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THE HYPERPLANES OF $DW(5, q)$ WITH NO OVOIDAL QUAD

BART DE BRUYN*
Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, B-9000 Gent, Belgium
e-mail: bdb@cage.ugent.be

and HARM PRALLE
Institute Computational Mathematics, Technische Universität Braunschweig, Pockelsstr. 14, 38106 Braunschweig, Germany
e-mail: H.Pralle@tu-bs.de

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Abstract. Let Δ be one of the dual polar spaces $DW(5, q)$ or $DH(5, q^2)$. We consider a class of subspaces of Δ, each member of which carries the structure of a near hexagon, and classify all these subspaces. Using this classification, we determine all hyperplanes of $DW(5, q)$ without ovoidal quads.

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1. Introduction. Let Δ be a finite thick dual polar space of rank 3. It is the dual geometry of the singular subspaces of a non-degenerate finite polar space Π of rank 3 with at least 3 points on every line and at least 3 planes through every line. By Tits' classification of polar spaces ([**12**]), Π is either a symplectic polar space $W(5, q)$, one of the orthogonal polar spaces $Q(6, q)$ and $Q^-(7, q)$, or one of the hermitian polar spaces $H(5, q^2)$ or $H(6, q^2)$, for some prime power q. Accordingly, one denotes Δ by $DW(5, q)$, $DQ(6, q)$, $DQ^-(7, q)$, $DH(5, q^2)$ or $DH(6, q^2)$.

The elements of type 1, 2 and 3 of Δ are the planes, lines and points, respectively, of Π. The point-line residue of a type 3-element Q of Δ consists of the singular planes and lines of Π through the corresponding point Q of Π, whence they form a generalized quadrangle. Therefore, the elements of type 1, 2 and 3 of Δ are called *points*, *lines* and *quads*. We denote the collinearity of Δ by $\perp$. The dual polar space Δ is a near hexagon (Shult and Yanushka [**11**]) which means that for every point p and every line L, there exists a unique point on L nearest to p.

A *hyperplane* of Δ is a proper subspace meeting every line. If H is a hyperplane of Δ then, for every quad Q of Δ, either $Q \subset H$ or $Q \cap H$ is a hyperplane of Q. Hence, one of the following possibilities occurs (see Payne and Thas [**6**, 2.3.1]).

- $Q \subset H$: in this case Q is called a *deep quad*.
- $Q \cap H = p^\perp \cap Q$ for some point p of Q: in this case Q is called a *singular quad* with *deep point* p.
- $Q \cap H$ is an ovoid: in this case Q is called an *ovoidal quad*.
- $Q \cap H$ is a proper subquadrangle of Q: in this case Q is called a *subquadrangular quad*.

*Postdoctoral Fellow of the Research Foundation – Flanders.

If H is a hyperplane such that all quads not contained in H are of the same kind, then H is called *uniform*, otherwise *nonuniform*. A uniform hyperplane is called *locally singular*, *locally subquadrangular* or *locally ovoidal* if all its quads not contained in H are singular, subquadrangular or ovoidal, respectively.

The locally singular hyperplanes of dual polar spaces of rank 3 have been classified in Shult [**10**] in the finite case and Pralle [**8**] without the finiteness hypothesis. One class of locally singular hyperplanes are the split Cayley hexagons in the orthogonal dual polar space $DQ(6, \mathbb{F})$ for a field $\mathbb{F}$. The only other class of locally singular hyperplanes is the *singular hyperplane with deepest point p* consisting of the points of the dual polar space at non-maximal distance from p.

For finite dual polar spaces, the locally subquadrangular hyperplanes have been classified in Pasini and Shpectorov [**5**]. The locally ovoidal hyperplanes are precisely the ovoids. Ovoids do not exist in the dual polar spaces $DH(5, q^2)$, $DQ(6, q)$ for q odd, and $DH(6, 4)$, since their quads do not have ovoids. We refer to [**6**] for the nonexistence of ovoids in $Q^-(5, q)$ and $W(q)$, q odd; Brouwer showed by computer that $DH(4, 4)$ has no ovoid. The nonexistence of ovoids in $DW(5, q)$, q even, readily follows from [**6**, 1.8.5], as has been noticed by Shult. See e.g. [**5**, Proposition 2.8]. The nonexistence of ovoids in $DW(5, q)$, q odd, has been shown in Cooperstein and Pasini [**2**]. The existence of ovoids in the dual polar spaces $DH(6, q^2)$, $q \geq 3$, and $DQ^-(7, q)$ is still an open problem.

