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ON CONVERGENCE OF VECTOR VALUED PRAMARTS AND SUBPRAMARTS

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0. Introduction. In [**15**] Millet and Sucheston introduced the notion of pramart and subpramart indexed by directed sets, generalizing that of martingale and submartingale, and studied their properties. In particular convergence theorems were proved. In this note we obtain convergence theorems for analogous Banach-valued processes.

Let E be a Banach lattice with the Radon-Nikodym property. Let $(X_t, \mathscr{F}_t, t \in J)$ be an E^+-valued subpramart of class (d). Precise definitions are given below (Section 1).

In [**10**] for $J = \mathbf{N}$, Egghe proved a subpramart convergence theorem under the additional assumption that there is a subsequence $\{n_k\} \subseteq \mathbf{N}$ such that $(\int_A X_{n_k} dP)$ converges weakly for each $A \in U_n \mathscr{F}_n$.

In [**19**] for $J = \mathbf{N}$, Slaby proved a subpramart convergence theorem assuming that the Banach lattice E has an unconditional basis and no subspace of E is isomorphic to c_0. In [**20**] he drops the assumption about the unconditional basis.

In the present work, independently of [**20**], we prove the convergence result in the original, more general, setting of subpramarts adapted to a stochastic basis $(\mathscr{F}_t, t \in J)$ satisfying the Vitali condition (Theorem 2.5).

The method also provides a partial answer to the pramart convergence problem (Theorem 3.2). Specifically we prove that pramarts of class (d) taking values in a Banach space E converge if E is a separable dual. The same result for E with the Radon-Nikodym property and weakly sequentially complete was obtained in [**20**].

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1. Definitions and basic notations. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and J a directed set filtering to the right with a countable cofinal subset. The relations are modulo sets of measure zero. The words almost surely may or may not be omitted. Let $(\mathscr{F}_t, t \in J)$ be an increasing family of sub-σ-fields of $\mathscr{F}$. A function $\tau:\Omega \rightarrow J$ is a *simple stopping time* of $(\mathscr{F}_t)$ if it takes finitely many values, and $\{\tau = t\} \in \mathscr{F}_t$ for all $t \in J$. Let T denote the set of all simple stopping times; under the

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natural order T is a set filtering to the right. For $\sigma \in T$, write E^{σ} for $E^{\mathscr{F}_{\sigma}}$.

Let $(E, \|\cdot\|)$ be a Banach lattice and $(X_t, t \in J)$ a family of E-valued, Bochner integrable random variables adapted to $(\mathscr{F}_t)$. The stochastic (in probability) limit is denoted by s-limit. The family (X_t) is called a *subpramart* if

$$s - \lim_{\sigma \leqq \tau; \sigma, \tau \in T} (X_{\sigma} - E^{\sigma}(X_{\tau}))^{+} = 0,$$

i.e., for every $\epsilon > 0$ there exists $\sigma_0 \in T$ such that $\sigma_0 \leqq \sigma \leqq \tau$ implies

$$P(\{ \| (X_{\sigma} - E^{\sigma}(X_{\tau}))^{+} \| > \epsilon \}) \leqq \epsilon.$$

Clearly for $J = \mathbf{N}$ or $\mathbf{R}^{+}$ every submartingale is a subpramart. Moreover, every positive adapted family $(Z_t, t \in J)$ with

$$s - \lim_{\tau \in T} Z_{\tau} = 0$$

is obviously a subpramart.

Let $(E')^{+}$ denote the positive cone of the dual E' of E. If (X_t) is a positive subpramart, then $(\|X_t\|)$ is a real-valued positive subpramart, i.e., satisfies

$$s - \lim_{\sigma \leqq \tau; \tau \in T} (\|X_{\sigma}\| - E^{\sigma}\|X_{\tau}\|)^{+} = 0.$$

Indeed if $\sigma \leqq \tau$, $\sigma, \tau \in T$, then for every $\epsilon > 0$

$$\begin{aligned} \{ \|(X_{\sigma} - E^{\sigma}(X_{\tau}))^{+}\| < \epsilon \} &\subseteq \{ (\|X_{\sigma}\| - \|E^{\sigma}(X_{\tau})\|)^{+} < \epsilon \} \\ &\subseteq \{ (\|X_{\sigma}\| - E^{\sigma}\|X_{\tau}\|)^{+} < \epsilon \}. \end{aligned}$$

Also $(\chi(X_t))$ is a real-valued subpramart for all $\chi \in (E')^{+}$.

