ON ROTOR CALCULUS

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Summary

It is known that to every proper homogeneous Lorentz transformation there corresponds a unique proper complex rotation in a three-dimensional complex linear vector space, the elements of which are here called "rotors". Equivalently one has a one-one correspondence between rotors and self-dual bi-vectors in space-time (w-space). Rotor calculus fully exploits this correspondence, just as spinor calculus exploits the correspondence between real world vectors and hermitian spinors; and its formal starting point is the definition of certain covariant connecting quantities τ_{Akl} which transform as vectors under transformations in rotor space (r-space) and as tensors of valence 2 under transformations in w-space. In the present paper, the first of two, w-space is taken to be flat. The general properties of the τ_{Akl} are established in detail, without recourse to any special representation. Corresponding to any proper Lorentz transformation there exists an image in r-space, i.e. an r-transformation such that the two transformations carried out jointly leave the τ_{Akl} numerically unchanged. Nevertheless, all relations are written in such a way that they are invariant under arbitrary w-transformations and arbitrary r-transformations, which may be carried out independently of one another. For this reason the metric tensor α_{AB} in r-space may be chosen arbitrarily, except that it shall be non-singular and symmetric. A large number of identities involving the basic quantities of the calculus is presented, including some which relate to complex conjugated rotors $\tau_{\dot{A}kl}$, $\alpha_{\dot{A}\dot{B}}$. The properties of the tensor equivalents of some simple irreducible rotors are investigated, after which the r-image of improper Lorentz transformations is considered. Since symmetric spinors of valence 2 are also in one-one correspondence with self-dual w-tensors one can also establish a direct correspondence between simple rotors and symmetric spinors of valence 2 by means of appropriate connecting quantities; and simple relations exist between the three kinds of connecting quantities now in hand. Finally, a particular representation is presented in detail. In a subsequent paper this work will be extended to curved w-spaces.

1. Introduction

The manifest covariance of physical laws is assured when they are exhibited as equations between vectors and tensors in world-space, that is to say in Minkowskian or Riemannian space-time, as the case may be. In saying this the phenomenon of intrinsic spin has been disregarded. If the latter are to be taken into account one has to introduce additionally the vectors and tensors of a two-dimensional linear complex vector space, those objects being collectively called "spinors". One further requires certain connecting quantities between world-tensors and spin-tensors, namely the Pauli matrices or their generalizations. These matrices transform in a certain way under transformations in world-space on the one hand, and in spin-space on the other; and when the world space[1] is flat the two kinds of transformations may be so geared to each other as to ensure the *numerical* invariance of the connecting matrices (e.g. Corson 1953). The reason for this state of affairs is ultimately group-theoretical: it arises from the homomorphism of the proper homogeneous Lorentz group and the two-dimensional unimodular group. In curved w-space one cannot in general find an s-image of any given w-transformation so as to maintain the functional form of the $\sigma^{k\dot{\mu}\nu}$; not, at any rate, unless the w-space has some kind of symmetry. This feature aside, spinor calculus in flat w-space extends naturally to curved w-space; and this extension finds its most beautiful expression in the work of Infeld and van der Waerden (1933).

Now, the proper homogeneous Lorentz group is isomorphic with the three-dimensional proper complex orthogonal group (e.g. Jordan, Ehlers and Kundt 1960); and just as there is a one-one correspondence between hermitian s-tensors of valence two and real w-vectors, so self-dual w-tensors can be brought into one-one correspondence with the vectors of a three-dimensional complex linear vector space. Such a space I shall call "rotor-space" or more often simply r-space; its vectors are "rotors" or r-vectors, its tensors r-tensors. (The name "rotor" is perhaps not very attractive but one may think of a rotor as being naturally connected with the "orbital rotation" of a particle, — i.e. it represents its relativistic angular momentum, — just as a simple spinor is connected with intrinsic spin, i.e. its "intrinsic rotation". Again, a general infinitesimal rotation in w-space is directly determined by a rotor; cf. the end of Section 4a). Then the object of the present work is to develop a rotor-calculus which is closely

[1] For reasons which will become clear later world-space will hereafter generally be referred to as w-space, and world vectors and tensors will be called w-vectors and w-tensors respectively. Spin-space will similarly be referred to as s-space, so that spinors will be s-vectors or s-tensors, as the case may be. The connecting matrices $\sigma^{k\dot{\mu}\nu}$ are thus at the same time w-vectors and hermitian s-tensors of valence 2. Linear transformations in w-space and s-space can now be called w-transformations and s-transformations respectively.

analogous to the two-spinor calculus though in principle quite independent of it. The two main stages of this development are (i) rotor-calculus in flat w-space, (ii) rotor-analysis in curved w-space.

Considerations involving r-space have in recent years played an important part in the local classification of gravitational fields (e.g. Petrov 1962). Moreover, rotors have, in one guise or another, been considered previously by many authors. Thus Peres (1962), who quotes earlier references, calls them "three-component spinors". Andrews (1964) introduces three skew-symmetric 4×4 matrices for the purpose of writing Maxwell's equations as a single matrix equation in which the electromagnetic field tensor is replaced by a rotor. Using a special representation for these matrices he derives commutation relations for them by explicit numerical calculation. To go over to curved w-space he introduces orthonormal tetrads: a procedure which has a historical precedent in the context of spinor equations. Kurşunoğlu (1961) discusses rotors with a view to introducing them explicitly into the theory of elementary particles. Synge (1964) has recently reviewed some aspects of the theory of euclidean complex three-spaces, whilst two complex 3×3 matrices are introduced by Rastall (1964) in a paper dealing with a characterization of space-time. At any rate, as far as I am aware, the theory of r-spaces appears to have been dealt with in a somewhat piecemeal and often ad hoc manner, a situation which the present work is intended to remedy.

With regard to notation and terminology, in as far as it concerns tensor calculus in general, I shall usually follow Schouten (1954); whilst in the context of spinor calculus I shall closely follow Infeld and van der Waerden (1933). In particular w-indices will be in small Roman type (range $1, \cdots, 4$), and s-indices in small Greek type (ranges 1,2 and $\dot{1},\dot{2}$ respectively. As for r-indices, these will be denoted by capital Roman type (ranges 1, 2, 3 and $\dot{1}, \dot{2}, \dot{3}$ respectively).

