C}^2$ of pairs. An object of $\mathscr{C}^2$ is a map $f: X \to Y$ of $\mathscr{C}$, and given objects $f: X_1 \to X_2$, $g: Y_1 \to Y_2$ of $\mathscr{C}^2$, a map

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$(u, v): f \to g$ is a pair of maps $u: X_1 \to Y_1$, $v: X_2 \to Y_2$ such that $gu = vf$. We have covariant functors $D_0, D_1: \mathscr{C}^2 \to \mathscr{C}$ given by $D_0(f) = Y, D_1(f) = X$, where $f: X \to Y$. Furthermore, given $(u, v): f \to g$, we have $D_0(u, v) = v$, $D_1(u, v) = u$. We have a natural transformation $G: D_1 \to D_0$ given by $G(f) = f$, where $f \in \mathscr{C}^2$. Given a right structure system $\mathscr{R} = (R, P, T; d, j)$ over $\mathscr{C}$, we have a right structure system

$$\mathscr{R}^2 = (RD_1, PD_0, TD_0; (dD_0)(RG), jD_0)$$

over $\mathscr{C}^2$. Given an object $f \in \mathscr{C}^2$, we shall say that f is $\mathscr{R}$-structured if f is $\mathscr{R}^2$-structured. It is easily seen that if $f: X \to Y$ is an object of $\mathscr{C}^2$, and X is $\mathscr{R}$-structured or Y is $\mathscr{R}$-structured, then f is $\mathscr{R}$-structured.

Let $\mathscr{R} = (R, P, T; d, j)$ be a right structure over $\mathscr{C}$. We may consider $j: T \to P$ as a natural fibration. Let $q: P \to Q$ be the cofibre of j and let $j_w: T_w \to P$ be the fibre of q. Then we have an associated weak structure $\mathscr{R}_w = (R, P, T_w; d, j_w)$ over $\mathscr{C}$. If $X \in \mathscr{C}$, we say that X can be weakly $\mathscr{R}$-structured if X can be $\mathscr{R}_w$-structured. It is easily seen that if $X \in \mathscr{C}$ can be $\mathscr{R}$-structured, then it can be weakly $\mathscr{R}$-structured, and that X can be weakly $\mathscr{R}$-structured if and only if $q(X)d(X) \simeq *$.

We now consider the n-cat structure $\mathscr{K}_n$ over $\mathscr{T}$. We have:

$$\mathscr{K}_n = \left(\mathrm{Id}, \prod_{i=1}^{n+1}, T_1; \Delta, j\right),$$

where Id is the identity functor of $\mathscr{T}$, $\prod_{i=1}^{n+1}$ is the cartesian product functor, T_1 is the "fat wedge" functor, Δ is the diagonal natural transformation, and j is the natural transformation induced by the inclusion of the fat wedge into the cartesian product. Thus, given a space X, $\operatorname{cat} X \leqq n$ if there is a map $\phi: X \to T_1(X, \ldots, X)$ such that $j\phi \simeq \Delta: X \to X^{n+1}$. Given a map $f: X \to Y$, we have $\operatorname{cat} f \leqq n$ if there exists a map $\phi: X \to T_1(Y, \ldots, Y)$ such that $j\phi \simeq \Delta f: X \to Y^{n+1}$. Furthermore, $\mathrm{w}\operatorname{cat} f \leqq n$ if $q\Delta f \simeq *$, where

$$q: Y^{n+1} \to \wedge_{i=1}^{n+1} Y$$

is the projection onto the smash product. Similarly, $\Sigma\,\mathrm{w}\operatorname{cat} f \leqq n$ if and only if there exists a map $\phi: \Sigma X \to \Sigma T_w(Y)$ such that

$$(\Sigma j_w)\phi \simeq \Sigma(\Delta f): \Sigma X \to \Sigma Y^{n+1}.$$

It is easily seen that $\Sigma\mathrm{w}\operatorname{cat} f \leqq \mathrm{w}\operatorname{cat} f \leqq \operatorname{cat} f$. Furthermore, $\mathrm{w}\,\Sigma\operatorname{cat} f \leqq n$ if and only if $\Sigma(q\Delta f) \simeq *$, and hence $\mathrm{w}\,\Sigma\operatorname{cat} f \leqq \Sigma\,\mathrm{w}\operatorname{cat} f$. Finally, given spaces X, Y and a map $f: X \to Y$, we have an H′-map $\Sigma f: \Sigma X \to \Sigma Y$. We shall write conil f for conil Σf; see (**1**) for definitions. We observe that in the above if we take $f =$ identity map, then the structures for f are just the structures for the space involved.

