PRIME DUAL IDEALS IN BOOLEAN ALGEBRAS

L. J. HEIDER

1. Introduction. Let $\mathfrak{B}$ denote an arbitrary Boolean algebra. Let Latin letters $a, b, \ldots$ denote general elements of $\mathfrak{B}$ while the symbols 0, 1 denote the special smallest and largest elements. Let Greek letters $\alpha, \beta, \ldots$ denote various prime dual ideals of elements of $\mathfrak{B}$. It is recalled that a prime dual ideal of $\mathfrak{B}$ is a proper subset of $\mathfrak{B}$ closed under finite intersections of its elements and maximal with respect to those properties. Every prime dual ideal includes the element 1 and for each element a of $\mathfrak{B}$ includes either a or $\bar{a}$ (complement of a in $\mathfrak{B}$) but not both. Occasional reference will be made to principal dual ideals of $\mathfrak{B}$. These are subsets of $\mathfrak{B}$ composed of all elements of $\mathfrak{B}$ majorizing some fixed non-zero element of $\mathfrak{B}$. Finally, let $X(\mathfrak{B})$ denote the collection of all prime dual ideals of $\mathfrak{B}$. Then, with the subsets $X(a) = [\alpha \in X(\mathfrak{B}) | a \in \alpha]$, $a \in \mathfrak{B}$, being used as a basis for open sets, the collection $X(\mathfrak{B})$ becomes (homeomorphic to) the Stone representation space for $\mathfrak{B}$.

The collection $X(\mathfrak{B})$, with its field of open-and-closed subsets, is primarily representative of the Boolean algebra $\mathfrak{B}$. Special field-related properties of particular algebras $\mathfrak{B}$ as, for example, the ability of $\mathfrak{B}$ to be represented as a quotient-field of sets, appear as special properties of the field $X(\mathfrak{B})$. However, the same collection $X(\mathfrak{B})$, with its compact, zero-dimensional, Hausdorff topology, may, with equal ease, be regarded as the Stone-Čech compactification space βY of a completely regular topological space Y. In this case, the algebra $\mathfrak{B}$ is provided by a basis of open-and-closed subsets of Y, and special properties of Y appear as special properties of $X(\mathfrak{B})$ and $\mathfrak{B}$.

In either case, it is the points of $X(\mathfrak{B})$ that matter. These points are not undefined terms, but complex structures, that is, prime dual ideals of a Boolean algebra $\mathfrak{B}$. Any prime dual ideal α of $\mathfrak{B}$ has the property that if a finite union element $\bigvee_{i=1}^{n} a_i$ of $\mathfrak{B}$ is in α, then some component element a_i of this union is likewise in α. This universal property of prime dual ideals may obviously be generalized. Let $\mathfrak{M}$ denote an infinite cardinal, and let I denote an index set of cardinality $\mathfrak{M}$. Assume that a union element $a_0 = \bigvee_{i \in I} a_i$ exists in $\mathfrak{B}$. In general, a prime dual ideal of $\mathfrak{B}$ containing a_0 may or may not contain a component element of this union.

This paper discusses the presence in $X(\mathfrak{B})$ of prime dual ideals that contain along with a union element $a_0 = \bigvee_{i \in I} a_i$ also a component element a_i of that union. The first result of this discussion is a unified theory of the use of $X(\mathfrak{B})$ in the representation of Boolean algebras $\mathfrak{B}$. Since the parts of this theory

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have been developed by many authors, the present treatment is in outline form. The emphasis is on the unity of theory achieved by use of the above special property of prime dual ideals. The second result is a characterization of the Boolean algebras $\mathfrak{B}$ for which the spaces $X(\mathfrak{B})$ may be regarded as the Stone-Čech compactification spaces βY associated with three special types of completely regular spaces Y, namely, the P-, P'- and U-spaces of **(3, 4)**. These special spaces were introduced because of the interest of the algebraic features of their associated rings of real-valued continuous functions. Our interest arose from the fact that for each space Y of any of these types the corresponding space βY is zero-dimensional and thus homeomorphic to the representation space $X(\mathfrak{B})$ of a Boolean algebra $\mathfrak{B}$. In the cases of the P- and P'-spaces, the points of $\beta Y = X(\mathfrak{B})$ corresponding to points in Y involve intriguing properties of prime dual ideals.

2. Boolean algebras and fields of sets. Let $\mathfrak{M}$ denote an arbitrary cardinal number. Let the concepts of a field of sets, an $\mathfrak{M}$-field of sets and an $\mathfrak{M}$-complete Boolean algebra be understood in the usual sense. An $\mathfrak{M}$-complete Boolean algebra is called *$\mathfrak{M}$-representable* if it is isomorphic to an $\mathfrak{M}$-field of sets modulo an $\mathfrak{M}$-complete ideal of that field. An $\mathfrak{M}$-complete Boolean algebra $\mathfrak{B}$ is called *$\mathfrak{M}$-distributive* if

$$\bigvee_{i\in I}\bigwedge_{j\in J} a_{ij} = \bigvee_{h\in J^I}\bigwedge_{i\in J} a_{i,h(i)}$$

for each doubly-indexed family $\{a_{ij}\}$, $i \in I, j \in J$, of elements of $\mathfrak{B}$ for which the cardinalities $\tilde{I}$, $\tilde{J}$ of the index sets do not exceed $\mathfrak{M}$. Here J^I indicates the family of all maps h with domain I and range J.

For any element a_0 of a given Boolean algebra $\mathfrak{B}$ let $a_0 = \bigvee_{i\in I} a_i$, $\tilde{I} \leqslant \mathfrak{M}$, be called an *$\mathfrak{M}$-representation* of the element a_0. Let

$$a_0 = \bigvee_{j\in J_i} a_{ij},\ i \in I,\ \tilde{I} \leqslant \mathfrak{M},\ \tilde{J}_i \leqslant \mathfrak{M},$$

be called an $\mathfrak{M}$-family of $\mathfrak{M}$-representations of a_0. With this terminology and these concepts at hand, the principal parts of the theory may be presented in three statements.