In rough outline the plan of this paper is as follows. The basic "tensor-rotors", or connecting matrices, τ_{Akl} are defined in Section 2, and an arbitrary symmetric non-singular metric tensor α_{AB} in r-space is introduced. Then r-indices as well as w-indices can be juggled freely and a preliminary set of relations obeyed by τ_{Akl} is derived. Finite and infinitesimal proper Lorentz transformations and their r-images are the subject of Section 3. After this, the integrability conditions on the Lorentz group are used in Section 4 to obtain more general relations, obeyed by the basic rotors, in particular their commutation relations. Given some linear (homogeneous) transformation $\Lambda^{A}{}_{A'}$, in r-space one can consider also quantities which transform under the transformation $\Lambda^{\dot{A}}{}_{A'}$, complex conjugate to this, and so one is naturally led to complex conjugated rotors (Section 5). A sizeable collection of miscellaneous identities involving the τ_{Akl} and τ_{Akl} is presented

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in Section 6. All these, like those of Sections 2, 4 and 5 are covariant under arbitrary transformations in w-space and r-space: where these transformations can be carried out independently of one another, i.e. the transformation in w-space need *not* be accompanied by its image in r-space. There is often a certain formal resemblance between the general relations of the rotor calculus on the one hand, and of the spinor calculus on the other. The spinor identities in question may be found in an earlier paper (Buchdahl 1962), which will hereafter be referred to by the letter S. Section 7 revolves about the question of the character of the w-tensor equivalents of some irreducible r-tensors. Improper Lorentz transformations are considered in Section 8. Spinors appear for the first time in Section 9, where basic "spinor-rotors" $\lambda_{A\mu\nu}$ are introduced to establish a *direct* link between simple rotors and symmetric spinors of valence 2; and a number of formal relations involving the $\lambda_{A\mu\nu}$ are obtained. In Section 10 a specific representation ("standard representation") is chosen for the first time, and the standard representatives of various quantities are exhibited in explicit form, i.e. as matrices with specific numerical elements. This section ends with some brief comments on the relation between the forms of certain specific representations and the canonical forms of the electromagnetic field tensor.

2. The connecting matrices τ_{Akl}

(a) Let ε^{klmn} be the contravariant numerical Δ-density of Levi-Civita, and define the tensor

$$e^{klmn} = (-g)^{-\frac{1}{2}}\varepsilon^{klmn}, \tag{2.1}$$

where g is the determinant of the metric tensor (of signature -2). Strictly speaking e^{klmn} is a w-tensor rather than a true tensor on account of the fact that one agrees always to take the positive square root of $-g$. However, for the time being only proper transformations will be considered, so that the distinction just drawn is irrelevant. If f_{kl} is a skew-symmetric tensor, let its *dual* ${}^{\dagger}f_{kl}$ be defined by

$$ {}^{\dagger}f^{kl} = -\tfrac{1}{2}ie^{klmn}f_{mn}. \tag{2.2}$$

Note that the dual is more often defined without the factor $-i$ on the right of (2.2). Then if

$$\eta^{klmn} = g^{k[m}g^{n]l} - \tfrac{1}{2}ie^{klmn}, \tag{2.3}$$

the tensor

$$F^{kl} = \eta^{klmn}f_{mn} = f^{kl} + {}^{\dagger}f^{kl} \tag{2.4}$$

is self-dual, i.e.

$$^{\dagger}F^{kl} = F^{kl}.$$

The definition (2.2) evidently entails the involutory nature of the operation of forming the dual. Now

$$\bar{\eta}^{klmn}\eta_{mnpq} = 0, \tag{2.5}$$

where a bar denotes complex conjugation. The self-duality of F^{kl} may therefore be expressed in the form

$$\bar{\eta}^{klmn}F_{mn} = 0. \tag{2.6}$$

The correspondence between self-dual tensors and rotors will now be exhibited in the form

$$F_{kl} = \tfrac{1}{2}\tau_{Akl}\phi^A, \tag{2.7}$$

where ϕ^A is contravariant rotor, and the three connecting matrices are skew-symmetric,

$$\tau_{A(kl)} = 0. \tag{2.8}$$

An explicit distinction is being drawn between covariant and contravariant r-indices since it is not necessary that the metric of r-space should have the form diag $(+1, +1, +1)$.

The right hand member of (2.7) is self-dual for every ϕ^A, so that, recalling (2.6),

$$\bar{\eta}^{klmn}\tau_{Amn} = 0, \tag{2.9}$$

i.e. each τ_{Amn} is self-dual. Explicitly

$$\tau_{Akl} = -\tfrac{1}{2}ie_{klmn}\tau_A{}^{mn}, \tag{2.10}$$

whence

$$\begin{aligned}\tau_{Akm}\tau_B{}^{lm} &= -\tfrac{1}{4}e_{kmpq}e^{lmst}\tau_A{}^{pq}\tau_{Bst}\\ &= \tfrac{1}{4}\delta^{lst}_{kpq}\tau_A{}^{pq}\tau_{Bst}.\end{aligned} \tag{2.11}$$

The generalized Kronecker delta in the right is a determinant of simple Kronecker deltas and so one arrives at

$$\tau_{(A}{}^{km}\tau_{B)lm} = \tfrac{1}{4}\delta^k{}_l\tau_A{}^{mn}\tau_{Bmn}. \tag{2.12}$$

On account of its self-duality F_{kl} gives rise to only one quadratic invariant $F_{kl}F^{kl}$. Likewise, if one introduces a non-singular but otherwise arbitrary symmetric tensor α_{AB} into r-space, ϕ^A gives rise to just one quadratic invariant $\phi_A\phi^A$. These two invariants are now to be identified, so that

$$\alpha_{AB} = \tfrac{1}{4}\tau_{Akl}\tau_B{}^{kl}. \tag{2.13}$$

The contravariant metric r-tensor is defined in the usual way by

(2.14) $$\alpha_{AB}\alpha^{BC} = \delta_A{}^C.$$

(2.12) now reads

(2.15) $$\tau_{(A}{}^{km}\tau_{B)lm} = \delta^k{}_l\alpha_{AB}.$$

Transvection of (2.7) with τ^{Bkl} then inverts this relation, i.e.

(2.16) $$\phi^A = \tfrac{1}{2}\tau^{Akl}F_{kl}.$$

Now insert (2.16) in (2.7) and use (2.4). One sees that

$$(\tfrac{1}{4}\tau_{Amn}\tau^{Akl}-\delta^k{}_m\delta^l{}_n)\eta_{klst}f^{st}$$

must vanish for arbitrary f^{st}. It follows that

(2.17) $$\tau_{Akl}\tau^A{}_{mn} = 2\eta_{klmn},$$

since, by (2.10),

(2.18) $$\eta_{klmn}\tau^{Akl} = 2\tau^A{}_{mn}.$$

(b) Contemplate now the r-tensor

(2.19) $$f^{ABC} = \tau^{Akl}\tau^B{}_{km}\tau^C{}_l{}^m.$$

Then, in view of (2.15),

$$f^{BAC} = (-\tau^{Akl}\tau^B{}_{km}+2\alpha^{AB}\delta^l{}_m)\tau^C{}_l{}^m = -f^{ABC}.$$

In the same way one shows easily that f^{ABC} is skew-symmetric in its second pair of indices also. In other words, f^{ABC} is completely skew-symmetric and must therefore be a scalar multiple of the e-tensor in r-space, i.e. of

(2.20) $$e^{ABC} = \alpha^{-\frac{1}{2}}\varepsilon^{ABC},$$

where

$$\alpha = \det \alpha_{AB}.$$

Thus now

$$f^{ABC} = fe^{ABC},$$

where f is a scalar. To determine it, form the invariant

(2.21) $$f^{ABC}f_{ABC} = 6f^2 = 8\eta^{kl}{}_{pq}\eta_{km}{}^{ps}\eta_l{}^{mq}{}_s,$$

in view of (2.17). One easily confirms that

(2.22) $$\eta^{kl}{}_{pq}\eta_{km}{}^{ps} = \tfrac{3}{4}\delta^s{}_q\delta^l{}_m+\eta^l{}_{mq}{}^s.$$