2. Let $f: X \to Y$, $g: Y \to Z$ be maps. Then it is easily seen that

$$\operatorname{cat}(gf) \leqq \min\{\operatorname{cat} f, \operatorname{cat} g\} \text{ and } \mathrm{w}\operatorname{cat}(gf) \leqq \min\{\mathrm{w}\operatorname{cat} f, \mathrm{w}\operatorname{cat} g\}.$$

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THEOREM 1. *Let* $f\colon X \to Y$ *be a map, where* Y *is an* H-*space. Then* $\operatorname{cat} f = \operatorname{cat}(e'f)$, $\operatorname{w\,cat} f = \operatorname{w\,cat}(e'f)$, *where* $e'\colon Y \to \Omega\Sigma Y$ *is the embedding.*

Proof. We need only show that $\operatorname{cat} f \leqq \operatorname{cat}(e'f)$, $\operatorname{w\,cat} f \leqq \operatorname{w\,cat}(e'f)$. Since Y is an H-space, there is a map $\gamma\colon \Omega\Sigma Y \to Y$ such that $\gamma e' \simeq 1_Y$. Then $\operatorname{cat} f = \operatorname{cat}(\gamma e' f) \leqq \operatorname{cat}(e'f)$ and $\operatorname{w\,cat} f = \operatorname{w\,cat}(\gamma e' f) \leqq \operatorname{w\,cat}(e'f)$.

THEOREM 2. *Let* $f\colon X \to Y$ *be a map. Then*

$$\operatorname{conil} f = \operatorname{w}\Sigma\operatorname{cat}(e'f) \leqq \Sigma\operatorname{w\,cat}(e'f).$$

Proof. The fact that $\operatorname{w}\Sigma\operatorname{cat}(e'f) \leqq \Sigma\operatorname{w\,cat}(e'f)$ follows from the definition of these structures. We need only show that $\operatorname{conil} f = \operatorname{w}\Sigma\operatorname{cat}(e'f)$. Suppose that $\operatorname{w}\Sigma\operatorname{cat}(e'f) \leqq n$. Then we have:

$$\Sigma(q\Delta e'f) \simeq *\colon \Sigma X \to \Sigma\left(\bigwedge_{i=1}^{n+1}\Omega\Sigma Y\right).$$

Let $c'\colon \Sigma Y \to \bigvee_{i=1}^{n+1}\Sigma Y$ be the cocommutator map of weight $(n+1)$ for ΣY. Then we can form a map $\bar{c}'\colon Y^{n+1} \to \Omega(\bigvee_{i=1}^{n+1}\Sigma Y)$ such that $\bar{c}'\Delta = \tau(c')$; see (**4**). Since $\Sigma(q\Delta e'f) \simeq *$, applying τ we have $\Omega\Sigma(q\Delta)e'f \simeq *$. Consider the following diagram, where each square is homotopy-commutative:

$$\begin{array}{ccccc}
X & & & & \\
\downarrow f & & & & \\
Y & \xrightarrow{\Delta} & Y^{n+1} & \xrightarrow{q} & \bigwedge_{i=1}^{n+1} Y \\
\downarrow e' & & \downarrow e' & & \downarrow e' \\
\Omega\Sigma Y & \xrightarrow[\Omega\Sigma\Delta]{} & \Omega\Sigma(Y^{n+1}) & \xrightarrow[\Omega\Sigma q]{} & \Omega\Sigma\left(\bigwedge_{i=1}^{n+1} Y\right)
\end{array}$$

We then have that $e'q\Delta f \simeq *$. Using (**4**, Lemmas 4.1_k and 4.2_k), it follows that $\bar{c}'\Delta f \simeq *$, that is, $\tau(c')f \simeq *$. Hence, $c'(\Sigma f) \simeq *$. Hence, $\operatorname{conil} f \leqq n$. This proves that $\operatorname{conil} f \leqq \operatorname{w}\Sigma\operatorname{cat}(e'f)$. The proof that $\operatorname{w}\Sigma\operatorname{cat}(e'f) \leqq \operatorname{conil} f$ is exactly the same, using again (**4**, Lemmas 4.1_k and 4.2_k).

Remark. If we take $f = 1_X$, we have $\operatorname{conil} X = \operatorname{w}\Sigma\operatorname{cat} e' \leqq \Sigma\operatorname{w\,cat} e' \leqq \operatorname{w\,cat} e' \leqq \operatorname{cat} e'$. In (**4**), it is shown that $\operatorname{conil} X \leqq \operatorname{w\,cat} e'$. Our paper is motivated by an attempt to obtain a suitable modified form for the dual of Stasheff's criterion; see (**4**; **6**). The exact dual of Stasheff's criterion would read: $\operatorname{conil} X \leqq 1$ if and only if $\operatorname{cat} e' \leqq 1$. This is false, as shown by an example in (**4**).

THEOREM 3. *Let* $f\colon X \to Y$ *be a map, and let* Y *be* $(m-1)$-*connected. If* $\dim X \leqq 2m(\operatorname{conil} f + 1) - 2$, *then* $\operatorname{conil} f = \operatorname{w}\Sigma\operatorname{cat}(e'f) = \Sigma\operatorname{w\,cat}(e'f) = \operatorname{w\,cat}(e'f)$.

