CONTRIBUTIONS TO THE CELL GROWTH PROBLEM

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Introduction. The cell growth problem is a combinatorial problem which may be stated as follows: A plane animal is made up of cells, each of which is a square of unit area. It starts as a single cell, and grows by adding cells one at a time in such a way that the new cell has at least one side in contact with a side of a cell already present in the animal. The problem is to find the number of different animals of area n, it being understood that animals which can be transformed into each other by reflections or rotations of the plane will be regarded as the same animal. An account of the history of this problem, including the information displayed in Table I, has been given by Harary **(2)**.* Examples of animals are given by the shaded portions of Figures 1, 2, 3, and 4.

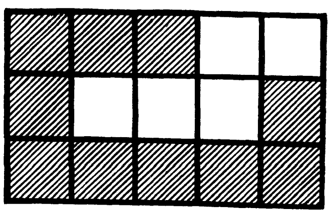

FIGURE 1

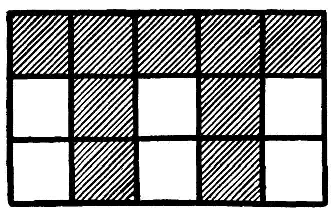

FIGURE 2

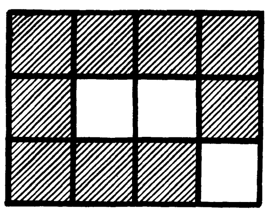

FIGURE 3

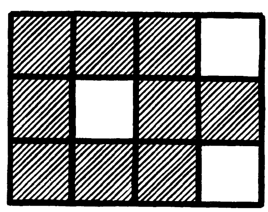

FIGURE 4

By definition these animals are connected. In enumerating them we may or may not include animals which are multiply connected as in Figures 3 and 4; in this paper we shall, unless otherwise stated, include them. The present state of knowledge concerning the numbers of animals is shown by Table I. The values for $n = 7, 8$ were computed by the MANIAC II computer at Los Alamos in 1959 (see Harary **(2)**).

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*Another name for these "animals" is *polyominoes* (being a generalization of dominoes—see, for example, Golomb **(1)**).

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TABLE I

Area n	1	2	3	4	5	6	7	8
No. of simply-connected animals	1	1	2	5	12	35	107	363
No. of multiply-connected animals	0	0	0	0	0	0	1	6

In this paper shall attack the problem indirectly, considering first some related topics rather more amenable to study, and then returning to the original cell growth problem.

1. Nomenclature and notation. If an animal can be drawn inside a $p \times q$ rectangle, that is, a rectangle having p rows and q columns of unit squares, in such a way that every column contains at least one cell of the animal, then it will be called a "$(p \times q)$-animal." If, in addition, every row of the $p \times q$ rectangle contains at least one cell of the animal we shall say that the animal is a "proper $(p \times q)$-animal." A $p \times q$ rectangle is thus the smallest rectangle inside which a proper $(p \times q)$-animal can be drawn. A $(p \times q)$-animal which is not proper will be called "improper." Figure 5 shows an improper (4×5)-animal.

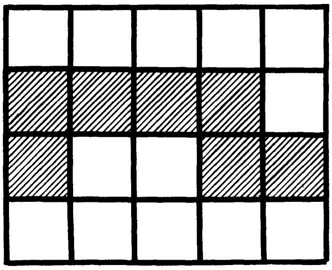

FIGURE 5

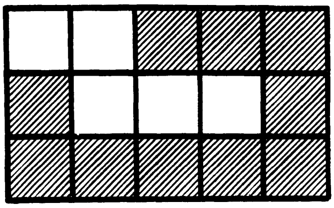

FIGURE 6

The first problem that we shall consider is that of finding, for given p and q, the number of proper $(p \times q)$-animals having a given number, n, of cells. In doing this we shall regard the rectangle as being fixed in the plane, so that the animals of Figures 1 and 6 will be regarded as *different* animals. Animals counted in this way will be called "fixed animals." Our second problem will be the same, except that we shall not regard the rectangle as fixed. Consequently, any two animals that can be transformed into each other by rotation or reflection, such as those in Figures 1 and 6, will now be regarded as the same. Animals counted in this way will be called "free animals."