Pralle [**7**] showed that every nonuniform hyperplane of Δ must contain at least one singular quad. All nonuniform hyperplanes without subquadrangular quads have been determined in Pralle [**8**]. Among the finite thick dual polar spaces of rank 3 only the quads of $DW(5, q)$ and $DH(5, q^2)$ have subquadrangles as hyperplanes. All hyperplanes of $DH(5, q^2)$ have been classified in De Bruyn and Pralle [**3**] and [**4**]. In Proposition 4.2, we determine all nonuniform hyperplanes of $DW(5, q)$ without ovoidal quads. From [**2**], [**5**], [**7**], [**8**], [**10**] and Proposition 4.2 of the present paper, our main result follows.

THEOREM 1.1. *If H is a hyperplane of $DW(5, q)$, then precisely one of the following holds.*

- *H is a singular hyperplane.*
- *There exist a quad Q and a subgrid G of Q such that $H = \bigcup_{x \in G} x^\perp$.*
- *There exist a quad Q and an ovoid O in Q such that $H = \bigcup_{x \in O} x^\perp$.*
- *The order q is even and the points and lines of H build a split Cayley hexagon.*
- *H is the (up to isomorphism) unique locally subquadrangular hyperplane of* $DW(5, 2)$.
- *There are a point p and a set O of points at distance 3 from p which meets every line at distance 2 from p, such that $H = p^\perp \cup O$.*
- *We have $q = 2$ and H is a hyperplane on 81 points mentioned in (e) of Proposition 4.2.*
- *There exist a singular, a subquadrangular and an ovoidal quad.*

In Section 2, we define a class $\mathcal{C}$ of subspaces of $DW(5, q)$ and $DH(5, q^2)$. Each member S of $\mathcal{C}$ satisfies the following properties: (i) S contains two disjoint quads, (ii) S is the union of a certain family of quads, (iii) the points and lines in S define a near hexagon. Section 3 is devoted to the classification of the subspaces of $\mathcal{C}$ and several new examples of near hexagons are developed. Using this classification in Section 4, we determine all hyperplanes of $DW(5, q)$ without ovoidal quads.

2. A class of subspaces in $DW(5, q)$ and $DH(5, q^2)$ containing two disjoint quads. The following lemma is straightforward. See Brouwer et al. [**1**, Lemma 3.1].

LEMMA 2.1. *Let S be a subspace of a dual polar space Δ of rank 3 with the property that, for every point $x \in S$, there exists a quad $Q_x \subseteq S$ through x. Then the points and lines contained in S define a near polygon.*

Several nice classes of near hexagons arise in the way described in Lemma 2.1. See [**1**]. In the present paper, we determine all subspaces of $DW(5, q)$ and $DH(5, q^2)$ belonging to a certain class $\mathcal{C}$ of subspaces. All of them satisfy the conditions of Lemma 2.1 and give rise to near hexagons.

If S is a subspace of a dual polar space Δ of rank 3 such that for every point $x \in S$, there exists a quad $Q_x \subseteq S$ through x, then one of the following two possibilities occurs.

- There exists a point in Δ that is contained in every quad $Q \subseteq S$.
- There exist two disjoint quads Q_1 and Q_2 contained in S.

The union of a number of quads through a given point is always a subspace. Lemmas 2.2, 2.3 and Proposition 3.1 determine all subspaces of $DW(5, q)$ containing two disjoint quads. All of these subspaces belong to the afore-mentioned class $\mathcal{C}$.