The right hand member of (2.21) then reduces to

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$$8\eta^{lmqs}\eta_{lmqs} = 96,$$

whence

$$f^2 = 16.$$

The sign of f may at this stage be chosen freely and I take

(2.23) $$f = -4.$$

Thus one now has the relation

(2.24) $$\tau^{Akl}\tau^{B}{}_{km}\tau^{C}{}_{l}{}^{m} = -4e^{ABC},$$

and by transvecting this throughout with τ_{Cst} one obtains

(2.25) $$\eta_{klmn}T^{mnAB} = -\,2e^{ABC}\tau_{Ckl},$$

where

(2.26) $$T^{mn}{}_{AB} = \tau_{A}{}^{[m}{}_{s}\tau_{B}{}^{n]s}.$$

3. Lorentz transformations

(a) Let w-vectors and r-vectors undergo the transformations

(3.1) $$'u_k = L^l{}_k u_l, \quad '\phi_A = \Lambda^B{}_A\phi_B.$$

(Exceptionally the kernel-index notation is not being used here in order to avoid the appearance of a large number of mixed Kronecker deltas of the type $\delta^k{}_{k'}$ and $\delta^A{}_{A'}$.) The transformations inverse to those which appear in (3.1) will be denoted by the corresponding lower case kernel symbols. Then

(3.2) $$'\tau_{Akl} = \Lambda^B{}_A L^m{}_k L^n{}_l \tau_{Bmn}.$$

Now if $L^l{}_k$ is a Lorentz transformation then $\Lambda^B{}_A$ is its "r-image" if

(3.3) $$'\tau_{Akl} = \tau_{Akl}.$$

From (3.2) and (3.3) one then has

(3.4) $$\Lambda^B{}_A = \tfrac{1}{4}l^k{}_m l^l{}_n \tau_{Akl}\tau^{Bmn}.$$

Hence under these joint w- and r-transformations

(3.5) $$'\alpha_{AB} = \Lambda^C{}_A\Lambda^D{}_B\alpha_{CD} = \tfrac{1}{8}\eta^{mnst}l^k{}_m l^l{}_n l^p{}_s l^q{}_t \tau_{Akl}\tau_{Bpq},$$

where (2.17) has been used. Now recall that $l^k{}_m$ is a Lorentz transformation, so that the g_{kl} $(=\text{diag}\,(-1, -1, -1, +1))$ are numerically invariant, i.e.

(3.6) $$l^k{}_m l^l{}_n g^{mn} = g^{kl}.$$

(3.5) therefore becomes

(3.7) $$'\alpha_{AB} = \tfrac{1}{16}(2g^{kp}g^{lq} - ie^{mnst}l^k{}_m l^l{}_n l^p{}_s l^q{}_t)\tau_{Akl}\tau_{Bpq}.$$

Since $(-g)^{\frac{1}{2}} = 1$ here, one has

$$e^{mnst}l^k{}_m l^l{}_n l^p{}_s l^q{}_t = ie^{klpq} \det l^s{}_t = ie^{klpq}\boldsymbol{l},$$

say. However, a proper Lorentz transformation is characterized by the condition $\boldsymbol{l} = 1$, and so (3.7) becomes

(3.8) $$'\alpha_{AB} = \tfrac{1}{8}\eta^{klpq}\tau_{Akl}\tau_{Bpq} = \alpha_{AB},$$

by (2.17) and (2.13). The α_{AB} thus remain numerically invariant, which shows explicitly that $\Lambda^B{}_A$, as given by (3.4) is indeed an orthogonal transformation in $\boldsymbol{r}$-space; and it is not difficult to see that it must be proper, i.e. a rotation.

(b) When the Lorentz transformation is infinitesimal write

(3.9) $$L^k{}_l = \delta^k{}_l - \omega^k{}_l, \quad \Lambda^B{}_A = \delta^B{}_A + \gamma^B{}_A, \quad \omega_{(kl)} = 0.$$

Then (3.4) at once yields

(3.10) $$\gamma^B{}_A = \tfrac{1}{2}\omega_{mn}T^{mn}{}_A{}^B,$$

so that, of course, $\gamma_{(AB)} = 0$; (see also eq. (4.11)).

4. Commutation relations and defining relation for τ_{Akl}

(a) In any representation of the Lorentz group the infinitesimal transformation

$$l^k{}_m = \delta^k{}_m + \omega^k{}_m$$

is represented by (Cf. Corson 1953, p. 11)

(4.1) $$1 + \tfrac{1}{2}\omega^{st}\boldsymbol{T}_{st},$$

where $\boldsymbol{T}_{st}$ are the representative matrices of the six infinitesimal transformations. They must satisfy the integrability conditions on the Lorentz group

(4.2) $$[\boldsymbol{T}_{kl}, \boldsymbol{T}_{mn}] = 4g_{[n[k}\boldsymbol{T}_{l]m]}.$$

Comparison of (4.1) with (3.10) shows that

(4.3) $$\boldsymbol{T}_{kl} = T_{klA}{}^B.$$

For the time being it suffices to use the contracted form

(4.4) $$\boldsymbol{T}_{[k}{}^n\boldsymbol{T}_{l]n} = -\boldsymbol{T}_{kl}$$

of (4.2), obtained from it by transvection with g^{ln}. In view of (4.3) this becomes [2]

[2] Symmetrizing and anti-symmetrizing brackets always act on only one kind of indices the character of which is determined by that of the indices next to the brackets and within them.

(4.5) $$T_{[k}{}^{n}{}_{A}{}^{B}T_{l]nB}{}^{C} = -T_{klA}{}^{C}.$$

On the left there appears the sum of products of τ-matrices, four at a time, and in each such product the index B is repeated, so that (2.17) may be used. Thus, for instance

(4.5) $$T_{k}{}^{n}{}_{A}{}^{B}T_{lnB}{}^{C} = \tfrac{1}{2}(\eta_{tmln}\tau_{Ak}{}^{m}\tau^{Ctn}+\eta_{tmkn}\tau_{A}{}^{tn}\tau^{C}{}_{l}{}^{m} -\eta_{kmln}\tau_{A}{}^{tm}\tau^{C}{}_{t}{}^{n}-\tfrac{3}{2}\tau_{Ak}{}^{n}\tau^{C}{}_{ln}).$$

But

(4.6) $$\eta_{tmln}\tau^{Ctn} = -\tfrac{1}{2}\tau^{C}{}_{lm},$$

on account of (2.10), and

(4.7) $$\eta_{kmln}\tau_{A}{}^{tm}\tau^{C}{}_{t}{}^{n} = \tfrac{1}{8}g_{kl}\delta_{A}{}^{C}-\eta_{klmn}T^{mn}{}_{A}{}^{C}+T_{klA}{}^{C}-\tfrac{1}{2}\tau_{A}{}^{t}{}_{l}\tau^{C}{}_{tk},$$

where (2.13) alone has been used. Inserting (4.6) and (4.7) in (4.5) that part of the latter which is skew-symmetric in k and l reduces to $-2T_{klA}{}^{C}+\frac{1}{2}\eta_{klmn}T^{mn}{}_{A}{}^{C}$. (4.4) therefore finally gives rise to

(4.8) $$\bar{\eta}_{klmn}T^{mnAB} = 0.$$

T^{mnAB} is thus also self-dual. This may now be combined with (2.25) to give the explicit *commutation relation* for the τ-matrices

(4.9) $$T_{kl}{}^{AB} = -e^{ABC}\tau_{Ckl}.$$

One may further combine this with (2.15):

(4.10) $$\tau_{Ak}{}^{m}\tau_{Blm} = g_{kl}\alpha_{AB}-e_{ABC}\tau^{C}{}_{kl}.$$

(4.10) may be looked upon as a *defining relation* for the τ-matrices.