We shall sometimes need to consider figures that resemble animals but which are not connected. We shall use the term "configuration" in place of "animal" whenever there is a possibility that the condition of connectedness is not satisfied.

We shall need to use rather a lot of symbols of various kinds, so that uniformity in their use is desirable. The letters A, B, C, D, and E will be associated with the values 1, 2, 3, 4, and 5 of p. Gothic letters will be used for the columns from which the animals will be built up. Capital letters (Latin or Greek) will refer to fixed animals; lower case letters will refer to free animals. An asterisk will denote that we are dealing with both proper and improper animals, while its absence will denote that we are dealing only with proper animals.

2. A general theorem. We now state and prove a theorem which will be used several times in what follows. We shall state it in very general terms, because it has relevance to problems other than those we are now considering.

Suppose we are given a certain finite number, say k, of objects, and that associated with each object is a positive integer, which we may call its "content." We are interested in constructing finite sequences of these objects, repetitions allowed, subject to certain rules concerning the manner in which the objects may follow each other in the sequence. These rules will be of three kinds.

(i) those that specify which objects may be chosen to start a sequence;

(ii) those that specify which objects may be chosen to finish a sequence, and

(iii) those which specify, for each object, the objects which may follow it.

Thus the set of objects which may occupy a given position in the sequence will depend on which object occupies the previous position. A sequence constructed according to the above rules will be called a "permissible sequence."

The problem is then to determine, for given integers q and n, the number of permissible sequences of length q and content n, where the length of a sequence is defined as the number of objects it contains, and the content of a sequence is the sum of the contents of its constituent objects.

Let the objects be denoted by $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_k$, and let σ_i be the content of $\mathfrak{M}_i$. The "start vector" is defined as the column vector $\mathbf{S} = \{s_1, s_2, \ldots, s_k\}$ where $s_j = y^{\sigma_j}$ if M_j may start a sequence, and $= 0$ if it may not, y being an indeterminate. The "stop vector" is defined as the row vector $\mathbf{P} = [p_1, p_2, \ldots, p_k]$ where $p_i = 1$ if $\mathfrak{M}_i$ may end a sequence, and $= 0$ if it may not.

The "transition matrix" is defined as the matrix

$$\mathbf{\Psi}(y) = [m_{ij}y^{\sigma_j}]$$

where $m_{ij} = 1$ if $\mathfrak{M}_i$ may follow $\mathfrak{M}_j$ and $= 0$ if it may not. Clearly the start and stop vectors and the transition matrix embody the rules mentioned above, and suffice to determine which sequences are permissible.

By the "qth transition vector" we shall mean the column vector whose jth element is the polynomial $\sum_n M_j(q, n)y^n$, where $M_j(q, n)$ is the number of sequences of length q and content n which end with the object $\mathfrak{M}_j$. We shall denote this vector by $\mathbf{M}(q)$. We see that $\mathbf{M}(1) = \mathbf{S}$.

If $M(q, n)$ is the number of permissible sequences of length q and content n, the polynomial $\sum_n M(q, n)y^n$ will be called the "qth counting polynomial." The series $\sum_q \sum_n M(q, n)x^q y^n$, where x is an indeterminate, will be called the "sequence counting series." This series is a formal one, so that there is no question of convergence or divergence.

We now state our general theorem.

THEOREM 1. *The* qth *counting polynomial is*

$$\mathbf{P}[\mathbf{\Psi}(y)]^{q-1}\mathbf{S}; \tag{2.1}$$

the sequence counting series is

$$x\mathbf{P}[\mathbf{I} - x\mathbf{\Psi}(y)]^{-1}\mathbf{S}, \tag{2.2}$$

where $\mathbf{I}$ *is the unit matrix the same size as* $\mathbf{\Psi}(y)$.