For an arbitrary Banach space $(E, \|\cdot\|)$, the family $(X_t, t \in J)$ is called a *pramart* if

$$s - \lim_{\sigma \leqq \tau; \sigma, \tau \in T} (X_{\sigma} - E^{\sigma}(X_{\tau})) = 0,$$

i.e., for every $\epsilon > 0$ there exists $\sigma_0 \in T$ such that $\sigma_0 \leqq \sigma \leqq \tau$ implies

$$P(\{ \|X_{\sigma} - E^{\sigma}(X_{\tau})\| > \epsilon \}) < \epsilon.$$

Clearly the class of pramarts is closed under linear combinations and contains the class of uniform amarts [**2**, **7**].

If (X_t) is a pramart then $(\|X_t\|)$ is a real-valued positive subpramart. Indeed, if $\sigma \leqq \tau$, $\sigma, \tau \in T$, then for every $\epsilon > 0$

$$\{ \|X_{\sigma} - E^{\sigma}(X_{\tau})\| < \epsilon \} \subseteq \{ (\|X_{\sigma}\| - E^{\sigma}\|X_{\tau}\|)^{+} < \epsilon \}.$$

Also $(\chi(X_t))$ is a real-valued pramart for all $\chi \in E'$.

A stochastic basis $(\mathscr{F}_t, t \in J)$ satisfies the *Vitali condition* V $(= \mathrm{V}_\infty)$ if for every adapted family of sets (A_t) and for every $\epsilon > 0$, there exists a simple stopping time $\tau \in T$ such that

$$P(e - \limsup_J A_t \backslash A_\tau) < \epsilon.$$

(The above formulation of the Vitali condition was given by Millet and Sucheston [**16**] page 344; see also [**18**] page 99 for an equivalent form not involving stopping times, first introduced by Krickeberg [**13**].)

We now state some known results for reference.

THEOREM 1.1. ([**15**], Theorem 3.3). *Let $(X_t, t \in J)$ be a real-valued, positive, integrable process. Then (X_t) is a subpramart if and only if there exists a positive submartingale $(R_\tau, \mathscr{F}_\tau, T)$ such that for every t, $R_t \leqq X_t$* a.s. *and if $Z_t = X_t - R_t$ then*

$$s - \lim_{\tau \in T} Z_\tau = 0.$$

R_t is given by $\inf_{\tau \geqq t} E^t(X_\tau)$.

Remark. It was observed in [**15**] that to say $(R_\tau, \mathscr{F}_\tau, T)$ is a submartingale it is the same as saying that $(R_t, \mathscr{F}_t, J)$ is a submartingale with the optional sampling property.

THEOREM 1.2. ([**15**], Theorem 3.7). *Let (X_t) be a real-valued subpramart which satisfies the assumption* (d):

(d) $\liminf_J E(X_t^+) + \liminf_J E(X_t^-) < \infty.$

Then the net $(X_\tau, \tau \in T)$ converges stochastically to an integrable random variable.

THEOREM 1.3. ([**15**], Theorem 4.1, see also [**17**] page 45). *Let E be a Banach space. Let $f(\sigma, \tau)$ be an E-valued family of $\mathscr{F}_\sigma$ measurable random variables defined for $\sigma, \tau \in T$, $\sigma \leqq \tau$. Assume that for every $t \in J$*

$$1_{\{\sigma = t\}} f(\sigma, \tau) = 1_{\{\sigma = \tau\}} f(t, \tau) \text{ and}$$

if $A \in \mathscr{F}_s$ and $\tau = \tau'$ on A,

$$1_A f(s, \tau) = 1_A f(s, \tau').$$

If $(f(\sigma, \tau))$ converges stochastically to f_∞ and V *holds then, $(f(\sigma, \tau))$ converges almost surely to f_∞.*

2. Convergence theorems for subpramarts. The following proposition is an extension of Neveu's Lemma (see [**18**], page 109) to directed index sets. The proof is analogous and therefore omitted. We use Theorems 1.2 and 1.3 above.