It is of interest to note that (3.10) and (4.9) together give

(4.11) $$\gamma_{AB} = \tfrac{1}{2}e_{ABC}\tau^{Cmn}\omega_{mn} = e_{ABC}\omega^{C},$$

say; so that the image of an infinitesimal Lorentz transformation is a rotation in r-space, characterized by the infinitesimal rotor ω^{C}.

(b) So far only the contracted concomitant (4.3) of (4.2) has been used. One may now insert (4.9) into the full integrability conditions (4.2), i.e.

(4.12) $$T_{kl}{}^{AB}T_{mnBC}-T_{mn}{}^{AB}T_{klBC} = 4g_{[n[k}T_{l]m]}{}^{A}{}_{C}.$$

The first term on the left becomes

$$e^{ABD}e_{BCF}\tau_{Dkl}\tau^{F}{}_{mn} = \delta^{DA}_{CF}\tau_{Dkl}\tau^{F}{}_{mn} = \tau_{Ckl}\tau^{A}{}_{mn}-2\eta_{klmn}\delta^{A}{}_{C}.$$

There follows the relation

(4.13) $$\tau^{[A}{}_{kl}\tau^{B]}{}_{mn} = 2e^{ABC}g_{[n[k}\tau_{Cl]m]}.$$

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5. Complex conjugate rotors

If ϕ_A is any rotor then the rotor complex conjugate to it will be denoted by $\phi_{\dot A}$; and if ϕ_A transforms according to

$$\phi_{A'} = \Lambda^{A}{}_{A'}\phi_A$$

then $\phi_{\dot A}$ transforms according to

$$\phi_{\dot A'} = \Lambda^{\dot A}{}_{\dot A'}\phi_{\dot A}, \tag{5.1}$$

where

$$\Lambda^{\dot A}{}_{\dot A'} = \overline{\Lambda^{A}{}_{A'}}. \tag{5.2}$$

More generally there will be r-tensors with any number of dotted and undotted indices. In particular one has r-tensors of valence 2, $\Gamma^{\dot A B}$ say; and $\Gamma^{\dot A B}$ is hermitian if

$$\Gamma^{\dot A B} = \Gamma^{\dot B A}. \tag{5.3}$$

Given any relation between r-tensors, the complex conjugate relation is obtained by dotting all undotted indices and omitting the dot from all previously dotted indices. (All this is quite familiar from spinor calculus.)

To begin with, consider the hermitian r-tensor

$$\hat{T}^{\dot A B} = \tau^{\dot A kl}\tau^{B}{}_{kl}. \tag{5.4}$$

On account of (2.10)

$$\hat{T}^{\dot A B} = -\tfrac{1}{2}ie_{klmn}\tau^{\dot A kl}\tau^{Bmn} = -\tau^{\dot A}{}_{mn}\tau^{Bmn} = -\hat{T}^{\dot A B}.$$

Hence

$$\hat{T}^{\dot A B} = 0. \tag{5.5}$$

A tensor which frequently occurs is the following:

$$\hat{T}^{kl\dot A B} = \tau^{\dot A km}\tau^{Bl}{}_{m}; \tag{5.6}$$

and according to (5.5) it is trace-free. Moreover, using (2.10),

$$\tau^{\dot A kl}\tau^{B}{}_{mn} = -\tfrac{1}{4}\delta^{klst}_{mnpq}\tau^{\dot A}{}_{st}\tau^{Bpq}.$$

Writing the Kronecker delta as a determinant of simple Kronecker deltas as usual, there follows

$$\tau_{\dot A kl}\tau_{Bmn}+\tau_{\dot A mn}\tau_{Bkl} = 4g_{[k[m}\hat{T}_{n]l]\dot A B}. \tag{5.7}$$

In particular, transvecting throughout with g^{ln},

$$\hat{T}_{[kl]\dot A B} = 0, \tag{5.8}$$

so that $\hat{T}_{kl\dot A B}$ is not only trace-free but is also symmetric in w-space, or hermitian in r-space, (cf. Section 10d). The relation (5.7) may now be used to exhibit the outer product (no indices paired) of an undotted and a dotted τ-matrix in terms of such products once contracted, i.e. in terms of $\hat{T}_{klAB}$. For this purpose transvect (5.7) throughout with $\frac{1}{2}ie^{kl}{}_{pq}$ and use (2.10). One gets at once

$$\tau_{Apq}\tau_{Bmn}-\tau_{Amn}\tau_{Bpq} = 2ig_{[k[m}\hat{T}_{n]l]AB}e^{kl}{}_{pq}.$$

If this be combined by (5.7) one obtains the desired relation at once, viz.

$$\tau_{Akl}\tau_{Bmn} = -2\bar{\eta}_{klp[m}\hat{T}_{n]}{}^{p}{}_{AB}. \tag{5.9}$$

6. Miscellaneous identities

(a) In this section some identities are derived which are useful in reducing expressions containing products of a large number of τ-matrices to tractable form. Consider therefore first the product of three such matrices with all but two w-indices paired. Thus when all r-indices are undotted one gets, on repeatedly using (4.10),

$$\begin{aligned}\tau_{Akl}\tau_{B}{}^{ml}\tau_{Cmn} &= (\delta^{m}{}_{k}\alpha_{AB}-e_{ABD}\tau^{D}{}_{k}{}^{m})\tau_{Cmn}\\ &= \alpha_{AB}\tau_{Ckn}+e_{ABD}(g_{kn}\delta^{D}{}_{C}-\alpha_{CE}e^{DGF}\tau_{Fkn})\\ &= \alpha_{AB}\tau_{Ckn}+\alpha_{BC}\tau_{Akn}-\alpha_{CA}\tau_{Bkn}+e_{ABC}g_{kn}.\end{aligned} \tag{6.1}$$

Now let one of the r-indices be dotted. Then

$$\begin{aligned}\tau_{Akl}\tau_{B}{}^{ml}\tau_{Cmn} &= \tau_{Akl}(\delta^{l}{}_{n}\alpha_{BC}-e_{BCD}\tau^{Dl}{}_{n})\\ &= \alpha_{BC}\tau_{Akn}-e_{BCD}\hat{T}_{knA}{}^{D},\end{aligned} \tag{6.2}$$

and that is as far as one can go. Note, however, that

$$\tau_{Akl}\tau_{B}{}^{ml}\tau_{Cm}{}^{k} = 0. \tag{6.3}$$

If on the left the second r-index had been dotted instead of the first one can still proceed in this way, first using (5.8). The cases of two or three dotted r-indices are covered by the identities complex conjugate to (6.2) and (6.1) respectively.