Let Π be a polar space isomorphic to either $W(5, q)$ or $H(5, q^2)$ embedded in a 5-dimensional projective space $\mathbb{P}$ and let Δ denote the corresponding dual polar space. For every quad Q of Δ and every point x not in Q, let $\pi_Q(x)$ denote the unique point of Q collinear with x. Let Q_1 and Q_2 denote two disjoint quads of Δ. The quad Q_i, $i \in \{1, 2\}$, corresponds with a point x_i of the polar space Π. Let $x_1, \dots, x_{q+1}$ denote the $q + 1$ points of Π on the line x_1x_2 of $\mathbb{P}$ and let Q_i, $i \in \{3, \dots, q + 1\}$, denote the quad of Δ corresponding with x_i. Every line meeting Q_1 and Q_2 also meets every Q_i, $i \in \{3, \dots, q + 1\}$.

Let A denote the set of all quads meeting Q_1 and Q_2. It corresponds to the set of points of Π contained in $(x_1x_2)^\zeta$ where ζ denotes the polarity defining Π. If $R \in A$, then R meets each Q_i, $i \in \{1, \dots, q + 1\}$, in a line and $R \cap (Q_1 \cup \cdots \cup Q_{q+1})$ is a subgrid G_R of R. If x is a point of Δ not contained in $Q_1 \cup \cdots \cup Q_{q+1}$, then x is contained in the unique element of A that is the quad containing x and the line through $\pi_{Q_1}(x)$ meeting Q_2. We have seen that the quads of A partition the set of points of Δ not contained in $Q_1 \cup \cdots \cup Q_{q+1}$. Every element R of A corresponds with a point x_R of Π that is contained in the three dimensional subspace $\mathbb{P}' = (x_1x_2)^\zeta$ of $\mathbb{P}$. Two quads R_1 and R_2 of A meet if and only if $x_{R_1} \in x_{R_2}^\zeta$. For a subset B of A, we define

$$S_B := (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) \cup \bigcup_{R \in B} R,$$

and X_B denotes the set of all points x_R, where R is an element of B.

LEMMA 2.2. *If $\Delta \cong DW(5, q)$, then S_B is a subspace of Δ if and only if $x_1x_2 \subseteq X_B$, for all $x_1, x_2 \in X_B$ with $x_1 \notin x_2^\zeta$. Let $\Delta \cong DH(5, q^2)$. Then S_B is a subspace of Δ if and only if $x_1x_2 \cap H(5, q^2) \subseteq X_B$, for all $x_1, x_2 \in X_B$ with $x_1 \notin x_2^\zeta$.*

Proof. Let L denote an arbitrary line of Δ. There are four possibilities.

- L is contained in $Q_1 \cup \cdots \cup Q_{q+1}$ and so in S_B.
- L intersects $Q_1 \cup \cdots \cup Q_{q+1}$ in a unique point x, and there exists a unique quad R of A containing L. If R is contained in B, then $L \subset S_B$. If R is not contained in B, then $L \cap S_B = \{x\}$.

• L is disjoint from $Q_1 \cup \cdots \cup Q_{q+1}$ and contained in a certain quad R of A. This case can only occur if $\Delta \cong DH(5, q^2)$. If R is contained in B, then $L \subseteq S_B$. If R is not contained in B, then L is disjoint from S_B.

• L is disjoint from $Q_1 \cup \cdots \cup Q_{q+1}$ and not contained in any quad of A. Then the points of L are contained in mutually disjoint elements of A that correspond with a hyperbolic line of $\mathbb{P}'$. Conversely, every hyperbolic line of $\mathbb{P}'$ corresponds with a set of $q+1$ mutually disjoint quads of A and there always exists a line outside $Q_1 \cup \cdots \cup Q_{q+1}$ meeting these quads.

Thus S_B is a subspace if and only if every hyperbolic line of $\mathbb{P}'$ has either 0, 1 or $q+1$ points in common with X_B. This proves the lemma. □

DEFINITION. Let $\mathcal{C}$ denote the set of all subspaces of the form S_B, where B is a subset of A.

LEMMA 2.3. *Suppose that $\Delta \cong DW(5, q)$. If S is a subspace of Δ containing the disjoint quads Q_1 and Q_2, then S belongs to $\mathcal{C}$.*

Proof. Let R denote an arbitrary element of A. The subgrid G_R of R is contained in S and $R \cap S$ is a subspace of R. If $R \setminus G_R$ contains a point of S, then R is contained in S, and S is of the form S_B, for some subset B of A. □

In the following section, we determine all subspaces of $\mathcal{C}$, or equivalently, all sets X_B satisfying the conditions of Lemma 2.2.