(A) The Boolean algebras that are isomorphic to $\mathfrak{M}$-fields of sets are the $\mathfrak{M}$-complete algebras that have for every non-zero element a prime dual ideal that contains a component of each $\mathfrak{M}$-representation of that element.

(B) The $\mathfrak{M}$-complete and $\mathfrak{M}$-distributive Boolean algebras are exactly those $\mathfrak{M}$-complete algebras that have for each non-zero element and for each $\mathfrak{M}$-family of $\mathfrak{M}$-representations of that element a principal dual ideal containing a component of each member of that family.

(C) The $\mathfrak{M}$-complete and $\mathfrak{M}$-representable Boolean algebras are exactly those $\mathfrak{M}$-complete algebras that have for each non-zero element and for each $\mathfrak{M}$-family of $\mathfrak{M}$-representations of that element a prime dual ideal containing a component of each member of that family.

These statements are made without proof. Their intended value lies in the unified treatment of diverse subjects that they provide. Statement (A) is an observation of Sikorski **(10)** in dual form. Enomoto's theorems **(2)** regarding $\mathfrak{M}$-fields of sets *in the wider sense* involve but slight rephrasing of this statement. Statement (B) is well known **(9, 11)**, but attention is here called to the position of $\mathfrak{M}$-distributive algebras midway between $\mathfrak{M}$-fields of sets and quotients of such fields by $\mathfrak{M}$-complete ideals. Statement (C) was suggested by work of Chang **(1)**, but is new at least in its simplicity.

An apparent addition to the existing literature on the subject matter of statement (C) may well be made here. Let $\mathfrak{B}$ be an $\mathfrak{M}$-complete Boolean algebra with representation space $X(\mathfrak{B})$. Let $\mathfrak{F}(\mathfrak{B})$ denote the $\mathfrak{M}$-field of subsets of $X(\mathfrak{B})$ generated by the subsets of $X(\mathfrak{B})$ of the type $X(a) = [\alpha \in X(\mathfrak{B}) \mid a \in \alpha]$, $a \in \mathfrak{B}$. Let an element of $\mathfrak{F}(\mathfrak{B})$ of the form $\bigcap_{j\in J} X(a_j)$ with $\bar{J} \leqslant \mathfrak{M}$ and $\bigwedge_{j\in J} a_j = 0$ in $\mathfrak{B}$ be called an $\mathfrak{M}$-nowhere dense subset of $X(\mathfrak{B})$. Let $\mathfrak{J}(\mathfrak{B})$ denote the $\mathfrak{M}$-complete ideal in $\mathfrak{F}(\mathfrak{B})$ generated by these $\mathfrak{M}$-nowhere dense subsets. Attention is now called to the fact that, for each $\mathfrak{M}$-complete and $\mathfrak{M}$-representable Boolean algebra $\mathfrak{B}$, the quotient $\mathfrak{F}(\mathfrak{B})/\mathfrak{J}(\mathfrak{B})$ is a specific example of an isomorphic representation of $\mathfrak{B}$ as the quotient of a $\mathfrak{M}$-field of sets modulo an $\mathfrak{M}$-complete ideal.

3. Fields of sets and topological spaces. The concept of a field of sets stands midway between that of a Boolean algebra and that of a topological space with a basis of open-and-closed subsets. Let $\mathfrak{M}$ denote an arbitrary cardinal number. Let $\mathfrak{F}(X)$ be an $\mathfrak{M}$-field of subsets of a set X. It will be assumed that $\mathfrak{F}(X)$ is *reduced*, that is, for $p \neq q$ in X there is an element O of $\mathfrak{F}(X)$ with $p \in O$ and $q \notin O$. Let $(X, \mathfrak{T})$ denote the set X as under the topology $\mathfrak{T}$ obtained by using the subsets of X in $\mathfrak{F}(X)$ as a basis for open sets. Any subset of X in $\mathfrak{F}(X)$ is open-and-closed in $(X, \mathfrak{T})$. However, there might be subsets of X not in $\mathfrak{F}(X)$ that are open-and-closed in $(X, \mathfrak{T})$. They would be of the form

$$A = \bigcup_{i\in I} O_i = \bigcap_{j\in J} O_j$$

where the index sets I, J are arbitrary and each O_i and O_j is an element of $\mathfrak{F}(X)$. This introduction of alien open-and-closed subsets will be undesirable for our purpose. Hence, a reduced $\mathfrak{M}$-field of sets $\mathfrak{F}(X)$ will be called *union-intersection closed* if every subset A of X as described above is an element of $\mathfrak{F}(X)$. With each reduced, $\mathfrak{M}$-field there is associated a minimal, reduced, union-intersection closed, $\mathfrak{M}$-field including the given field. It consists of all subsets A as described above.

We now turn to the very special topological spaces described in **(3, 4, 8)**. As usual, for any topological space Y, $C(Y)$ will denote the collection of all real-valued functions, defined and continuous on Y. For each element f of $C(Y)$, let $P(f) = [p \in Y \mid f(p) > 0]$ and $Z(f) = [p \in Y \mid f(p) = 0]$. Let βY and υY denote, respectively, the Stone-Čech compactification space and the

Hewitt Q-space associated with a completely regular space Y. The first special completely regular spaces to be considered are the P-spaces.