Next consider products of four τ-matrices. When all r-indices are undotted one may simply transvect (6.1) throughout with $\tau_{Ds}{}^{n}$. Once again using (4.10) one obtains

$$\begin{aligned}\tau_{Akl}\tau_{B}{}^{ml}\tau_{Cmn}\tau_{Ds}{}^{n} &= (\alpha_{AB}\alpha_{CD}-\alpha_{AC}\alpha_{BD}+\alpha_{AD}\alpha_{BC})g_{sk}\\ &\quad+(\alpha_{AB}e_{CDE}+\alpha_{BC}e_{ADE}+\alpha_{AC}e_{DBE}+\alpha_{DE}e_{ABC})\tau^{E}{}_{sk}\end{aligned} \tag{6.4}$$

In particular

$$\tau_{Akl}\tau_{B}{}^{ml}\tau_{Cmn}\tau_{D}{}^{kn} = 4(\alpha_{AB}\alpha_{CD}-\alpha_{AC}\alpha_{BD}+\alpha_{AD}\alpha_{BC}). \tag{6.5}$$

When one r-index is dotted nothing much can be achieved: in effect all one can do is to transvect (6.1) with $\tau_{Ds}{}^{n}$. Note, however, that

(6.6) $$\tau_{Akl}\tau_{\dot{B}}{}^{ml}\tau_{Cmn}\tau_{\dot{D}}{}^{kn} = 0.$$

When two r-indices are dotted one gets

(6.7)
$$\begin{aligned}\tau_{Akl}\tau_{\dot{B}}{}^{ml}\tau_{Cmn}\tau_{Ds}{}^{n} &= \hat{T}_{kmAC}\hat{T}^{m}{}_{s\dot{B}D}\\ &= \alpha_{A\dot{B}}\alpha_{CD}g_{ks}-\alpha_{CD}e_{A\dot{B}\dot{E}}\tau^{\dot{E}}{}_{ks}-\alpha_{A\dot{B}}e_{CDE}\tau^{E}{}_{ks}-e_{A\dot{B}\dot{E}}e_{CDF}\hat{T}_{ks}{}^{\dot{E}F}.\end{aligned}$$

In particular

(6.8) $$\tau_{Akl}\tau_{\dot{B}}{}^{ml}\tau_{Cmn}\tau_{D}{}^{kn} = 4\alpha_{A\dot{B}}\alpha_{CD}.$$

(b) Different kinds of identities arise when one considers transvection of products of τ-matrices with the e-tensors. Thus consider

$$e^{klmn}e^{pqst}\tau_{Akp}\tau_{Blq} = -\delta^{klmn}_{pquv}g^{su}g^{tv}\tau_{Ak}{}^{p}\tau_{B}{}^{lq}.$$

As a next step write δ^{klmn}_{pquv} as a determinant of simple Kronecker deltas. One is left with fourteen separate terms involving the product of two τ-matrices. Two of these terms may be reduced by means of (2.13), and another eight by means of (4.10). The end result of all this is

(6.9) $$e^{klmn}e^{pqst}\tau_{Akp}\tau_{Blq} = 4\tau_{A}{}^{[s[m}\tau_{B}{}^{n]t]}+8e_{ABC}\tau^{C[s[m}g^{n]t]}.$$

In much the same way one obtains

(6.10) $$e^{klmn}e^{pqst}\tau_{Akp}\tau_{Blq} = 4g^{[m[s}\hat{T}^{t]n]}{}_{AB}.$$

Now replace the dummy index C on the right of (6.9) by some other symbol and transvect throughout with $\tau_{Cms}\tau_{Dnt}$. After some reduction one obtains the result

(6.11) $$e^{klmn}e^{pqst}\tau_{Akp}\tau_{Blq}\tau_{Cms}\tau_{Dnt} = -24\alpha_{A(B}\alpha_{CD)}.$$

Suppose now that the four r-indices all have the same fixed value J. Then, bearing (2.1) in mind, the left hand member of (6.11) is simply $-24g^{-1}\det\tau_{Jkl}$. Thus

(6.12) $$\det\tau_{Jkl} = g(\alpha_{JJ})^{2}.$$

(c) An argument analogous to that used at the beginning of Section 2b may also be used in w-space. Thus the expression $\tau^{A}{}_{[kl}\tau^{B}{}_{mn]}$ is completely skew-symmetric in its w-indices so that one has

$$\tau^{A}{}_{[kl}\tau^{B}{}_{mn]} = e_{klmn}f^{AB},$$

where f^{AB} is an r-tensor. By transvection with e^{klmn} it follows that

(6.13) $$\tau^{A}{}_{[kl}\tau^{B}{}_{mn]} = -\tfrac{1}{3}i\alpha^{AB}e_{klmn}.$$

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If one writes this out in full one easily convinces oneself that it may be written in the equivalent form

$$\tau^{(A}{}_{k[l}\tau^{B)}{}_{mn]} = -\tfrac{1}{3}i\alpha^{AB}e_{klmn}. \tag{6.14}$$

7. Tensor equivalents

The correspondence, or equivalence, of rotors ϕ_A on the one hand and self-dual w-tensors F_{kl} on the other is basic to the present calculus. Both ϕ_A and F_{kl} are irreducible, and likewise there will correspond to any irreducible r-tensor an irreducible w-tensor. A general r-tensor θ_{AB} of valence 2 (both indices undotted) gives rise to the irreducible tensors θ $(=\theta_C{}^C)$, $\theta_{[AB]}$, $\theta_{(AB)}-\tfrac{1}{3}\alpha_{AB}\theta$, with 1, 3 and 5 distinct (complex) components respectively. The scalar θ is of no interest here, and this is true also of the skew tensor $\theta_{[AB]}$ since this is not essentially distinct from the vector $e^{ABC}\theta_{AB}$. This leaves only the symmetric trace-free tensor

$$\Theta_{AB} = \theta_{(AB)}-\tfrac{1}{3}\alpha_{AB}\theta \tag{7.1}$$

to be contemplated. It defines a w-tensor of valence 4:

$$t_{klmn} = \tfrac{1}{4}\tau^A{}_{kl}\tau^B{}_{mn}\Theta_{AB}. \tag{7.2}$$

This relation is reversible, and one has

$$\Theta_{AB} = \tfrac{1}{4}\tau_A{}^{kl}\tau_B{}^{mn}t_{klmn}. \tag{7.3}$$

From (7.2)

$$\begin{gathered} t_{(kl)mn} = 0, \quad t_{kl(mn)} = 0, \quad t_{klmn} = t_{mnkl}, \\ t_{k[lmn]} = 0, \quad g^{lm}t_{klmn} = 0. \end{gathered} \tag{7.4}$$

The first and second of these follow from (2.8), the third from the symmetry of Θ_{AB}, the fourth and fifth from (7.14) and (2.15) respectively, bearing in mind in each case that Θ_{AB} is both symmetric and trace-free. Accordingly t_{klmn} has all the algebraic properties of the conformal curvature tensor of Riemannian geometry.