REMARK. If S is a subspace of $\Delta \cong DH(5, q^2)$ containing the disjoint quads Q_1 and Q_2, then the set X_B with B the set of all quads that are contained in S and intersect Q_1 and Q_2 in lines, still satisfies the conditions of Lemma 2.2.

3. The classification of the subspaces of $\mathcal{C}$.

The classification follows from Lemma 2.2 and Propositions 3.1 and 3.2.

PROPOSITION 3.1. *Let ζ denote a symplectic polarity of* PG(3, q) *and let X be a set of points of* PG(3, q) *with the property that $x_1x_2 \subseteq X$ $(*)$ for every two points x_1 and x_2 of X with $x_1 \notin x_2^\zeta$. Then one of the following cases occurs:*
(a) *$X \subseteq L$, for some totally isotropic line L;*
(b) *$X = L$, for some hyperbolic line L;*
(c) *$X = L \cup \{x\}$, for some hyperbolic line L and some point $x \in L^\zeta$;*
(d) *$X = L \cup L^\zeta$, for some hyperbolic line L;*
(e) *$X = p^\zeta \setminus \{p\}$, for some point p of* PG(3, q)*;*
(f) *$X = p^\zeta$, for some point p of* PG(3, q)*;*
(g) *$X =$* PG(3, q)*;*
(h) *$q = 2$ and X is the complement of an ovoid of the generalized quadrangle $Q_\zeta \cong W(2)$ associated with ζ.*

Proof. If $x_1 \in x_2^\zeta$, for all points x_1 and x_2 of X, then X is as in (a). Suppose that there exist points $x_1, x_2 \in X$ such that $x_1 \notin x_2^\zeta$, and let L denote the line through x_1 and x_2. If X has no points outside $L \cup L^\zeta$, then one of the cases (b), (c) or (d) occurs.

If there exists a point $x_3 \in X$ not contained in $L \cup L^\zeta$, then let p be the point of PG(3, q) such that $p^\zeta = \langle x_1, x_2, x_3 \rangle$. Property $(*)$ implies that every point of $p^\zeta \setminus \{p\}$ is contained in X. If all points of X are contained in p^ζ, then either case (e) or (f) occurs. Now, suppose that there exists a point x_4 in X not contained in p^ζ. If $p \in X$, then

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every point of $\mathrm{PG}(3, q) \setminus x_4^\zeta$ belongs to X by property $(*)$. Also every point of x_4^ζ is contained in X by property $(*)$, since through every point $y \in x_4^\zeta$ there exists a line not contained in $x_4^\zeta \cup y^\zeta$. Hence, if $p \in X$, then $X = \mathrm{PG}(3, q)$. Suppose therefore that $p \notin X$ and consider the complement $X^C := \mathrm{PG}(3, q) \setminus X$ of X. By property $(*)$, every point of X^C is contained in either the line $x_4 p$ or the plane x_4^ζ. Now, let K be a line through p different from $x_4 p$ and not contained in p^ζ. Then K contains at most two points of X^C (namely p and $K \cap x_4^\zeta$). By property $(*)$, it then follows that (i) $q + 1 \leq 3$ or $q = 2$, and (ii) $K \cap x_4^\zeta \in X^C$. One easily verifies that $X^C = (x_4^\zeta \setminus ((x_4^\zeta \cap p^\zeta) \cup \{x_4\})) \cup (px_4 \setminus \{x_4\})$ and that the set X^C is an ovoid of $Q_\zeta \cong W(2)$. □

Proposition 3.2. *Let H denote a hermitian variety in* $\mathrm{PG}(3, q^2)$ *and let ζ denote the hermitian polarity of* $\mathrm{PG}(3, q^2)$ *associated with H. Let X be a set of points of H with the property that $x_1 x_2 \cap H \subseteq X$ $(*)$, for every two points x_1 and x_2 of X with $x_1 \notin x_2^\zeta$. The lines and points lying in $H(3, q^2)$ define a generalized quadrangle $Q(5, q)$ and X corresponds with a set X' of lines of $Q(5, q)$. One of the following cases occurs.*