The P-spaces may be characterized in a number of different ways (**3**, Theorem 5.3). For one thing, a completely regular space Y is a P-space if, and only if, every countable intersection of open sets of Y is itself open in Y. From this it follows that each P-space Y is a zero-dimensional Hausdorff space in which each countable intersection of open-and-closed subsets is open-and-closed. Hence, dually, in a P-space any countable union of open-and-closed subsets is likewise open-and-closed. Thus, if Y is a P-space and $\mathfrak{F}(Y)$ is the field of open-and-closed subsets of Y, then $\mathfrak{F}(Y)$ is a reduced, union-intersection closed, σ-field of sets in the sense explained above.

Conversely, let $\mathfrak{F}(Y)$ be a reduced, union-intersection closed, σ-field of sets. Use the subsets of Y in $\mathfrak{F}(Y)$ as the basis of a topology $\mathfrak{T}$ on Y, and let $(Y, \mathfrak{T})$ denote Y with this topology.

THEOREM 3.1. *If $\mathfrak{F}(Y)$ is a reduced, union-intersection closed, σ-field of sets, then $(Y, \mathfrak{T})$ is a P-space and every P-space may be thus described.*

Proof. With $\mathfrak{F}(Y)$ and $(Y, \mathfrak{T})$ as described, it is obvious that $(Y, \mathfrak{T})$ is a zero-dimensional Hausdorff space and thus completely regular. Consider, moreover, the intersection $\bigcap U_n$ of a countable family $\{U_n\}$ of sets open in $(Y, \mathfrak{T})$. If p_0 is a point of Y in this intersection, then there exists a family $\{O_n\}$ of sets in $\mathfrak{F}(Y)$ with $p_0 \in O_n \leqslant U_n$ for each n. Hence, with $\mathfrak{F}(Y)$ a σ-field, there exists an element O_0 of $\mathfrak{F}(Y)$ with $p_0 \in O_0 \subseteq U_n$ for each n. Thus any countable intersection of open subsets of $(Y, \mathfrak{T})$ is open, so that $(Y, \mathfrak{T})$ is a P-space.

If, conversely, one begins with a P-space Y and then forms $\mathfrak{F}(Y)$ and $(Y, \mathfrak{T})$ as described, clearly $(Y, \mathfrak{T})$ is homeomorphic to Y.

With the P-spaces thus firmly linked to reduced, union-intersection closed, σ-fields of sets, attention is turned elsewhere for the moment. First, two additional facts (**3**, Theorem 5.3, (2) and (3)) concerning P-spaces are needed: if Y is a P-space, so likewise is νY; if Y is a P-space, then the zero-set $Z(f)$ is open-and-closed in Y for each element f of $C(Y)$.

Now, for any completely regular space Y and for any point p_0 in Y, let p_0 be called a P-point of Y if for each element f of $C(Y)$ there exists a neighbourhood U of p_0 in Y such that $f(p) = f(p_0)$ for each point p in U. Then, from the facts cited just above, it follows that for any P-space Y each point of υY is a P-point of υY. Next consider $\beta Y = \beta(\upsilon Y)$. It is rather obvious that each P-point of υY as imbedded in βY becomes a P-point of βY. On the other hand, no point $\bar{p}$ of $\beta Y - \upsilon Y$ as in βY is a P-point of βY. Thus, for each point $\bar{p}$ of this type, there is an element f of $C(\beta Y)$ with $f(\bar{p}) = 0$ while $f(p) > 0$ for all points p of υY (**5**, Example 2.3). This, of course, excludes the local constancy of f at $\bar{p}$ since the points of νY are dense in βY. Thus, for any P-space Y, the points of υY as imbedded in βY are identified with the P-points of βY.

The fact that each zero-set $Z(f)$ associated with a P-space Y is open-and-closed in Y indicates that for such spaces the sets $\bar{P}(f)$ are likewise open-and-closed in Y. Thence it follows (**4**, Theorem 8.3) that for any P-space Y the lattice $C(Y)$ is conditionally countably complete, so that $\beta Y = X(\mathfrak{B})$ where $\mathfrak{B}$ is a σ-complete Boolean algebra (**12**). This algebra may, of course, be identified with the Boolean algebra of all open-and-closed subsets of βY or, equivalently, of υY or even of Y itself.

4. P-spaces and Boolean algebras. Interest now turns to the P-points of a space $X(\mathfrak{B})$ where $\mathfrak{B}$ is a σ-complete Boolean algebra. Each point of $X(\mathfrak{B})$ is a prime dual ideal of $\mathfrak{B}$. Let α be such an ideal while $\mathfrak{M}$ is a cardinal number and I is an index set with $\tilde{I} \leqslant \mathfrak{M}$. We introduce two conditions:

$(I - \mathfrak{M})$ If $\vee_{i \in I} a_i$ exists and is in α, $\tilde{I} \leqslant \mathfrak{M}$, then some a_i is in α.

$(II - \mathfrak{M})$ If $\{a_i, i \in I\} \subseteq \alpha$, $\tilde{I} \leqslant \mathfrak{M}$, then $\wedge_{i \in I} a_i$ exists and is non-zero.

For $\mathfrak{M}$-complete Boolean algebras the two conditions are equivalent. For any Boolean algebra, if condition $II - \mathfrak{M}$ is satisfied with respect to a particular prime dual ideal, then condition $I - \mathfrak{M}$ is satisfied also.

A Boolean algebra $\mathfrak{B}$ will be called a $\mathfrak{B}(II - \mathfrak{M}, D)$ algebra if the prime dual ideals of $\mathfrak{B}$ satisfying condition $II - \mathfrak{M}$ are dense in $X(\mathfrak{B})$ or, equivalently, each element of $\mathfrak{B}$ is contained in a prime dual ideal of $\mathfrak{B}$ satisfying this condition. For each $\mathfrak{B}(II - \mathfrak{M}, D)$ algebra $\mathfrak{B}$, let D also denote the subspace of $X(\mathfrak{B})$ consisting of all points (prime dual ideals) satisfying condition $II - \mathfrak{M}$. A similar definition and notation can be used for $\mathfrak{B}(I - \mathfrak{M}, D)$ algebras. Although reference is made to an arbitrary cardinal number $\mathfrak{M}$, interest centers on the first infinite cardinal number $\aleph_0 = \sigma$. Two lemmas are now in order.