PROPOSITION 2.1. *Assume that* V *holds. Let* $\{(X_t^i, t \in J), i \in I\}$ *be a countable family of real-valued integrable submartingales that are subpramarts (e.g., submartingales with the optional sampling property). Assume also that*

$$\sup_J E[\sup_I X_t^i] < \infty.$$

Then each of the submartingales converges a.s. *to an integrable limit* $X^i (i \in I)$ *and*

$$\liminf_J (\sup_I X_t^i) = \sup_I X^i \text{ a.s.}$$

If E is a separable Banach lattice, then there exists a countable set D in $B(E') \cap (E')^+$ such that for all x in E^+ $\|x\| = \sup_D \chi(x)$, and $\sup_D \chi(x) \leqq \|x\|$ for any x in E, where $B(E')$ denotes the unit ball of E'. The set D is called the *norming subset* of $(E')^+$.

The case $J = \mathbf{N}$ of the following lemma is due to L. Egghe [**10**]. Here we give a slightly different proof.

LEMMA 2.2. *Let* E *be a separable Banach lattice, and* $(X_t, t \in J)$ *an* E*-valued positive subpramart. For each* $\chi \in D$ *let*

$$R_t^\chi + Z_t^\chi = \chi(X_t)$$

be the decomposition of the real positive subpramart $(\chi(X_t))$ *(Theorem* 1.1).

Then

(u) $\quad s - \lim_T \sup_D Z_\sigma^\chi = 0.$

If V *holds then* $\lim_T \sup_D Z_\sigma^\chi = 0$ a.s.

Proof. Suppose that (u) fails. Then given any stopping time σ_0 there exists $\epsilon > 0$ and $\sigma \geqq \sigma_0$ such that

$$P(\{\sup_D Z_\sigma^\chi \geqq \epsilon\}) > 2\epsilon.$$

Now we can find a finite subset $\bar{D}$ of D, say $\bar{D} = \{\chi_1, \ldots, \chi_k\}$, such that

[A] $\quad P(\{\sup_{\bar{D}} Z_\sigma^\chi \geqq \epsilon\}) = P(\{\sup_{\bar{D}} (\chi(X_\sigma) - R_\sigma^\chi) \geqq \epsilon\}) \geqq \epsilon.$

By [**18**], VI-1-1, p. 121, and using Theorem 1.1 there exists a sequence (σ_n^χ) in T such that

$$R_\sigma^\chi = \inf_{n \in \mathbf{N}} E^\sigma(\chi(X_{\sigma_n^\chi})), \text{ a.s.}$$

$$= \lim_{n \to \infty} \inf_{k=1,\ldots,n} E^\sigma(\chi(X_{\sigma_k^\chi})).$$

Using a classical localization procedure (see [**15**], p. 93) we find a sequence (τ_n^χ) in T such that

$$R_\tau^\chi = \lim_{n\to\infty}\downarrow E^\sigma(\chi(X_{\tau_n^\chi})), \text{ a.s.}$$

for each $\chi \in (E')^+$. Thus for each $\chi \in \bar{D}$ there exists a stopping time $\tau_n^\chi = \tau_\chi$ such that

$$P\left(\left\{E^\sigma(\chi(X_{\tau_\chi}) - R_\sigma^\chi > \frac{\epsilon}{2}\right\}\right) < \frac{\epsilon}{2k}.$$

Therefore

$$[\text{B}] \quad P\left(\left\{\sup_{\bar{D}}(E^\sigma(\chi(X_{\tau_\chi})) - R_\sigma^\chi) \geqq \frac{\epsilon}{2}\right\}\right) \leqq \frac{\epsilon}{2}.$$

But

$$\{\sup_D(\chi(X_\sigma) - R_\sigma^\chi) \geqq \epsilon\}$$
$$\subseteq \left\{\sup_{\bar{D}}(\chi(X_\sigma) - E^\sigma(\chi(X_{\tau_\chi}))) \geqq \frac{\epsilon}{2}\right\}$$
$$\cup \left\{\sup_{\bar{D}}(E^\sigma(\chi(X_{\tau_\chi})) - R_\sigma^\chi) \geqq \frac{\epsilon}{2}\right\}.$$

Because of [A] and [B] we have

$$\frac{\epsilon}{2} \leqq P\left(\left\{\sup_{\bar{D}}(\chi(X_\sigma) - E^\sigma(\chi(X_{\tau_\chi}))) \geqq \frac{\epsilon}{2}\right\}\right).$$