(a) *X is a (possibly empty) set of points on a line of H. X' is a (possibly empty) set of lines through a given point of $Q(5, q)$.*

(b) *$X = L \cap H$ for some secant line L (i.e. $|L \cap H| = q + 1$). X' is a regulus of $Q(5, q)$.*

(c) *$X = (L \cap H) \cup \{x\}$, for some secant line L and some point $x \in L^\zeta \cap H$. X' consists of a regulus of $Q(5, q)$ together with a line of its opposite regulus.*

(d) *$X = (L \cup L^\zeta) \cap H$, for some secant line L. X' consists of the lines contained in a subgrid of order $(q, 1)$ of $Q(5, q)$.*

(e) *$X = \alpha \cap H$, for some nontangent plane α. X' is a regular spread of $Q(5, q)$.*

(f) *$X = \alpha \cap H$, for some tangent plane α. X' is the set of lines of $Q(5, q)$ having nonempty intersection with a given line of $Q(5, q)$.*

(g) *$X = (p^\zeta \cap H) \setminus \{p\}$, for some point p of H. X' is the set of lines of $Q(5, q)$ intersecting a given line in a unique point.*

(h) *$X = B$ where B is a Baer-subplane which is contained in $p^\zeta \cap H$, for some point p of B. There exists a subquadrangle $Q \cong Q(4, q)$ in $Q(5, q)$, and X' consists of all lines of Q having nonempty intersection with a given line of Q.*

(i) *$X = B \setminus \{p\}$ with p a point of H and B a Baer-subplane of p^ζ through p completely contained in H. There exists a subquadrangle $Q \cong Q(4, q)$ in $Q(5, q)$ and X' consists of all lines of Q intersecting a given line of Q in a unique point.*

(j) *X is a 3-dimensional Baer-subspace of* $\mathrm{PG}(3, q^2)$ *contained in H. X' consists of all lines contained in a subquadrangle $Q \cong Q(4, q)$ of $Q(5, q)$.*

(k) *$X = H$. X' is the whole set of lines of $Q(5, q)$.*

(l) *$q = 2$ and X' consists of all 18 lines which are contained in three mutually disjoint grids of $Q(5, 2)$.*

(m) *$q = 2$ and there exist a subquadrangle $Q \cong Q(4, 2)$ of $Q(5, 2)$ and a spread S in Q such that X' consists of all lines of Q that are not contained in S.*

In the remainder of this section, we prove Proposition 3.2.

Lemma 3.3. *If α is a nontangent plane such that $\alpha \cap H$ contains three noncollinear points of X, then $\alpha \cap H$ is completely contained in X.*

Proof. This follows from property $(*)$ and the fact that every subspace of a Steiner system $S(2, q + 1, q^3 + 1)$ is a point, a line or the whole Steiner system. For, if U is a proper subspace of $S(2, q + 1, q^3 + 1)$ containing a line L and a point not contained in

L, then $|U| \geq 1 + q \cdot |L| = 1 + q + q^2$, and if there is a point not belonging to U, then $q^3 + 1 \geq 1 + q \cdot |U|$, a contradiction. □

LEMMA 3.4. *If α is a tangent plane such that $\alpha \cap X = \alpha \cap H$, then X is of type* (f) *or* (k).

Proof. If all points of X are contained in α, then X is of type (f). Suppose that there exists a point $x \in X$ not contained in α. Let x' denote an arbitrary point of H not contained in x^ζ and let x'' denote the unique point of α on the line xx'. If $x'' \in H$, then by property $(*)$, also $x' \in H$. Suppose $x'' \notin H$. There are at least $q^2 - q - 1 \geq 1$ nontangent planes through xx' not containing the point α^ζ. If γ is such a plane, then every point of $\gamma \cap H$ is contained in X, by Lemma 3.3. In particular, also x' belongs to X, and every point of H outside x^ζ belongs to X. Now, let y denote an arbitrary point of $x^\zeta \cap H$. There exists a line through y not contained in $x^\zeta \cup y^\zeta$ and, by property $(*)$, it follows that $y \in X$. Hence $X = H$. □

LEMMA 3.5. *If α is a tangent plane such that $\alpha \cap X = (\alpha \cap H) \setminus \{\alpha^\zeta\}$, then X is of type* (g).