LEMMA 4.2. *Every $\mathfrak{B}(II - \mathfrak{M}, D)$ Boolean algebra is $\mathfrak{M}$-complete.*

LEMMA 4.3. *For any $\mathfrak{B}(II - \sigma, D)$ Boolean algebra $\mathfrak{B}$, the P-points of the space $X(\mathfrak{B})$ are the prime dual ideals satisfying condition $I - \sigma = II - \sigma$.*

The proof of Lemma 4.2 is brief. Let $\{a_i, i \in I\}$, $\tilde{I} \leqslant \mathfrak{M}$, be a subset of elements of a $\mathfrak{B}(II - \mathfrak{M}, D)$ algebra $\mathfrak{B}$. If $\vee_{i \in I} a_i \neq 1$, there exists element a_0 of $\mathfrak{B}$, $a_0 \neq 0$, with $a_0 \leqslant \bar{a}_i$ for all i in I. However, for each non-zero element a_0 of a $\mathfrak{B}(II - \mathfrak{M}, D)$ algebra, there exists a prime dual ideal α_0 of that algebra containing a_0 and in which condition $II - \mathfrak{M}$ is verified. Then $\{\bar{a}_i, i \in I\} \subseteq \alpha_0$, so that $\wedge_{i \in I} \bar{a}_i$ and thus $\vee_{i \in I} a_i$ exists and the lemma is proved. Referring to statement (A) of the second section, it is now clear that the Boolean algebras isomorphic to $\mathfrak{M}$-fields of sets are exactly the $\mathfrak{B}(II - \mathfrak{M}, D)$ algebras and that, for each such algebra, the associated $\mathfrak{M}$-field of sets may be taken as the field of open-and-closed subsets of the subspace D of $X(\mathfrak{B})$.

Lemma 4.3 is a particular instance of a more general statement (**3**, Theorem 4.2 (3)) and returns us to the subject of P-spaces. From it one sees that for each P-space Y the Boolean algebra $\mathfrak{B}$ of all open-and-closed subsets of Y

is a $\mathfrak{B}(II - \sigma, D)$ algebra with $\beta Y = X(\mathfrak{B})$ and that the space υY may be identified with the subspace D of the representation space of this algebra. However, such $\mathfrak{B}(II - \sigma, D)$ algebras $\mathfrak{B}$ are still of a special character in that $\beta D = X(\mathfrak{B})$. This may be cared for in the following way.

Henceforth, a *P-Boolean algebra* will be understood as any $\mathfrak{B}(II - \sigma, D)$ algebra $\mathfrak{B}$ in which the following completeness condition obtains: every collection $\{a_i, i \in I\}$ of elements of $\mathfrak{B}$ such that each prime dual ideal in D either contains an element of that collection or contains an element of $\mathfrak{B}$ disjoint from every element of the collection has a least upper bound $\bigvee_{i\epsilon I} a_i$ in $\mathfrak{B}$.

The significance of this completeness condition is explained in two steps. Let $\mathfrak{F}(D)$ denote the field of open-and-closed subsets of the subspace D of the representation space $X(\mathfrak{B})$ of a $\mathfrak{B}(II - \sigma, D)$ algebra $\mathfrak{B}$. As noted in reference to Lemma 4.3, $\mathfrak{F}(D)$ is a reduced, σ-complete field of sets isomorphic to the algebra $\mathfrak{B}$. As the first step, it is shown that $\mathfrak{F}(D)$ is union-intersection closed exactly when the given $\mathfrak{B}(II - \sigma, D)$ algebra satisfies the stated completeness condition. Recall that elements a of $\mathfrak{B}$ are in $1 - 1$ order preserving correspondence with elements O of $\mathfrak{F}(D)$ through the relationship $X(a) \cap D = O$. Then, for any subset $A = \bigcup_{i\epsilon I} O_i = \bigcap_{j\epsilon J} O_j$ of D, the elements a_i of $\mathfrak{B}$ corresponding to the elements O_i in $\bigcup_{i\epsilon I} O_i$ are such that each prime dual ideal of D in A contains one of the a_i, while each prime dual ideal of D in $D - A$ contains an element b_j of $\mathfrak{B}$ disjoint from each of the a_i, namely, an element b_j of $\mathfrak{B}$ corresponding to the complement in D of some O_j in $\bigcap_{j\epsilon J} O_j$. Then, with $a = \bigvee_{i\epsilon I} a_i$ existing in $\mathfrak{B}$, it is clear that $X(a) \cap D = A$, so that $\mathfrak{F}(D)$ is union-intersection closed. Conversely, if the set $\mathfrak{F}(D)$ is union-intersection closed and $\{a_i, i \in I\}$ is a family of elements of $\mathfrak{B}$ such that each prime dual ideal in D either contains an element a_i of this family or an element b_j of $\mathfrak{B}$ disjoint from every member of the family, then, with $A = \bigcup_{i\epsilon I}[X(a_i) \cap D]$, one has $D - A = \bigcup_{j\epsilon J}[X(b_j) \cap D]$. Then $A = \bigcup_{i\epsilon I}[X(a_i) \cap D] = \bigcap_{j\epsilon J}[X(\bar{b}_j) \cap D]$. Finally, with a_0 in $\mathfrak{B}$ such that $X(a_0) \cap D = A$, it easily follows that $a_0 = \bigvee_{i\epsilon I} a_i$ in $\mathfrak{B}$, so that the completeness condition follows.