The proof of Theorem 1 is straightforward. From the definition of m_{ij} we have

$$M_i(q, n) = \sum_j m_{ij} M_j(q - 1, n - \sigma_j) \tag{2.3}$$

so that

$$\sum_n M_i(q, n)y^n = \sum_j \sum_n m_{ij} y^{\sigma_j} M_j(q - 1, n - \sigma_j) y^{n-\sigma_j}$$
$$= \sum_j \left\{ m_{ij} y^{\sigma_j} \sum_{n'} M_j(q - 1, n')y^{n'} \right\}$$

where $n' = n - \sigma_j$, since we may take $M_j(q - 1, n') = 0$ for $n' < 1$. Hence

$$\mathbf{M}(q) = \mathbf{\Psi}(y)\mathbf{M}(q - 1)$$

and thus

$$\mathbf{M}(q) = [\mathbf{\Psi}(y)]^{q-1}\mathbf{S}. \tag{2.4}$$

The total number of sequences is given by

$$M(q, n) = \sum_i p_i M_i(q, n),$$

so that the qth counting polynomial is

$$\sum_n M(q, n)y^n = \mathbf{P}\mathbf{M}(q)$$
$$= \mathbf{P}[\mathbf{\Psi}(y)]^{q-1}\mathbf{S}.$$

The sequence counting series is

$$\sum_q \sum_n M(q, n)x^q y^n = \sum_q \mathbf{P}[\mathbf{\Psi}(y)]^{q-1}\mathbf{S}x^q$$
$$= x\mathbf{P}\left\{ \sum_q [x\mathbf{\Psi}(y)]^{q-1} \right\}\mathbf{S}$$
$$= x\mathbf{P}[I - x\mathbf{\Psi}(y)]^{-1}\mathbf{S}.$$

This completes the proof of the theorem.

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If the matrix $\mathbf{\Psi}(y)$ is not too large, then the right-hand side of (2.2) can be found explicitly, and a complete answer to the problem is then available. When the inversion of the matrix $\mathbf{I} - x\mathbf{\Psi}(y)$ is impracticable, equation (2.1) may be used to find the counting polynomials, and hence to compile tables of values of $M(q, n)$.

It is sometimes possible to use a matrix smaller than $\mathbf{\Psi}(y)$. Suppose that $s_\alpha = s_\beta$ for integers α, β, and that rows α and β of $\mathbf{\Psi}(y)$ have the same elements in all columns except columns α and β. Suppose further that

$$m_{\alpha\alpha} + m_{\alpha\beta} = m_{\beta\alpha} + m_{\beta\beta} = 1.$$

Then it is readily verified that $M_\alpha(q, n) = M_\beta(q, n)$ for all values of q and n. Hence one of these functions, say $M_\beta(q, n)$ need not be introduced at all. The effect of this will be that the element s_β of $\mathbf{S}$ can be deleted; that the element p_β of $\mathbf{P}$ can be deleted and p_α replaced by $p_\alpha + p_\beta$; and that column β of $\mathbf{\Psi}(y)$ can be deleted, and column α replaced by the sum of columns α and β.

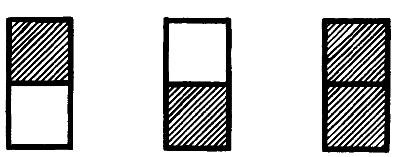
We shall then have smaller vectors $\mathbf{S}$ and $\mathbf{P}$, and a smaller matrix $\mathbf{\Psi}(y)$, for which formulae (2.1) and (2.2) will yield the same result as before. This process can be continued until no further reduction in size is possible. The resulting vectors and matrices will be said to be "reduced."

Theorem 1 and equation (2.4) hold equally well for the reduced vectors and matrices as for the original ones.