Let s_{klmn} be any tensor of valence 4 of which it is required only that it be skew-symmetric in the first and in the second pair of indices. Then one defines its right dual and left dual as

$$s^\dagger{}_{klmn} = -\tfrac{1}{2}ie_{mn}{}^{pq}s_{klpq}, \quad {}^\dagger s_{klmn} = -\tfrac{1}{2}ie_{kl}{}^{pq}s_{pqmn}. \tag{7.5}$$

Its "double dual" is then

$${}^\dagger s^\dagger{}_{klmn} = -\tfrac{1}{4}e_{kl}{}^{pq}e_{mn}{}^{st}s_{pqst}.$$

Manipulating as usual on the right (as after eq. (6.10) for instance) one gets

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(7.6) $$^{\dagger}s^{\dagger}_{klmn} = s_{klmn} + 4g_{[k[m}\hat{s}_{n]l]}$$

where

(7.7) $$\hat{s}_{kl} = s_{kl} - \tfrac{1}{4}g_{kl}s^{n}{}_{n}, \quad s_{kl} = s^{n}{}_{kln}.$$

It follows incidentally that s_{klmn} is self-double dual if and only if the contraction $\hat{s}_{kl}$ vanishes. In the case of the tensor of eq. (7.2) one has at once

(7.8) $$t^{\dagger}{}_{klmn} = {}^{\dagger}t_{klmn} = t_{klmn},$$

because of (2.10). This is consistent with (7.6) since t_{klmn} is entirely trace-free.

Amongst rotors of valence 2 it remains to consider those with just one dotted index $\theta_{\dot{A}B}$ say, and it suffices to take $\theta_{\dot{A}B}$ as hermitian. Then, since according to (5.8) $\hat{T}_{kl\dot{A}B}$ is hermitian, symmetric and trace-free, the w-tensor

(7.9) $$t_{kl} = \hat{T}_{kl\dot{A}B}\,\theta^{\dot{A}B}$$

is real, symmetric and trace-free. The inverse relation is

(7.10) $$\theta^{\dot{A}B} = \tfrac{1}{2}\hat{T}^{kl\dot{A}B}t_{kl},$$

on account of (6.8). It may be noted that the tensor

(7.11) $$t_{klmn} = \tfrac{1}{4}\tau_{\dot{A}kl}\tau_{Bmn}\theta^{\dot{A}B}$$

is only superficially more general that the tensor t_{kl} in the sense that it expresses itself in terms of t_{kl} and the metric tensor alone. This follows at once from (5.9), because of which (7.11) becomes

(7.12) $$t_{klmn} = -\bar{\eta}_{klp[m}t_{n]}{}^{p}.$$

One may contemplate r-tensors of valence greater than 2, for example a completely symmetric trace-free tensor of valence 3. This has in general $10-3=7$ distinct components. The w-tensor equivalent, defined in analogy with (7.2), is of valence 6, and it has rather complicated symmetries. However, this is not the place to go further with these questions.

8. Improper Lorentz transformations

An improper Lorentz transformation ($l = -1$) cannot have as its image a linear transformation in r-space. It is instructive to see what would be the formal consequences in Section 3a if one inadvertently ignored the condition $l = 1$ there. It suffices to consider inversions:

(8.1) $$l^{k}{}_{m} = e_{m}\delta^{k}{}_{m}, \quad (e_1 = e_2 = e_3 = -1,\ e_4 = 1),$$

where m is fixed on the right. Then (3.4) becomes

$$\begin{aligned} \Lambda^B{}_A &= \tfrac{1}{4}\sum_{k,l} e_k e_l \tau_{Akl}\tau^{Bkl} \\ &= \tfrac{1}{4}[\tau_{Akl}\tau^{Bkl} - \sum(1-e_k e_l)\tau_{Akl}\tau^{Bkl}] \\ &= \delta^B{}_A - \tau_{Ak4}\tau^{Bk4} = 0, \end{aligned} \tag{8.2}$$

in view of (2.13) and (4.10); and this is absurd.

The appropriate r-transformation associated with an improper Lorentz transformation is *antilinear*, i.e. in place of the second member of (3.1) the associated transformation of a rotor ϕ_A is

$$'\phi_A = \Lambda^B{}_A \bar{\phi}_B. \tag{8.3}$$

On the right there appears the rotor $\bar{\phi}_B$ whose components are the complex conjugates of those of ϕ_B; but the index B remains undotted. Any other barred rotor is defined analogously. Then one has in place of (3.4)

$$\Lambda^B{}_A = \tfrac{1}{4} l^k{}_m l^l{}_n \tau_{Akl}\bar{\tau}^{Bmn}, \tag{8.4}$$

and this is the explicit form of the r-image of the improper Lorentz transformation. As for α_{AB}, one gets an equation just like (3.7) except that the sign of the second term in the brackets is reversed. This just makes up for the change in sign of l and therefore

$$'\alpha_{AB} = \alpha_{AB} \tag{8.5}$$

again.

9. Rotor-spinors

In the spinor calculus one is familiar with so-called tensor-spinors, i.e. covariant quantities which possess both s-indices and w-indices and so transform as tensors under transformations in s-space and transformations in w-space, whether these be complex or not. The analogous quantities in the rotor calculus are those which have both r-indices and w-indices. The most important of these are of course the basic matrices τ_{Akl}, which are the analogues of the Pauli matrices $\sigma_{k\dot{\mu}\nu}$.

Now on the one hand the preceding work arises out of the one-one correspondence between self-dual w-tensors and rotors, whilst on the other hand there is a one-one correspondence between self-dual w-tensors and symmetric s-tensors of valence 2. Thus one may also contemplate the formal consequences of the one-one correspondence between rotors ϕ_A and symmetric spinors $f_{\mu\nu}$, — thus closing the circle, as it were. (See also the remarks following eq. (9.13).) The basic rotor-spinors (connecting matrices) will be denoted by $\lambda_{A\mu\nu}$, so that

(9.1) $$\phi_A = \tfrac{1}{2}\lambda_{A\mu\nu}f^{\mu\nu},$$

with

(9.2) $$\lambda_{A[\mu\nu]} = 0.$$

Together with (9.1) one has (2.7) or its inverse (2.16), and the connecting equation between $f_{\mu\nu}$ and F_{kl}, i.e.

(9.3) $$F_{kl} = \tfrac{1}{2}S_{kl\mu\nu}f^{\mu\nu}$$

or its inverse

(9.4) $$f_{\mu\nu} = S^{kl\mu\nu}F_{kl},$$

(cf. *S*, Section 2). Insert (9.3) in (2.16) and compare with (9.1). Then, since $\lambda_{A\mu\nu}$ and $S_{kl\mu\nu}$ are symmetric, it follows that

(9.5) $$\lambda_{A\mu\nu} = \tfrac{1}{2}\tau_A{}^{mn}S_{mn\mu\nu}.$$

Each of the connecting matrices may thus be exhibited as a transvection of those of the other two types, i.e.

(9.6) $$S_{kl\mu\nu} = \tfrac{1}{2}\tau^A{}_{kl}\lambda_{A\mu\nu},$$

and

(9.7) $$\tau_{Akl} = \lambda_A{}^{\mu\nu}S_{kl\mu\nu}.$$

Various identities obeyed by the $\lambda_{A\mu\nu}$ may now be obtained by using those obeyed by the other two kinds of connecting matrices. Thus, using (2.17) and the identity

(9.8) $$S^{kl\mu\nu}S_{kl\alpha\beta} = 2\delta^\alpha{}_{(\mu}\delta^\beta{}_{\nu)}$$

it follows from (9.5) that

(9.9) $$\lambda_{A\mu\nu}\lambda^{A\alpha\beta} = 2\delta^\alpha{}_{(\mu}\delta^\beta{}_{\nu)}.$$

By means of this (9.1) may be inverted, viz.