As the second step, it is now shown that the demand that $\mathfrak{F}(D)$ be union-intersection closed is equivalent to the demand that $\beta D = X(\mathfrak{B})$. First assume that $\mathfrak{F}(D)$ is union-intersection closed. The space $(D, \mathfrak{T})$ consisting of the set D and the topology $\mathfrak{T}$ derived from the field $\mathfrak{F}(D)$ is homeomorphic to the space D as a subspace of $X(\mathfrak{B})$. Hence $\beta(D, \mathfrak{T}) = \beta D$. However, $(D, \mathfrak{T})$ is a P-space so that $\beta(D, \mathfrak{T})$ is the representation space of the algebra of all open-and-closed subsets of $(D, \mathfrak{T})$. With $\mathfrak{F}(D)$ union-intersection closed, this latter algebra is isomorphic to the algebra $\mathfrak{F}(D)$ and thus to the given $\mathfrak{B}(II - \sigma, D)$ algebra $\mathfrak{B}$. Hence $\beta(D, \mathfrak{T}) = X(\mathfrak{B})$. Thus, if $\mathfrak{F}(D)$ is union-intersection closed, then $\beta D = X(\mathfrak{B})$. Conversely, if $\beta D = X(\mathfrak{B})$ so that each open-and-closed subset of D in its relative topology is of the form $X(a) \cap D$, then $\mathfrak{F}(D)$ is obviously union-intersection closed.

The preceding observations are now summarized.

THEOREM 4.4. *The class of all P-Boolean algebras is identical with the class of all algebras of the open-and-closed subsets of the P-spaces. For any P-space Y, the spaces βY and vY are homeomorphic to the spaces $X(\mathfrak{B})$ and D associated with the P-Boolean algebra of all open-and-closed subsets of Y. Two P-spaces Y and Z correspond to the same P-Boolean algebra if, and only if, $\beta Y = \beta Z$.*

With P-spaces characterized as completely regular spaces in which countable intersections of open sets are open, it seems proper to ask concerning completely regular spaces in which any $\mathfrak{M}$-intersection of open sets is open, $\mathfrak{M}$ being a cardinal number presumably larger than $\aleph_0 = \sigma$. Such spaces may be referred to as P-$\mathfrak{M}$-spaces. Let a $\mathfrak{B}(II - \mathfrak{M}, D)$ algebra satisfying the additional completeness condition cited above for P-Boolean algebras be called a P-$\mathfrak{M}$-Boolean algebra. An exact analogue of Theorem 4.4. may then be stated concerning the relationship of P-$\mathfrak{M}$-spaces and P-$\mathfrak{M}$-Boolean algebras.

5. The P'-spaces.

The P'-spaces form the second class of completely regular spaces to be discussed here. Their characterization embodied a slight weakening of that of the P-spaces. However, the most enlightening characteristic of the P'-spaces is the following: for each element f of $C(Y)$ and for each point p_0 of $Z(f)$, if there is no neighbourhood U of p_0 in Y such that $f(p) = 0$ throughout U, then there is a deleted neighbourhood U' of p_0 such that $f(p) > 0$ throughout U' or $f(p) < 0$ throughout U'. It is this feature of P'-spaces that guides the next procedures. Use is also made of the fact (**4**, Theorem 8.4) that, for each P'-space Y, $\beta Y = X(\mathfrak{B})$ where $\mathfrak{B}$ is a σ-complete Boolean algebra.

Let a point p_0 of an arbitrary completely regular space Y be termed a P'-point of Y if it has the property cited just above.

LEMMA 5.1. *Let Y be a completely regular space such that $\beta Y = X(\mathfrak{B})$ where $\mathfrak{B}$ is a σ-complete Boolean algebra. Let each point p of Y as in $X(\mathfrak{B})$ be considered as a prime dual ideal α_p of $\mathfrak{B}$. Then a point $\tilde{p}$ of Y is a P'-point of Y if, and only if, the corresponding prime dual ideal $\alpha_{\tilde{p}}$ satisfies the following condition: for each countable union $1 = \vee a_n$ in $\mathfrak{B}$ of which no component element a_n is in $\alpha_{\tilde{p}}$, there exists a non-zero element a_0 of $\mathfrak{B}$ with a_0 in $\alpha_{\tilde{p}}$ and such that all other α_p containing a_0 contain likewise some component of the given union.*

Proof. Assume first that $\tilde{p}$ is a P'-point of Y. Let $1 = \vee a_n$ be a disjoint countable union of elements of $\mathfrak{B}$ of which no component a_n is in $\alpha_{\tilde{p}}$. Then, because of the σ-completeness of $\mathfrak{B}$, there exists an element f of $C(X[\mathfrak{B}])$ with $f(\alpha) = 1/n$ for each prime dual ideal (point) α containing a_n. Now let a_0 be any element of $\mathfrak{B}$ in $\alpha_{\tilde{p}}$. Then $a_0 \wedge a_n \neq 0$ for at least one element a_n of the union $1 = \vee a_n$ and, since α_p contains no element of this union, actually $a_0 \wedge a_n \neq 0$ for infinitely many subscripts n. From this it follows that $f(\alpha_{\tilde{p}}) = 0$. However, with $\tilde{p}$ a P'-point of Y, there exists a deleted neighbourhood U'

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of $\tilde{p}$ in Y and thus a particular element a_0 of $\mathfrak{B}$ in $\alpha_{\tilde{p}}$ such that $f(\alpha_p) > 0$ for all α_p containing $a_0, \alpha_p \neq \alpha_{\tilde{p}}$. However, $f(\alpha_p) > 0$ means $f(\alpha_p) = 1/n$ for some n. This, in turn, is easily seen to mean that $a_n \in \alpha_p$. Thus there exists an element a_0 of $\mathfrak{B}$ in $\alpha_{\tilde{p}}$ such that every α_p containing a_0, $\alpha_p \neq \alpha_{\tilde{p}}$, contains likewise some element a_n of the given countable union.