Proof. Suppose the contrary. Then there exists a point $x \in X$ not contained in α. The same argument, as in the proof of Lemma 3.4, shows that every point of H not contained in $x\alpha^\zeta \cup x^\zeta$ is contained in X. Let L denote a line through α^ζ not contained in α such that $L \cap x^\zeta \notin H$. The line L is a secant line and property $(*)$ implies that $\alpha^\zeta \in X$, a contradiction. □

LEMMA 3.6. *Suppose that $q = 2$. If α is a nontangent plane such that $\alpha \cap H = \alpha \cap X$, then X is of type* (e), (k) *or* (l).

Proof. The points of $\alpha \cap H$ correspond with a regular spread S of $Q(5, 2)$. The lines and reguli of S define an affine plane of order 3. If $X' = S$, then X is of type (e). Suppose now that there exists a line L in $X' \setminus S$. Then there exists a unique partition $\{G_1, G_2, G_3\}$ of $Q(5, 2)$ in three subgrids such that each line of $S \cup \{L\}$ is contained in some grid of the partition. Let T denote the set of 18 lines contained in one of the grids of the partition. By property $(*)$ it follows that $T \subseteq X'$. If $X' = T$, then X is of type (l). If T is a proper subset of X', then property $(*)$ implies that X' is the whole set of lines of $Q(5, 2)$. In this case X is of type (k). □

LEMMA 3.7. *Suppose that $q \neq 2$. If α is a nontangent plane such that $\alpha \cap H = \alpha \cap X$, then X is of type* (e) *or* (k).

Proof. If all points of X are contained in α, then X is of type (e). Now, suppose that there exists a point $x \in X$ not contained in α. Let x' denote an arbitrary point of H not contained in x^ζ and let x'' denote the unique point of α on the line xx'. If $x'' \in H$ then, by property $(*)$, also $x' \in H$. Suppose that $x'' \notin H$. There are at least $q^2 + 1 - 2(q + 1) \geq 1$ nontangent planes through xx' that intersect α in a secant line. If γ is such a plane, then every point of $\gamma \cap H$ is contained in X, by Lemma 3.3. In particular, also x' belongs to X, so that every point of H outside x^ζ is contained in X. A similar argument as in Lemma 3.4 shows that also every point of $x^\zeta \cap H$ belongs to X. Hence X is of type (k). □

In the sequel, we suppose that X is not of type (a), (b), (c), (d), (e), (f), (g), (k) or (l). Then there exist three points x_1, x_2 and x_3 such that $x_2 \notin x_1^\zeta$, $x_3 \notin x_1x_2$ and $x_3 \notin (x_1x_2)^\zeta$. By Lemmas 3.3, 3.6 and 3.7, the plane $\langle x_1, x_2, x_3 \rangle$ must be a tangent

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plane p^ζ. The plane p^ζ has a unique Baer-subplane B through p, x_3 and $x_1x_2 \cap H$, and B is completely contained in H. By property ($*$), every point of $B \setminus \{p\}$ is contained in X. The set $B \setminus \{p\}$ corresponds with a set Y' of lines of a subquadrangle $Q \cong Q(4, q)$ of $Q(5, q)$ intersecting a given line L of Q in a unique point. If all lines of X' are contained in Q, then X is of type (h), (i), (j) or (m) corresponding with the respective cases (f), (e), (g) and (h) of Proposition 3.1. Suppose therefore that X' contains a line L' not contained in Q. If L' meets L, then property ($*$) implies that every point of $p^\zeta \setminus \{p\}$ belongs to X (use e.g. similar countings as in the proof of Lemma 3.3), contradicting Lemmas 3.4 and 3.5. If L' intersects Q in a unique point not contained in L, then we can choose disjoint lines M and N in Y' such that L' is disjoint from any line of $\{M, N\}^{\perp\perp}$. Since there is no line in $Q(5, q)$ meeting L', M and N, they correspond with a set of three points on $H(3, q^2)$ generating a nontangent plane. This contradicts Lemmas 3.3, 3.6 and 3.7 and we have proved Proposition 3.2.