3. The enumeration of fixed animals; $p = 1,2$. There is clearly only one $(1 \times q)$-animal, and it has q cells. To conform to our general notation we shall let $A(q, n)$ be the number of proper $(1 \times q)$-animals having n cells. Thus

(3.1) $$A(q, n) = 1 \quad \text{if } n = q \qquad \text{and} \quad = 0 \quad \text{if } n \neq q.$$

We now consider a typical $(2 \times q)$-animal. We can imagine it being built up column by column starting with the left-hand column. These columns are of three kinds, as shown below, where the shading indicates those squares



$\mathfrak{B}_1$ $\mathfrak{B}_2$ $\mathfrak{B}_3$

that are cells of the animal. Since the animal is to be connected we see that

(3.2)
$$\begin{aligned}&\mathfrak{B}_1 \text{ may follow } \mathfrak{B}_1 \text{ and } \mathfrak{B}_3 \text{ only,}\\ &\mathfrak{B}_2 \text{ may follow } \mathfrak{B}_2 \text{ and } \mathfrak{B}_3 \text{ only,}\\ &\mathfrak{B}_3 \text{ may follow } \mathfrak{B}_1, \mathfrak{B}_2, \text{ and } \mathfrak{B}_3.\end{aligned}$$

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Thus our problem is a special case of the general problem of the previous section. The "objects" are the columns, the content of a column being the number of cells it contains. From (3.2) the transition matrix is

$$\begin{bmatrix} y & \cdot & y^2 \\ \cdot & y & y^2 \\ y & y & y^2 \end{bmatrix}$$

and, since any column may start or end the sequence, the start and stop vectors are $\{y, y, y^2\}$ and $[1, 1, 1]$. These may be reduced, since the first two rows of the transition matrix have the property described at the end of § 2, and the first two elements of the start vector are the same. We therefore write

$$\mathbf{\Psi}_2(y) = \begin{bmatrix} y & y \\ 2y^2 & y^2 \end{bmatrix}$$

$$\mathbf{S}_2 = \{y, y^2\} \text{ and } \mathbf{P}_2 = [2, 1].$$

If $B_j^*(q, n)$ is the number of fixed $(2 \times q)$-animals having n cells, and ending in $\mathfrak{B}_j$, then (2.4) gives

$$\mathbf{B}^*(q) = \begin{bmatrix} \sum B_1^*(q, n)y^n \\ \sum B_3^*(q, n)y^n \end{bmatrix} = \begin{bmatrix} y & y^2 \\ 2y & y^2 \end{bmatrix}^{q-1} \begin{bmatrix} y \\ y^2 \end{bmatrix};$$

while if $B^*(q, n)$ is the total number of fixed $(2 \times q)$-animals having n cells, Theorem I gives

$$\sum_n B^*(q, n)y^n = [2, 1] \begin{bmatrix} y & y \\ 2y^2 & y^2 \end{bmatrix}^{q-1} \begin{bmatrix} y \\ y^2 \end{bmatrix} \tag{3.3}$$

and

$$\sum_q \sum_n B^*(q, n)x^q y^n = [2, 1]x[I - x\mathbf{\Psi}_2(y)]^{-1} \begin{bmatrix} y \\ y^2 \end{bmatrix} \tag{3.4}$$

$$= [2, 1]x \begin{bmatrix} 1 - xy & -xy \\ -2xy^2 & -xy^2 \end{bmatrix}^{-1} \begin{bmatrix} y \\ y^2 \end{bmatrix}$$

which reduces to

$$\frac{1 + xy}{1 - xy - xy^2 - x^2y^3} - 1. \tag{3.5}$$

By omitting the multiplication on the left by [2, 1] in (3.4) we find that

$$\sum_q \sum_n B_1^*(q, n)x^q y^n = \frac{xy}{1 - xy - xy^2 - x^2y^3} \tag{3.6}$$

$$\sum_q \sum_n B_3^*(q, n)x^q y^n = \frac{xy^2(1 + xy)}{1 - xy - xy^2 - x^2y^3}. \tag{3.7}$$

The numbers $B_j^*(q, n)$ and $B^*(q, n)$ include some improper $(2 \times q)$-animals. These are readily excluded and we find that

$$B_1(q, n) = B_2(q, n) = B_1^*(q, n) - A(q, n) \tag{3.8}$$
$$B_3(q, n) = B_3^*(q, n)$$
$$B(q, n) = B^*(q, n) - 2A(q, n)$$

where the $B_j(q, n)$, etc., are the corresponding numbers of proper animals.