Conversely, assume that prime dual ideal $\alpha_{\tilde{p}}$ of σ-complete algebra $\mathfrak{B}$ corresponding to point $\tilde{p}$ of Y has the property with respect to countable unions stated in the theorem. Let element f of $C(Y)$ be such that $f(\tilde{p}) = 0$. Assume, for the moment, that f is non-negative throughout Y. Let $f_0 = f \wedge 1$ in the usual sense of function lattices. Let $\bar{f}_0$ or, for notational simplicity, simply f denote the extension of f_0 over $\beta Y = X(\mathfrak{B})$. Let $O_n = [\alpha \in X(\mathfrak{B}) | f(\alpha) < 1/n]$. Then, by reason of the σ-completeness of $\mathfrak{B}$, there exists element a_n of $\mathfrak{B}$ such that $X(a_n) = \bar{O}_n$. The sequence $\{a_n\}$ is obviously such that $a_{n+1} \leqslant a_n$. Form the element $a_0 = \wedge a_n$ in $\mathfrak{B}$. Finally, construct a new sequence $\{b_n\}$ in $\mathfrak{B}$ with: $b_0 = \bar{a}_0$, $b_1 = 1 \wedge \bar{a}_1$, $b_2 = a_1 \wedge \bar{a}_2, \ldots$.

Now $\vee_{n=0}^{\infty} b_n = 1$ and is a countable disjoint union. If some (non-zero) b_n is in $\alpha_{\tilde{p}}$, clearly this b_n is $b_0 = a_0$ and one concludes that $f(\alpha_p) = 0$ for all α_p with $a_0 \in \alpha_p$. Then $U = [p \in Y | a_0 \in \alpha_p]$ is a neighbourhood of $\tilde{p}$ in Y such that $f(p) = 0$ throughout U. If no b_n is in $\alpha_{\tilde{p}}$, then, by hypothesis, there is an element c_0 of $\mathfrak{B}$ in $\alpha_{\tilde{p}}$ such that every α_p containing c_0, $\alpha_p \neq \alpha_{\tilde{p}}$, contains some (non-zero) b_n. Since b_0 is here assumed as not contained in $\alpha_{\tilde{p}}$, this first c_0 may be replaced by $\bar{b}_0 \wedge c_0$. Denote this element also by the symbol c_0. Then each α_p containing c_0, $\alpha_p \neq \alpha_{\tilde{p}}$, contains also an element b_n of the countable disjoint union and this b_n is not the element b_0. However, with $b_n = a_{n-1} \wedge \bar{a}_n$ in α_p, $n \geqslant 1$, then $1/n \leqslant f(\alpha_p) \leqslant 1/(n-1)$ so that $f(\alpha_p)$ is non-zero. One concludes from this that $U' = [p \in Y | p \neq \tilde{p}$ and $c_0 \in \alpha_p]$ is a deleted neighbourhood of $\tilde{p}$ in Y such that $f(p) > 0$ throughout U'.

Finally, for an arbitrary element f of $C(Y)$ with $f(\tilde{p}) = 0$, first apply the above analysis to the elements f^+, f^- formed in the usual function-lattice sense. Note that if $f^+(p) > 0$ throughout a deleted neighbourhood, then $f^-(p) = 0$ throughout the same neighbourhood. With this in mind, this converse part of the theorem is easily seen to hold for all elements f of $C(Y)$ with $f(\tilde{p}) = 0$.

THEOREM 5.2. *Let $X(\mathfrak{B})$ be the Stone representation space of a σ-complete Boolean algebra $\mathfrak{B}$. Let Y be a subspace of $X(\mathfrak{B})$ such that $\beta Y = X(\mathfrak{B})$ and also such that for every countable union $\vee a_n = 1$ in $\mathfrak{B}$ each point (prime dual ideal) α_0 of Y either contains a component of this union or contains an element a_0 of $\mathfrak{B}$ such that every other point α of Y which contains a_0 contains an element of this union. Then Y is a P'-space and every P'-space may be thus described.*

For the sake of brevity, a Boolean algebra of the type described in Theorem 5.2 will be called a P'-Boolean algebra. The description of such algebras is very awkward. However, with $\mathfrak{B}$, $X(\mathfrak{B})$ and Y as described in that theorem, consider the field $\mathfrak{F}(Y)$ of open-and-closed subsets of Y. Obviously $\mathfrak{F}(Y)$ is reduced and union-intersection closed. In view of the σ-completeness of $\mathfrak{B}$, also $\mathfrak{F}(Y)$ is σ-complete in the sense that every countable set of elements of $\mathfrak{F}(Y)$ is contained in a smallest element of $\mathfrak{F}(Y)$. Finally, from Theorem 5.2, $\mathfrak{F}(Y)$ is seen to have an additional property that may be called the near-σ-field property; if O is the smallest element of $\mathfrak{F}(Y)$ including each of the elements $\{O_n\}$ and if point $\tilde{p}$ of Y is in O but in no O_n, then there exists element O_0 of $\mathfrak{F}(Y)$ with $\tilde{p} \in O_0$ while $p \in O_0$, $p \neq \tilde{p}$, implies $p \in O_n$ for some n. Thus for any P'-Boolean algebra $\mathfrak{B}$ as described in Theorem 5.2 the associated field $\mathfrak{F}(Y)$ is a reduced, union-intersection closed, σ-complete, near-σ-field of sets which, as a Boolean algebra, is isomorphic to $\mathfrak{B}$ while the space $(Y, \mathfrak{T})$ derived from $\mathfrak{F}(Y)$ is homeomorphic to the P'-space Y. Note that $\beta(Y, \mathfrak{T})$, as homeomorphic to $X(\mathfrak{B})$, is of dimension zero.