If

$$X_i(\sigma) \equiv \chi_i(X_\sigma) - E^\sigma(\chi_i(X_{\tau_{\chi_i}})), \quad i = 1, 2, \ldots, k,$$

we can then define the following sets

$$A_1 = \left\{X_1(\sigma) \geqq \frac{\epsilon}{2}\right\}$$

$$A_2 = \left\{X_1(\sigma) < \frac{\epsilon}{2}, X_2(\sigma) \geqq \frac{\epsilon}{2}\right\}$$

$$\vdots$$

$$A_k = \left\{X_1(\sigma) < \frac{\epsilon}{2}, \ldots, X_{k-1}(\sigma) < \frac{\epsilon}{2}, X_k(\sigma) \geqq \frac{\epsilon}{2}\right\}.$$

Then the A_i's, $i = 1, 2, \ldots, k$ are disjoint, each belongs to $\mathscr{F}_\sigma$ and

$$\bigcup_{i=1}^k A_i = \left\{\sup_{\bar{D}}(\chi(X_\sigma) - E^\sigma(\chi(X_{\tau_\chi}))) \geqq \frac{\epsilon}{2}\right\} = A.$$

Define

$$\tau = \begin{cases} \tau_{\chi_i} \text{ on } A_i \\ \sigma \text{ on } A^{C}. \end{cases}$$

Then by the localization property, $\tau \in T$, $\sigma \leqq \tau$ and

$$A = \left\{ \sup_{\bar{D}}(\chi(X_\sigma) - E^\sigma(\chi(X_\tau))) \geqq \frac{\epsilon}{2} \right\}.$$

Hence

$$\frac{\epsilon}{2} \leqq P\left(\left\{ \sup_{\bar{D}}(\chi(X_\sigma) - E_\sigma(\chi(X_\tau))) \geqq \frac{\epsilon}{2} \right\} \right)$$

$$= P\left(\left\{ \sup_{\bar{D}}\chi(X_\sigma - E^\sigma(X_\tau)) \geqq \frac{\epsilon}{2} \right\} \right)$$

$$\leqq P\left(\left\{ \sup_{D}\chi(X_\sigma - E^\sigma(X_\tau))^+ \geqq \frac{\epsilon}{2} \right\} \right)$$

$$= P\left(\left\{ \| (X_\sigma - E^\sigma(X_\tau))^+ \| \geqq \frac{\epsilon}{2} \right\} \right).$$

But then (X_t) cannot be a subpramart, a contradiction.

LEMMA 2.3. *Assume that* V *holds. Let E be a separable Banach lattice and D a countable norming subset of* $(E')^+$. *Let* $(X_t,\ t \in J)$ *be an E-valued positive subpramart which satisfies the assumption* (d):

(d) $\liminf_J E(\|X_t\|) < \infty$.

Suppose also that there exists an E-valued random variable X such that for all $\chi \in D$,

$$\lim_J \chi(X_t) = \chi(X) \text{ a.s.}$$

Then

$$\lim_J \|X_t\| = \|X\|.$$

Proof. Let $R_t^\chi + Z_t^\chi = \chi(X_t)$ be the decomposition of the real positive subpramart $(\chi(X_t))$, for each $\chi \in D$. Then $\{ (R_t^\chi,\ t \in J),\ \chi \in D\}$ is a countable family of real valued positive integrable submartingales with the optional sampling property (Theorem 1.1), such that

$$\liminf_J E(\sup_D R_t^\chi) \leqq \liminf_J E(\|X_t\|) < \infty.$$

The last relation implies that

$$\sup_J E(\sup_D R_t^\chi) < \infty$$

since $(\sup_D R_t^X, t \in J)$ is a submartingale. We also have

$$\lim_J R_t^X = \chi(X), \quad \chi \in D.$$

Therefore Proposition 2.1 gives

(1) $\liminf_J (\sup_D R_t^X) = \sup_D \chi(X) = \|X\|.$

On the other hand

$$0 \leqq \sup_D \chi(X_t) \leqq \sup_D R_t^X + \sup_D Z_t^X$$

thus

$$0 \leqq \|X_t\| - \sup_D R_t^X \leqq \sup_D Z_t^X.$$

By Lemma 2.2 we have

(2) $\lim_J (\|X_t\| - \sup_D R_t^X) = 0.$

The real valued subpramart $(\|X_t\|, t \in J)$, being of class (d), converges (Theorem 1.2). Using now (1) and (2) we have

$$\lim_J \|X_t\| = \|X\|.$$

LEMMA 2.4. *Let E be a Banach lattice which is a separable dual (i.e., $E = F'$, for some Banach lattice F), and $(x_t, t \in J)$ an E^+-valued net, such that*

$$\limsup_J \|x_t\| < \infty.$$

Assume that $(\chi(x_t), t \in J)$ converges for all χ in a countable dense subset of F^+. Then there exists an element x in E such that

$$\chi(x_t) \to \chi(x) \quad \textit{for all } \chi \in F.$$

Proof. This is straightforward. See e.g. **[18]**, page 108.