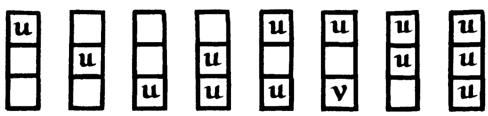
We note without proof the following explicit formula for $B^*(q, n)$ (and hence for $B(q, n)$ provided $n > q$):

$$B(q, n) = \sum_{\nu} \binom{n - q + 1}{2q - n - \nu} \binom{n - q + \nu}{n - q} \tag{3.9}$$

which may be deduced from (3.5).

4. Fixed animals; $p = 3$. When we consider the enumeration of fixed $(3 \times q)$-animals we encounter a new difficulty. This is seen if we consider the process of building up the animal of Figure 6. When the third column is added the resulting configuration is not connected. This remains true when the fourth column is added, but the addition of the fifth column restores the connectedness. Had the fifth column been the same as the third, for example, the final configuration would have remained disconnected. Thus we must devise a method whereby animals such as Figure 6 are counted, while ensuring that disconnected configurations are not counted.

To do this we take as our possible columns those shown in the diagram



$\mathfrak{C}_1$ $\mathfrak{C}_2$ $\mathfrak{C}_3$ $\mathfrak{C}_4$ $\mathfrak{C}_5$ $\mathfrak{C}_6$ $\mathfrak{C}_7$ $\mathfrak{C}_8$

where the squares containing a "u" or a "v" are the cells of the animal. It will be seen that $\mathfrak{C}_5$ and $\mathfrak{C}_6$ represent the same column. The difference lies in the way they are used. In building up a $(3 \times q)$-animal column by column we allow $\mathfrak{C}_6$ to follow columns $\mathfrak{C}_1$, $\mathfrak{C}_3$, $\mathfrak{C}_4$, and $\mathfrak{C}_7$ even though the configuration resulting at that stage is not connected. $\mathfrak{C}_6$ may be followed either by another $\mathfrak{C}_6$, which leaves the configuration disconnected, or by $\mathfrak{C}_8$, which connects it up again. Thus when all q columns have been chosen, the resulting configuration will be connected if and only if it does not end in $\mathfrak{C}_6$. We shall therefore put $p_6 = 0$ in our stop vector.

The column $\mathfrak{C}_5$ is used whenever the preceding column is such that the resulting configuration is connected. This will happen if and only if the preceding column is $\mathfrak{C}_5$ or $\mathfrak{C}_8$. By way of illustration, the animal of Figure 1

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would be built up as the sequence $\mathfrak{C}_8\mathfrak{C}_5\mathfrak{C}_5\mathfrak{C}_1\mathfrak{C}_7$; whereas that of Figure 6 would be given by $\mathfrak{C}_4\mathfrak{C}_3\mathfrak{C}_6\mathfrak{C}_6\mathfrak{C}_8$.

We cannot start a sequence of columns with $\mathfrak{C}_5$ so we put $s_5 = 0$ in our start vector.

Note that the labels "u" and "v" simply indicate that the two cells belong to different components of the configuration. Thus it is immaterial which cell is labelled "u" and which "v."