Conversely, let $\mathfrak{F}(Y)$ be a reduced, union-intersection closed, σ-complete, near-σ-field of sets and let $(Y, \mathfrak{T})$ be formed as usual. Then, by methods similar to those used in Theorem 3.1, it may be proved that $(Y, \mathfrak{T})$ is a P'-space, provided one has assurance that $\beta(Y,\mathfrak{T})$ is of dimension zero. Whether or not such assurance is contained in the stated assumptions regarding $\mathfrak{F}(Y)$, the present writer does not know. However, he has indicated elsewhere **(6)** how to state such assurance regarding $\beta(Y, \mathfrak{T})$ in purely set-theoretic language.

These observations are now summarized.

THEOREM 5.3. *The P'-Boolean algebras are identical with the algebras formed under the inclusion relation by elements of reduced, union-intersection closed, σ-complete, near-σ-fields of sets $\mathfrak{F}(Y)$ with $\beta(Y, \mathfrak{T})$ of dimension zero. Such fields, in turn, may be identified with the fields of open-and-closed subsets of the P'-spaces.*

6. The UF-Boolean algebras. We turn now to the U-spaces described in **(4)**. A completely regular space X is a U-space if, and only if, to each element f of $C(X)$ there is associated a unit element u in $C(X)$ such that $f = u \,.\, |f|$. For any completely regular space X, X is a U-space if, and only if, βX is a U-space **(4**, Theorem 5.2**)**. Finally, βX is a U-space if, and only if, it is zero-dimensional and for each element f of $C(\beta X)$ the sets $P(f)$ and $N(f)$ are completely separated in βX. The zero-dimensionality of such βX links the U-spaces to Boolean algebras.

Let $\mathfrak{B}$ again denote an arbitrary Boolean algebra. Let $\rho = \{a_n\}$ denote a monotone, non-decreasing sequence of elements of $\mathfrak{B}$. For the sake of brevity, refer to a sequence like ρ as a *tower* in $\mathfrak{B}$. Two towers $\rho = \{a_n\}$ and $\tau = \{b_n\}$ will be called disjoint if $a_n \wedge b_n = 0$ for each positive integer n. Finally, an element a_0 of $\mathfrak{B}$ will be called a *cap* of a tower ρ if $a_n \leqslant a_0$ for each element a_n of $\rho = \{a_n\}$.

Now define a Boolean algebra $\mathfrak{B}$ to be a *UF-Boolean algebra* if, and only if, disjoint towers in $\mathfrak{B}$ have disjoint caps in $\mathfrak{B}$. The UF-Boolean algebras have a close relationship to the U-spaces (and F-spaces) of **(3**; **4)**.

THEOREM 6.1. *The UF-Boolean algebras are exactly those Boolean algebras* $\mathfrak{B}$ *for which the sets* $P(f)$ *and* $N(f)$ *are completely separated in* $X(\mathfrak{B})$ *for each element* f *of* $C[X(\mathfrak{B})]$.

Proof. Assume that $\mathfrak{B}$ is a *UF*-Boolean algebra and let f be an element of $C[X(\mathfrak{B})]$. Let $F_n = [\alpha \in X(\mathfrak{B}) \mid f(\alpha) \geqslant 1/n]$ while $O_n = [\alpha \in X(\mathfrak{B}) \mid f(\alpha) > 1/(n + \frac{1}{2})]$. Then, using the compactness of F_n and the openness of O_n, one can conclude to the existence in $\mathfrak{B}$ of an element a_n such that $F_n \subseteq [\alpha \in X(\mathfrak{B}) \mid a_n \in \alpha] \subseteq O_n$. Moreover, since $F_n \subseteq O_n \subseteq F_{n+1} \subseteq O_{n+1}$, one has $a_n \leqslant a_{n+1}$ and the sequence $\rho = \{a_n\}$ is a tower in $\mathfrak{B}$. Similarly, with $F_n^* = [\alpha \in X(\mathfrak{B}) \mid f(\alpha) \leqslant -1/n]$ and $O_n^* = [\alpha \in X(\mathfrak{B}) \mid f(\alpha) < -1/(n + \frac{1}{2})]$, let a second tower $\tau = \{b_n\}$ be constructed with $F_n^* \subseteq [\alpha \in X(\mathfrak{B}) \mid b_n \in \alpha] \subseteq O_n^*$. The two towers thus formed are clearly disjoint and thus, by assumption, have disjoint caps a_0 and b_0. It is now but a small matter to verify that $P(f) \subseteq [\alpha \in X(\mathfrak{B}) \mid a_0 \in \alpha]$ and $N(f) \subseteq [\alpha \in X(\mathfrak{B}) \mid b_0 \in \alpha]$ so that the sets $P(f)$ and $N(f)$ are completely separated in $X(\mathfrak{B})$.

Conversely, assume that for each element f of $C[X(\mathfrak{B})]$ the sets $P(f)$ and $N(f)$ are completely separated in $X(\mathfrak{B})$. Let $\rho = \{a_n\}$ and $\tau = \{b_n\}$ be a pair of disjoint towers in $\mathfrak{B}$. Let f_n be the unique element of $C[X(\mathfrak{B})]$ with $f_n(\alpha) = 1$ for all α with $a_n \in \alpha$, with $f_n(\alpha) = -1$ for all α with $b_n \in \alpha$ and with $f_n(\alpha) = 0$ for all α containing $\bar{a}_n \wedge \bar{b}_n$. Finally, form $f_0 = \Sigma_{n=1}^{\infty} f_n/2^n$. Then f_0 is an element of $C[X(\mathfrak{B})]$ and, by assumption, the sets $P(f_0)$ and $N(f_0)$ are completely separated in $X(\mathfrak{B})$. In virtue of the zero-dimensionality of $X(\mathfrak{B})$, this implies that there exists elements a_0 of $\mathfrak{B}$ such that $P(f_0) \subseteq [\alpha \in X(\mathfrak{B}) \mid a_0 \in \alpha]$, while $N(f_0) \subseteq [\alpha \in X(\mathfrak{B}) \mid \bar{a}_0 \in \alpha]$. The element a_0 is now seen to cap the tower $\rho = \{a_n\}$ while its complement $\bar{a}_0$ caps the tower $\tau = \{b_n\}$. Thus the theorem is proved.