(9.10) $$f_{\mu\nu} = \lambda_{A\mu\nu}\phi^A.$$

Again, inserting (9.7) in (4.10) it follows without much difficulty that

(9.11) $$\lambda_{A\mu\nu}\lambda_{B\nu}{}^{\alpha} = \gamma_{\mu\nu}\alpha_{AB} + e_{ABC}\lambda^C{}_{\mu\nu},$$

if use be made of the identity *S* (2.18), i.e.

(9.12) $$S^{ks\mu\nu}S_{kt\alpha\beta} = \sigma^{s\dot{\lambda}(\mu}\sigma_{t\dot{\lambda}(\alpha}\delta_{\beta)}{}^{\nu)}.$$

The identity

(9.13) $$\lambda^{[A}{}_{\mu\nu}\lambda^{B]\alpha\beta} = e^{ABC}\lambda_C{}^{(\alpha}{}_{(\lambda}\delta^{\beta)}{}_{\nu)}$$

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follows similarly from (4.13). These examples will suffice to illustrate the general procedure.

Despite the many formal analogies between the identities of this section and those derived earlier it is apparent that the $\lambda_{A\mu\nu}$ are intrinsically less interesting that the $\sigma_{k\dot{\mu}\nu}$ and the τ_{Akl}. Thus there can be no relations analogous to those of Section 5 since no transvections can be formed of $\lambda_{A\mu\nu}$ with a conjugated quantity $\lambda^{\dot{B}\dot{\rho}\dot{\sigma}}$. Furthermore, given an infinitesimal Lorentz transformation, the infinitesimal operators acting on the indices of $\lambda_{A\mu\nu}$ are already known, so that no new infinitesimal operators arise: at best one might define

$$\lambda_{\mu\nu AB} = \lambda_{A(\mu}{}^{\alpha}\lambda_{B\nu)\alpha} \tag{9.14}$$

in analogy with T_{klAB} and $S_{kl\mu\nu}$. With (9.11) and (4.9) one has at once

$$\Lambda_{\mu\nu AB} = \tfrac{1}{2}S^{mn}{}_{\mu\nu}T_{mnAB}, \tag{9.15}$$

which on inversion gives

$$T_{klAB} = S_{kl}{}^{\mu\nu}\Lambda_{\mu\nu AB}, \tag{9.16}$$

so that $\Lambda_{\mu\nu AB}$ is the direct connecting link between the infinitesimal spinor and rotor operators.

10. Standard representation

(a) All relations hitherto considered have been arrived at without the aid of any particular representation of the various matrices involved. However, for purposes of calculation or for other reasons it is sometimes convenient to have the explicit form of these available in some particular representation. One such representation, hereafter called "standard representation", will now be examined, the metric tensors being taken as

$$g_{kl} \equiv \eta_{kl} = \mathrm{diag}\,(-1, -1, -1, +1), \quad \alpha_{AB} = \mathrm{diag}\,(1, 1, 1). \tag{10.1}$$

(See also Subsection (d) below). Evidently one need not distinguish now between covariant and contravariant r-indices, and they will therefore all be written as subscripts, though the summation convention will be retained for them. Furthermore, in this section indices u, v, w shall take values 1, 2, 3 only: and since only a particular representation is under consideration one can contemplate the formal appearance of a quantity such as ε_{Aklm} for instance, in which the fact that indices are *apparently* associated with different vector spaces is to be regarded as irrelevant. In other words, though in a general context, A would strictly speaking go over the range I, II, III and k over $1, \cdots, 4$, I, II, III are here not to be distinguished from 1, 2, 3.

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The infinitesimal transformations of the 3-dimensional group of spatial rotations in the representation of the group by itself (eq. Corson 1953, p. 43) give a particular choice of T_{uvAB}, viz.

$$T_{uvAB} = -2\delta^u{}_{[A}\delta^v{}_{B]},$$

bearing (4.3) in mind. Then, because of (4.9),

$$\tau_{Auv} = -\tfrac{1}{2}\varepsilon_{ABC}T_{uvBC} = \varepsilon_{Auv}. \tag{10.2}$$

Now in view of (2.9) one can always write

$$\tau_{Akl} = \eta_{klmn}\rho_A{}^{mn}, \tag{10.3}$$

where $\rho_A{}^{mn}$ is skew-symmetric in the superscripts but otherwise arbitrary. In particular

$$\begin{aligned}\tau_{Auv} &= \rho_{Auv}+\tfrac{1}{2}i\varepsilon_{uvmn}\rho_A{}^{mn}\\ &= \rho_{Auv}-i\varepsilon_{uvw4}\rho_{Aw4}.\end{aligned}$$

In the present instance, comparing this with (10.2) it follows that

$$\rho_{Auv} = \varepsilon_{Auv}, \quad \rho_{Au4} = 0,$$

and these may be combined into

$$\rho_{Akl} = \varepsilon_{Akl4}. \tag{10.4}$$

Standard representation (SR) is then that representation in which (granted (10.1)) the τ_{Akl} are given by (10.3), (10.4). Explicitly,

$$\tau_{Auv} = \varepsilon_{Auv}, \quad \tau_{Au4} = i\delta^A{}_u. \tag{10.5}$$

Writing these out fully in matrix form, (cf. Corson 1953, p. 99)

(10.6)

$$\rho_{1kl} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad \rho_{2kl} = \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad \rho_{3kl} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\tau_{1kl} = \begin{pmatrix} 0 & 0 & 0 & i\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -i & 0 & 0 & 0\end{pmatrix}, \quad \tau_{2kl} = \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & i\\ 1 & 0 & 0 & 0\\ 0 & -i & 0 & 0\end{pmatrix}, \quad \tau_{3kl} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & i\\ 0 & 0 & -i & 0\end{pmatrix}.$$

A special Lorentz transformation

$$l^k{}_m = \begin{pmatrix} \gamma & 0 & 0 & \gamma u\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \gamma u & 0 & 0 & \gamma\end{pmatrix}, \quad \gamma = (1-u^2)^{-\frac{1}{2}} \tag{10.7}$$

then has the image

$$(10.8)\qquad \Lambda^B{}_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & iu\gamma \\ 0 & -iu\gamma & \gamma \end{pmatrix},$$

whereas a spatial rotation through the angle θ about the x^3-axis has th image

$$(10.9)\qquad \Lambda^B{}_A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The T_{klAB} are effectively given by (10.6), on account of (4.9). Th $\hat{T}_{kl\dot{A}B}$ on the other hand remain to be evaluated. They may be exhibite either as symmetric 4×4 matrices or hermitian 3×3 matrices (cf. th remark following eq. (5.8).) The first choice gives

(10.10)

$$\hat{T}_{\dot{1}1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\quad \hat{T}_{\dot{1}2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix},\quad \hat{T}_{\dot{1}3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\hat{T}_{\dot{2}2} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\quad \hat{T}_{\dot{2}3} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},\quad \hat{T}_{\dot{3}3} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

whilst $\hat{T}_{\dot{2}1}$, $\hat{T}_{\dot{3}1}$, $\hat{T}_{\dot{3}2}$ follow from the second, third and fifth of these b complex conjugation.