We readily verify that

$$\begin{array}{ll} & \mathfrak{C}_1 \text{ may follow } \mathfrak{C}_1, \mathfrak{C}_5, \mathfrak{C}_7, \mathfrak{C}_8 \\ & \mathfrak{C}_2 \text{ may follow } \mathfrak{C}_2, \mathfrak{C}_4, \mathfrak{C}_7, \mathfrak{C}_8 \\ & \mathfrak{C}_3 \text{ may follow } \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5, \mathfrak{C}_8 \\ & \mathfrak{C}_4 \text{ may follow } \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5, \mathfrak{C}_6, \mathfrak{C}_8 \\ (4.1) & \mathfrak{C}_5 \text{ may follow } \mathfrak{C}_5, \mathfrak{C}_8 \\ & \mathfrak{C}_6 \text{ may follow } \mathfrak{C}_1, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_7 \\ & \mathfrak{C}_7 \text{ may follow } \mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_4, \mathfrak{C}_5, \mathfrak{C}_7, \mathfrak{C}_8 \\ & \mathfrak{C}_8 \text{ may follow } \mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5, \mathfrak{C}_6, \mathfrak{C}_7, \mathfrak{C}_8. \end{array}$$

We then have a straightforward problem of the type considered in § 3.

If $C_j^*(q, n)$ is the number of $(3 \times q)$-animals having n cells and ending with $\mathfrak{C}_j$, then, by symmetry (or by inspection of the transition matrix), we see that

$$(4.2) \qquad \begin{aligned} C_1^*(q, n) &= C_3^*(q, n), \\ C_4^*(q, n) &= C_7^*(q, n). \end{aligned}$$

The transition matrix may therefore be reduced from an 8×8 matrix to a 6×6 matrix, the start and stop vectors being similarly reduced. The reduced transition matrix is

$$(4.3) \qquad \mathbf{\Psi}_3(y) = \begin{bmatrix} y & \cdot & y & y & \cdot & y \\ \cdot & y & 2y & \cdot & \cdot & y \\ y^2 & y^2 & 2y^2 & y^2 & \cdot & y^2 \\ \cdot & \cdot & \cdot & y^2 & \cdot & y^2 \\ 2y^2 & \cdot & 2y^2 & \cdot & y & \cdot \\ 2y^3 & y^3 & 2y^3 & y^3 & y^3 & y^3 \end{bmatrix}$$

and the reduced start and stop vectors are

$$(4.4) \qquad \begin{aligned} \mathbf{S}_3 &= \{y, y, y^2, 0, y^2, y^3\}, \\ \mathbf{P}_3 &= [2, 1, 2, 1, 0, 1]. \end{aligned}$$

The qth transition vector

$$\boldsymbol{C}^*(q) = \{\sum C_j^*(q, n)y^n\} \qquad \text{where } j = 1, 2, 4, 5, 6, 8$$

is given by (2.4), that is, by

$$(4.5) \qquad \boldsymbol{C}^*(q) = \mathbf{\Psi}_3^{q-1}\mathbf{S}_3$$

from which the polynomials $\sum C_j^*(q, n)y^n$ and hence the values of $C_j^*(q, n)$ may be calculated. The total number $C^*(q, n)$ of $(3 \times q)$-animals having n cells can be calculated from

$$\sum C^*(q, n)y^n = \mathbf{P}_3\mathbf{\Psi}_3^{q-1}\mathbf{S}_3 \tag{4.6}$$

which follows from (2.1).

We could, in theory, use (2.2) to obtain a generating function for $C^*(q, n)$. It would be a rational function of x and y, but its denominator would not easily be expressible as a power series, so that it would not be of much use for finding the values of $C^*(q, n)$, even ignoring the tedium of inverting the matrix $\boldsymbol{I} - x\mathbf{\Psi}_3(y)$. The numerical results quoted later were obtained using (4.5) which is quite convenient for this purpose.