The observations of this section may now be summarized.

THEOREM 6.2. *Any UF-Boolean algebra is the algebra of all open-and-closed subsets of some U-space and any such algebra is a UF-Boolean algebra. Two U-spaces* Y *and* Z *correspond to the same UF-Boolean algebra if, and only if,* $\beta Y = \beta Z$.

7. Comments. This section begins with an observation concerning *F*-spaces **(4)**. A completely regular space Y is an *F*-space if, and only if, for each element f of $C(Y)$ the sets $\bar{P}(f)$ and $\bar{N}(f)$ are completely separated. Every *F*-space Y has the following property **(4**, Theorem 2.6**)** pertinent to our purpose: for each zero set Z of Y each element f of $C^*(Y - Z)$ has a continuous extension $\bar{f}$ in $C^*(Y)$. Here $C^*(Y)$ indicates the collection of bounded elements of $C(Y)$.

LEMMA 7.1. *Let* Y *be a completely regular F-space. Then* βY *is without* G_δ-*points other than isolated points. Moreover, a point* p *of* Y *is a non-isolated*

G_δ-point in Y if, and only if, every element f of $C^*(Y - \{p\})$ has a continuous extension at p while some element of $C(Y - \{p\})$ lacks such an extension.

Proof. As regards the first assertion, assume that p is a G_δ-point of βY. If p is not an imbedded point of Y in βY, then every element of $C^*(\beta Y - \{p\})$ has a continuous extension at p by definition of βY. If p is an imbedded point of Y in βY, then $\{p\}$ is a zero set in Y and, by the property of F-spaces cited above, one again concludes that every element of $C^*(\beta Y - \{p\})$ has a continuous extension at p. Hence $\beta(\beta Y - \{p\}) = \beta Y$ unless p is an isolated point of βY. However, for any completely regular space X the cardinality of a zero set contained in $\beta X - X$ is at least exp (exp $\aleph_0$) (**7**, Theorem 49). Thus the point p must be an isolated point in βY.

As to the second assertion, it is merely to be noted that if a point p of Y has the extension properties listed in the theorem, then $\beta(Y - \{p\}) = \beta Y$ while $p \notin v(Y - \{p\})$. From this it follows easily that such a point is a G_δ-point (**5**, Example 2.3).

Now the F-spaces X such that βX is zero-dimensional and thus of present interest are identical with the U-spaces (**4**, Theorem 5.5). With the U-spaces described in terms of Boolean algebras, attention may now be called to the following conclusion.

THEOREM 7.2. *The Stone representation spaces of Boolean σ-algebras and, more generally, of UF-Boolean algebras are without G_δ-points other than isolated points.*

This theorem cannot be extended to include all Boolean algebras. In a written communication, C. W. Kohls called the attention of the writer to the following example.

Example. Let N denote the set of all positive integers. Let $\mathfrak{B}(N)$ denote the class of all finite subsets of N along with their complements in N together with the empty set and the set N itself. As partially ordered by the inclusion relation, $\mathfrak{B}(N)$ is a Boolean algebra. In $X(\mathfrak{B}[N])$ there is only one prime dual ideal other than the point-principal dual ideals. That ideal consists of all the infinite subsets of N in $\mathfrak{B}(N)$. As a point of $X(\mathfrak{B}[N])$ this ideal is obviously a non-isolated G_δ-point. It is also easily seen that $\mathfrak{B}(N)$ is not a UF-Boolean algebra. Thus let $a_n = \{1, 3, \ldots, 2n - 1\}$ and $b_n = \{2, 4, \ldots, 2n\}$. Then, as elements of $\mathfrak{B}(N)$, $a_n \leqslant a_{n+1}$, $b_n \leqslant b_{n+1}$ and $a_n \wedge b_n = 0$. However, it is impossible to find in $\mathfrak{B}(N)$ elements a_0, b_0 with $a_0 \wedge b_0 = 0$ and such that $a_n \leqslant a_0$ and $b_n \leqslant b_0$ for all positive integers n.

References

1. C. C. Chang, *On the representation of α-complete Boolean algebras*, Trans. Amer. Math. Soc., *85* (1957), 208–18.
2. S. Enomoto, *Boolean algebras and fields of sets*, Osaka Math. J., *5* (1953), 99–115.
3. L. Gillman and M. Henriksen, *Concerning rings of continuous functions*, Trans. Amer. Math. Soc., *77* (1954), 340–62.
4. ——— *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc., *82* (1956), 362–91.
5. L. J. Heider, *A note concerning completely regular G_δ spaces*, Proc. Amer. Math. Soc., *8* (1957), 1060–6.
6. ——— *Compactifications of dimension zero*, Notices Amer. Math. Soc., 552–9.
7. E. Hewitt, *Rings of real-valued continuous functions, I*, Trans. Amer. Math. Soc., *64* (1948), 45–99.
8. J. R. Isbell, *Zero-dimensional spaces*, Tohoku Math. J., *7* (1955), 1–8.
9. R. S. Pierce, *Distributivity in Boolean algebras*, Pacific J. Math., *7* (1957), 983–93.
10. R. Sikorski, *On the representation of Boolean algebras as fields of sets*, Fund. Math., *35* (1948), 247–58.
11. E. C. Smith, Jr., *A distributivity condition for Boolean algebras*, Ann. Math., *64* (1956), 551–61.
12. M. H. Stone, *Boundedness properties in function lattices*, Can. J. Math., *1* (1949), 176–86.

Institute for Advanced Study
Marquette University