Proof. We need only show that $\operatorname{w\,cat}(e'f) \leqq \operatorname{conil} f$ under the restriction on $\dim X$. Suppose that $\operatorname{conil} f \leqq n$. Then $c'(\Sigma f) \simeq *$, where $c'\colon \Sigma Y \to \bigvee_{i=1}^{n+1}\Sigma Y$

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is the cocommutator map of weight $(n+1)$ for ΣY. Using the map $\bar{c}'$ above such that $\bar{c}'\Delta = \tau(c')$, we then have that $\bar{c}'\Delta f \simeq *$. Using (**4**, Lemma 4.2_k) and the same diagram as in Theorem 2 above, we have that $\Sigma(q\Delta e'f) \simeq *$. The dimension restrictions now shows that

$$\Sigma\colon \left[X, \bigwedge_{i=1}^{n+1} \Omega\Sigma Y\right] \to \left[\Sigma X, \Sigma\left(\bigwedge_{i=1}^{n+1} \Omega\Sigma Y\right)\right]$$

is an injection. Hence, $q\Delta e'f \simeq *$, that is, $\text{w cat}(e'f) \leqq n$. Hence, $\text{w cat}(e'f) \leqq \text{conil} f$.

Corollary. *If X is $(m-1)$-connected and* $\dim X \leqq 2m(\text{conil } X + 1) - 2$, *then* $\text{conil } X = \text{w } \Sigma \text{ cat } e' = \Sigma \text{ w cat } e' = \text{w cat } e'$.

Theorem 4. *Let Y be $(m-1)$-connected, where $m \geqq 2$. If*

$$\dim Y \leqq m(\text{conil } Y + 2) - 2,$$

we have:

$$\text{cat } Y = \text{conil } Y = \text{w cat } e' = \text{w cat } Y.$$

Proof. If $\dim Y \leqq m(\text{conil } Y + 2) - 2$ and $m \geqq 2$, then we have $\dim Y \leqq 2m(\text{conil } Y + 1) - 2$. Hence, by above, we have $\text{conil } Y = \text{w cat } e'$. Clearly, we now have that $H^\gamma(Y) = 0$ for $\gamma > m(\text{conil } Y + 2) - 2$. Hence, by (**4**, Theorem 4.3_k), $\text{cat } Y \leqq \text{conil } Y$. Hence, we now have $\text{cat } Y \leqq \text{conil } Y \leqq \text{w cat } e' \leqq \text{w cat } Y \leqq \text{cat } Y$. This proves the result.

Remark. We now give a form of Stasheff's criterion for maps. We recall that Stasheff's criterion states that $\text{nil } X \leqq 1$ if and only if $e\nabla\colon \Sigma\Omega X \vee \Sigma\Omega X \to X$ extends to $\Sigma\Omega X \times \Sigma\Omega X$, where $e\colon \Sigma\Omega X \to X$ is $\tau^{-1}(1_{\Omega X})$.

Let $f\colon X \to Y$ be a map. We shall write $\text{nil} f$ for $\text{nil } \Omega f$. Then we have the following theorem.

Theorem 5. $\text{nil} f \leqq 1$ *if and only if* $fe\nabla\colon \Sigma\Omega X \vee \Sigma\Omega X \to Y$ *extends to* $\Sigma\Omega X \times \Sigma\Omega X$.

Proof. We note that $\text{nil} f \leqq 1$ if and only if $(\Omega f)c \simeq *$, where $c\colon \Omega X \times \Omega X \to \Omega X$ is the basic commutator map for ΩX. Let $i_1, i_2\colon \Sigma\Omega X \to \Sigma\Omega X \vee \Sigma\Omega X$ be the inclusions in the first and second coordinates, respectively. Then we have a generalized Whitehead product $[i_1, i_2] \in [\Sigma(\Omega X \wedge \Omega X), \Sigma\Omega X \vee \Sigma\Omega X]$. Now, $fe\nabla$ extends to $\Sigma\Omega X \times \Sigma\Omega X$ if and only if $fe\nabla[i_1, i_2] = 0$, that is, if and only if $[fe, fe] = 0$. Now, we have a generalized Samelson product $\langle\ ,\ \rangle\colon [\Omega X, \Omega Y] \times [\Omega X, \Omega Y] \to [\Omega X \wedge \Omega X, \Omega Y]$. This satisfies the relation

$$\tau[fe, fe] = \langle \tau(fe), \tau(fe)\rangle = \langle \Omega f, \Omega f\rangle.$$

Hence, $fe\nabla$ extends if and only if $\langle \Omega f, \Omega f\rangle = 0$, that is, if and only if $q^{\#}\langle \Omega f, \Omega f\rangle = 0$, where $q\colon \Omega X \times \Omega X \to \Omega X \wedge \Omega X$. We note that $q^{\#}$ is a monomorphism. Now $q^{\#}\langle \Omega f, \Omega f\rangle = c(\Omega f \times \Omega f) = (\Omega f)c$, where the last c stands for the basic commutator map $\Omega X \times \Omega X \to \Omega X$ and the first c stands for the basic commutator map $\Omega Y \times \Omega Y \to \Omega Y$. Thus, $fe\nabla$ extends if and only if $\text{nil} f \leqq 1$.

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