Before stating the main theorem of this note we recall that a norm $\|\cdot\|$ on a Banach space is said to have the Kadec-Klee property (with respect to a countable and norming subset η of the dual), if, whenever $\chi(x_n) \to \chi(x)$ for all $\chi \in \eta$ and $\|x_n\| \to \|x\|$ then $x_n \to x$ strongly.

Davis, Ghoussoub and Lindenstrauss **[5]** proved the following fundamental renorming theorem for Banach lattices:

A Banach lattice, E, has an equivalent Kadec-Klee lattice norm $\|\cdot\|_1$, if (and only if) it has an order continuous norm.

THEOREM 2.5. *Assume that* V *holds. Let E be a separable Banach lattice. The following are equivalent.*

(i) *E has the Radon-Nikodym property.*

(ii) *Every E-valued positive subpramart $(X_t, t \in J)$ of class* (d), *i.e.*,

$$\liminf_J E\|X_t\| < \infty,$$

converges a.s. *in the norm topology to an E-valued integrable random variable.*

Proof. (i $\Rightarrow$ ii). Since E is separable with the Radon-Nikodym property by a theorem of Talagrand (Theorem 1 of [**21**]), E is isometrically the dual of a Banach lattice F, i.e., $E = F'$.

Since ($\|X_t\|$) is a real valued subpramart of class (d), it converges a.s. and therefore

$$\lim \sup_J \|X_t(\omega)\| = \lim_J \|X_t(\omega)\| < \infty$$

for all ω in some set Ω_0 with $P(\Omega_0) = 1$. Let now D be a countable dense subset of F^+. For each $\chi \in D$, $(\chi(X_t))$ is a real valued subpramart of class (d) and therefore $(\chi(X_t(\omega)))$ converges for all ω in some set Ω_χ with $P(\Omega_\chi) = 1$. Thus for

$$\omega \in \Omega_1 = \Omega_0 \cap (\cap_D \Omega_\chi)$$

we have

(i) $\lim \sup_J \|X_t(\omega)\| < \infty$.

(ii) $(\chi(X_t(\omega)), t \in J)$ converges for all $\chi \in D$.

Hence by Lemma 2.4 there exists an E-valued mapping X on Ω_1 such that for all $\omega \in \Omega_1$,

$$\lim_J \chi(X(\omega)) = \chi(X(\omega)) \quad \text{for every } \chi \in F.$$

Since E is order continuous, by the renorming theorem, it admits an equivalent lattice norm $\| \|_1$ which is Kadec-Klee with respect to a countable norming subset η of F^+. Then by Lemma 2.3 we have

$$\lim_J \|X_t\|_1 = \|X\|_1.$$

The Kadec-Klee property of the norm gives $\lim_J X_t = X$ a.s. in the norm topology.

Since X is the strong limit of $(X_t, t \in J)$, it is measurable and integrable.

(ii $\Rightarrow$ i). It was proved by Ghoussoub-Talagrand [**11**] that convergence of positive martingales in E implies the Radon-Nikodym property; now, every martingale is a subpramart.

Remark. The assumption that E is separable is not a loss of generality if $J = \mathbf{N}$ since each X_n being Bochner integrable is separable valued and so is the set $\{X_n, n \in \mathbf{N}\}$.

COROLLARY 2.6. *Let E be a Banach lattice with the Radon-Nikodym property. Let $(X_t, t \in J)$ be an E-valued, L_1-bounded positive subpramart. Then $(X_t, t \in J)$ converges in probability to an E-valued integrable random variable.*

Proof. Since convergence in probability is given by a complete metric, the proof follows from Theorem 2.5 and Lemma V-1-1 in [**18**].

3. Convergence theorems for pramarts. If E is a separable Banach space, then there exists a countable set D in $B(E')$ such that for all $x \in E$,

$$\|x\| = \sup_D \chi(x).$$

LEMMA 3.1. *Let E be a separable Banach space and $(X_t, t \in J)$ an E-valued pramart. For each $\chi \in D$, let*

$$R_t^\chi + Z_t^\chi = |\chi(X_t)|$$

be the decomposition of the real valued positive subpramart $(|\chi(X_t)|)$ *(Theorem* 1.1). *Then*

(u) $s - \lim_T \sup_D Z_\sigma^\chi = 0.$

If V *holds then*

$$\lim_T \sup_D Z_\sigma^\chi = 0.$$

Proof. The proof is identical to that of Lemma 2.2.