4. Application to hyperplanes of dual polar spaces.

PROPOSITION 4.1. *If H is a hyperplane of $DW(5, q)$ containing two disjoint deep quads, then one of the following cases occurs.*

(a) *There exists a quad Q and a subgrid G of order $(q, 1)$ in Q such that $H = \bigcup_{x \in G} x^\perp$.*

(b) *$q = 2$ and H is a locally subquadrangular hyperplane of $DW(5, 2)$.*

(c) *$q = 2$ and H is a hyperplane of $DW(5, 2)$ with 81 points corresponding with possibility* (d) *of Proposition* 3.1.

Proof. We use the same notations as in Section 2. Let Q_1 and Q_2 denote two disjoint quads contained in H. Since H is a subspace, it is of the form S_B, for some subset B of S; see Lemma 2.3. The set X_B must correspond with one of the 8 possibilities mentioned in Proposition 3.1. But there exist additional restrictions on the set X_B. Since H is a hyperplane, each line of Δ meets H, which implies that every hyperbolic line of PG(3, q) meets X_B. (See the proof of Lemma 2.2.) Possibility (f) of Proposition 3.1 gives rise to a hyperplane of type (a). Possibility (h) of Proposition 3.1 gives rise to a hyperplane of type (b). See also Pasini and Shpectorov [**5**]. Possibility (d) of Proposition 3.1 gives rise to a hyperplane only when q is equal to 2. All the remaining possibilities do not give rise to hyperplanes. □

REMARK. The example mentioned in (c) of Proposition 4.1 is Example 6 of [**9**].

PROPOSITION 4.2. *If H is a hyperplane of $DW(5, q)$ not containing ovoidal quads, then one of the following cases occurs.*

(a) *H is a singular hyperplane.*

(b) *There exist a quad Q and a subgrid G of order $(q, 1)$ in Q such that $H = \bigcup_{x \in G} x^\perp$.*

(c) *q is even and the points and lines contained in H define a split Cayley hexagon.*

(d) *$q = 2$ and H is a locally subquadrangular hyperplane of $DW(5, 2)$.*

(e) *$q = 2$ and H is a hyperplane of $DW(5, 2)$ with 81 points corresponding with possibility* (d) *in Proposition* 3.1.

Proof. If there exists no subquadrangular quad, then each quad is either deep or singular and either case (a) or (c) occurs by Shult [**10**]. If there exist two disjoint deep quads, then either case (b), (d) or (e) occurs by Proposition 4.1. Now, suppose that there exists a subquadrangular quad Q and that any two deep quads meet. Let G denote the subquadrangle $Q \cap H$. For every point x of G, let A_x denote the number of lines through x that are contained in H but not in Q. For every point x of G, we choose a line $L_x \subset Q$ through x not contained in G. There are q (nonovoidal) quads through L_x different from Q in each of which there exists a line through x contained in H. Hence $A_x \geq q$, for every point x of G. Now, consider two disjoint lines K_1 and K_2 in G. It is impossible that there exist deep quads through both K_1 and K_2, for otherwise we should have two disjoint deep quads. Without loss of generality, we may suppose that there exist no deep quads through K_1. Now, consider the number $N = \sum_{x \in K_1} A_x$ which is at least equal to $q(q+1)$. A quad through K_1 different from Q contributes q to N if it is singular and $q+1$ if it is subquadrangular. Since there are precisely q quads through K_1 different from Q, it follows that $A_x \leq q(q+1)$, and we conclude the following statements.

- Every quad through K_1 is subquadrangular.
- Every quad through L_x, where $x \in K_1$, that is different from Q is singular.

Now, let Q' denote an arbitrary subquadrangular quad through K_1 different from Q. Let x_1 and x_2 denote two different points of K_1 and let K_i', $i \in \{1, 2\}$, denote the unique line of $Q' \cap H$ through x_i different from K_1. As before there exists an $i \in \{1, 2\}$ such that every quad through K_i' is subquadrangular. But this is impossible since the quad $\langle L_{x_i}, K_i' \rangle$ is singular. □

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