(b) It may be noted that the introduction of ρ_{Akl} through (10.3) i hardly useful in the general case. Thus, even in a representation other tha SR the ρ_{Akl} might be real. Then if α_{AB} is also real one easily infers fron (4.10) that

$$(10.11)\qquad \rho_{Akl}\rho_B{}^{kl} = 2\alpha_{AB},\quad e^{klmn}\rho_{Akl}\rho_{Bmn} = 0,$$

and

$$(10.12)\qquad \rho_{[Ak}{}^m\rho_{B]lm} = -\tfrac{1}{2}e_{ABC}\rho^C{}_{kl}.$$

Yet one can easily convince oneself that in general the reality of the ρ_{Ak} will be destroyed under a coordinate transformation. Thus under th coupled infinitesimal transformations (3.9), (3.10) one finds (if ρ_{Akl} i real) that

$$(10.13)\qquad \mathrm{Im}\,({}'\rho_{Akl}) = \tfrac{1}{2}\omega^{mn}e_{mnpq}e^{ABC}\rho_{Bkl}\rho_C{}^{pq},$$

which will in general fail to vanish. In particular, if, for example, on takes SR on the right, one sees that the ρ_{Akl} will remain real only if the Lorentz transformation is simply a spatial rotation.

It may be mentioned in passing that the ρ_{Akl} define a symmetric tensor

(10.14) $$\zeta_{kl} = \tfrac{1}{2}\rho_{Akm}\rho^{A}{}_{l}{}^{m}.$$

In SR it has the components

(10.15) $$\zeta_{kl} = 2\eta_{k[l}\eta_{4]4} = \operatorname{diag}(-1, -1, -1, 0),$$

which under an arbitrary Lorentz transformation $L^{m}{}_{k}$ becomes

(10.16) $$'\zeta_{kl} = \eta_{kl} - L^{4}{}_{k}L^{4}{}_{l}.$$

(c) The term SR may be extended to cover some standard form of the basic s-tensors. The following will be adopted:

(10.17) $$\sigma_{1\mu\dot{\nu}} = \sqrt{\tfrac{1}{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_{2\mu\dot{\nu}} = \sqrt{\tfrac{1}{2}}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},$$
$$\sigma_{3\mu\dot{\nu}} = \sqrt{\tfrac{1}{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \sigma_{4\mu\dot{\nu}} = \sqrt{\tfrac{1}{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

which are exactly those of Infeld and van der Waerden (1933). Then the $S^{kl}{}_{\mu\nu}$ are

(10.18) $$S^{23} = \tfrac{1}{2}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}, \quad S^{31} = \tfrac{1}{2}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \quad S^{12} = \tfrac{1}{2}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},$$
$$S^{14} = \tfrac{1}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad S^{24} = \tfrac{1}{2}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}, \quad S^{34} = \tfrac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

which satisfy the conditions of self-duality

(10.19) $$\bar{\eta}_{klmn}S^{mn} = 0.$$

In view of (10.19) and (10.3), (9.5) becomes

(10.20) $$\lambda_{A\mu\nu} = \rho_{Amn}S^{mn}{}_{\mu\nu}.$$

Because of (10.4) one therefore has at once

(10.21) $$\lambda_1 = \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}, \quad \lambda_2 = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \quad \lambda_3 = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

(d) Since, for any (fixed) A τ_{Akl} is a skew-symmetric tensor, the problem of choosing some standard representation is formally equivalent to investigating canonical forms of the electromagnetic field tensor, as set out at length by Synge (1958) for instance. It is hardly necessary to go into detail here. It suffices to remark that the analogue of the energy momentum tensor is $\hat{T}_{kl\dot{A}A}$ (any A), whilst the invariants of the field are combined into α_{AA}. The eigenvalues of τ_{Akl} are $\pm(-\alpha_{AA})^{\frac{1}{2}}$, each of these appearing twice, and the "null field" corresponds to $\alpha_{AA} = 0$.

It will be observed that if the canonical tensors F_{rs}, $F_{rs}{}^*$ given by Synge (eq. (98), p. 336) be combined into the self-dual tensor $F_{rs}+iF_{rs}{}^*$ (Synge's notation), and due allowance is made for the use of an imaginary time coordinate one gets the consistent result that

$$F_{rs}+iF_{rs}{}^* \leftrightarrow \tau_{1rs} \tag{10.22}$$

provided one formally sets $H_1+iE_1 = 1$. The metric of r-space chosen above of course implies the absence of null τ_{Akl}. With a choice of the r-metric in which one or more of the α_{AA} are zero, one would of course have to adopt some alternative SR. Suppose, for the sake of illustration, that one took as a new SR that obtained from the one above by the transformation

$$\Lambda^B{}_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & -i \\ 0 & 1 & 1 \end{pmatrix}, \tag{10.23}$$

so that then

$$'\alpha_{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}. \tag{10.24}$$

Then, in particular

$$'\tau_2 = i\tau_2+\tau_3 = \begin{pmatrix} 0 & 1-i & 0 \\ -1 & 0 & 0-1 \\ i & 0 & 0 & i \\ 0 & 1-i & 0 \end{pmatrix}.$$

Taking $E_2 = 1$ in Synge's canonical tensors in the null case (eq. (97) p. 336), one arrives at the harmonious correspondence

$$F_{rs}+iF_{rs}{}^* \leftrightarrow {'\tau_{2rs}}. \tag{10.25}$$

In this sort of way a variety of results established by Synge can be transcribed into the present context.

11. Concluding remarks

At this point the development of the rotor calculus in flat w-space may be broken off since the stage is adequately set for its generalization to curved w-spaces, i.e. Riemann or Weyl spaces, a task to be undertaken in a second paper (Buchdahl 1966). Once this generalization has been achieved various special topics which might already have been treated above can be dealt with more generally, or more conveniently; and here duality rotations may serve as a suitable example.

Finally, I should like to express my warmest thanks to Dr. Mark Andrews for a number of stimulating and informative discussions, without which this work would most likely never have been done.

Note added in proof: Since the manuscript of this paper was completed Professors Debever and Cahen have kindly drawn my attention to their work involving *r*-space; see, for example, Debever, R., Cahier de Phys., 168–169, 303.

References

Andrews, M., 1964, Thesis, *Birmingham.*

Buchdahl, H. A., 1962, *Jour. Australian Math. Soc.* 2, 369.

Buchdahl, H. A., 1966, *Jour. Australian Math. Soc.* 6, 424.

Corson, E. M., 1953, *Introduction to tensors, spinors, and relativistic wave equations,* § 13. Blackie and Son.

Infeld, L. and van der Waerden, B. L., 1933, *Sitz. Preu. Akad. Wiss.*, 9, 380.

Jordan, P., Ehlers, J. and Kundt, W., 1960, *Akad. Wiss. Lit. Mainz. Abh. Math.-Nat. Kl.* No. 2, 29.

Kurşunoğlu, B., 1961, *Jour. Math. Phys.* 2, 22.

Peres, A., 1962, *Jour. Math. Mech.* 11, 61.

Petrov, A. Z., 1962, *Recent developments in general relativity,* p. 371. Pergamon.

Rastall, P., 1964, *Nuovo Cim.* 34, 1563.

Schouten, J. A., 1954, *Ricci Calculus* (*2nd ed.*). Springer.

Synge, J. L., 1958, *Relativity: the special theory.* Chap. 9, North-Holland.

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