We now observe that Lemma 2.3 remains true for Banach valued pramarts of class (d).

THEOREM 3.2. *Assume that* V *holds. Let E be a separable dual Banach space (i.e., $E = F'$). Let $(X_t, t \in J)$ be an E-valued pramart of class* (d). *Then the net (X_t) converges in the norm topology to an integrable random variable.*

Proof. The argument is similar to the one given by Neveu ([**18**], p. 108) for convergence of martingales. By Lemma 2.4 (see also the proof of Theorem 2.5) there exists an E-valued mapping X on Ω such that

$$\lim_J \chi(X_t) = \chi(X) \quad \text{for all } \chi \in D$$

(D now in $B(F)$). Lemma 2.3 applies and we have

$$\lim_J \|X_t\| = \|X\|.$$

If $\alpha \in E$, fixed, then the net $(X_t - \alpha, t \in J)$ is also a pramart of class (d), hence

$$\lim_J \|X_t - \alpha\| = \|X - \alpha\| \text{ a.s.}$$

If A is a countable dense subset of E we then have

$$\lim_J \|X_t - \alpha\| = \|X - \alpha\| \quad \text{for all } \alpha \in A.$$

Consequently

$$\lim_J \|X_t - \alpha\| = \|X - \alpha\| \quad \text{for all } \alpha \in E,$$

in particular

$$\lim_J \|X_t - X\| = 0 \text{ a.s.}$$

Remark. Notice that Theorem 3.2 still remains true if E is only a subspace of a separable dual.

Finally we prove the reversed pramart convergence theorem.

Write $-J$ for J with the reversed ordering. Given a stochastic basis $(\mathscr{F}_t, t - J)$ the set $(-T, \leqq)$ is filtering to the left. The family $(X_t, \mathscr{F}_t, t \in -J)$ of E-valued, Bochner integrable random variables is called a *reversed pramart* if

$$s - \lim_{\sigma \leqq \tau, \sigma, \tau \in -T} (X_\sigma - E^\sigma(X_\tau)) = 0.$$

THEOREM 3.3. *Let E be a Banach space and $(X_t, t \in -J)$ a reversed pramart. Then the net $(X_\tau, \tau \in -T)$ converges stochastically to an E-valued integrable random variable. If* V *holds then we have* a.s. *convergence.*

Proof. Let $\epsilon > 0$. From the definition of reversed pramart there exists $\tau_0 \in -T$ such that

$$P(\{\|X_\sigma - E^\sigma(X_\tau)\| < \epsilon\}) < \epsilon$$

for $\sigma \leqq \tau \leqq \tau_0$, $\sigma, \tau \in -T$. We observe now that the reversed martingale

$$(E^\sigma(X_{\tau_0}), \sigma \leqq \tau_0, \sigma \in -T)$$

converges in L_1 (e.g. [**18**]). Thus $(E^\sigma(X_\tau))$ converges in probability.

On the other hand

$$\|X_\sigma - X_\tau\| \leqq \|X_\sigma - E^\sigma(X_{\tau_0})\| + \|E^\sigma(X_{\tau_0}) - E^\tau(X_{\tau_0})\| + \|E^\tau(X_{\tau_0}) - X_\tau\|.$$

Therefore the net $(X_\sigma, \sigma \in -T)$ is Cauchy in probability and thus converges stochastically. Under V, there is a.s. convergence (Theorem 1.3).

It may be pointed out that this simple proof does not depend on the reduction to the real case.

Added in proof. Since this paper was submitted, M. Talagrand proved that pramarts indexed by **N** and of class (d) converge in every Banach space with the Radon-Nikodym property. See M. Talagrand, *Some structure results for martingales in the limit and pramarts* (to appear in Ann. of Prob.). Also the methods of the present work can establish this result for any index set J with a countable cofinal subset such that $(\mathscr{F}_t, t \in J)$ satisfies the Vitali condition V. It suffices to apply a recent result of

N. Goussoub and B. Maurey, *The asymptotic and the Radon-Nikodym properties are equivalent in separable Banach spaces* (preprint).

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