We are really interested in the numbers $C_j(q, n)$ and $C(q, n)$ of proper animals of the above kinds.* By examining the way in which improper animals may occur we see that

$$\begin{aligned}
C_1(q, n) &= C_3(q, n) = C_1^*(q, n) - B_1(q, n) - A(q, n)\\
C_2(q, n) &= C_2^*(q, n) - 2B_1(q, n) - A(q, n)\\
C_4(q, n) &= C_7(q, n) = C_4^*(q, n) - B_3(q, n)\\
C_5(q, n) &= C_5^*(q, n); \qquad C_8(q, n) = C_8^*(q, n)\\
C(q, n) &= C^*(q, n) - 2B(q, n) - 3A(q, n).
\end{aligned}$$

5. Fixed animals; $p = 4, 5$. For $p = 4$ the method is essentially the same as in the last section. The same device is used to ensure the counting of animals which, though ultimately connected are disconnected at some stage in their construction, and we take, as possible columns, those shown in the diagram.

u		u	u	u	u	u			u	u	u						u	u	u
	u	u					u	u			u				u	u	u	u	u
			u	v	u	v	u	u			u		u	u					u
					u	v	u		u	v	u	u		u	u	v	u	v	
$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{D}_4$	$\mathfrak{D}_5$	$\mathfrak{D}_6$	$\mathfrak{D}_7$	$\mathfrak{D}_8$	$\mathfrak{D}_9$	$\mathfrak{D}_{10}$	$\mathfrak{D}_{11}$	$\mathfrak{D}_{12}$	$\mathfrak{D}_{13}$	$\mathfrak{D}_{14}$	$\mathfrak{D}_{15}$	$\mathfrak{D}_{16}$	$\mathfrak{D}_{17}$	$\mathfrak{D}_{18}$	$\mathfrak{D}_{19}$	$\mathfrak{D}_{20}$

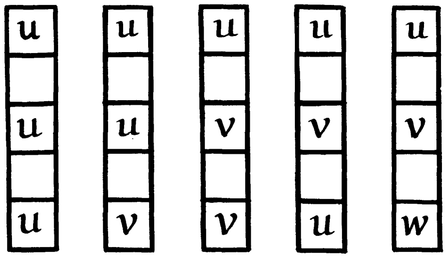
With the usual notation we note that $D^*_{j+12}(q, n) = D_j^*(q, n)$ for $j = 1, 2, \ldots, 8$. Hence our transition matrix may be reduced to a 12×12 matrix. This matrix, and the reduced start and stop vectors are readily constructed from the list of columns above, but we shall not take space to give them here. Despite the size of the matrix, equation (2.4) still gives a practical method of calculating the values of $D_j^*(q, n)$, even by hand, and it is, moreover, a method well suited to machine computation. The numbers $D_j(q, n)$ of proper

*The numbers $C_j(q, n)$ will be needed later when we consider the enumeration of free animals.

animals are expressible in terms of the $D_j^*(q, n)$ and other functions obtained in the previous sections. We shall not list these results, but we note the following:

$$(5.1) \qquad D(q, n) = D^*(q, n) - 2C(q, n) - 3B(q, n) - 4A(q, n)$$
$$= D^*(q, n) - 2C^*(q, n) + B^*(q, n).$$

Very little need be said about the case $p = 5$ except that it is now possible for a configuration resulting during the construction of an animal to have three components. This is allowed for by an obvious extension of the previous method; the column consisting of three separated cells will give rise to five different "objects," as shown in the diagram.



The transition matrix is 50×50, but can be reduced to a 31×31 matrix. Equation (2.4) can still be used to find the numbers $E_j^*(q, n)$ (defined in the obvious way) for $q \leqslant 6$ without excessive effort. From these the numbers $E^*(q, n)$, and $E_j(q, n)$ can be found. We note that

$$(5.2) \qquad E(q, n) = E^*(q, n) - 2D^*(q, n) + C^*(q, n).$$

The problems for $p \geqslant 6$ are probably beyond the range of convenient hand calculation, and will not be considered.

6. Some numerical results. We give below the results of the calculations described above for the functions $B(q, n)$, $C(q, n)$, $D(q, n)$, and $E(q, n)$ for $n \leqslant 10$.

$B(q, n)$

q \ n	1	2	3	4	5	6	7	8	9	10
1		1								
2			4	1						
3				8	6	1				
4					12	18	8	1		
5						16	38	32	10	1
6							20	66	88	50
7								24	102	192
8									28	146
9										32

$C(q, n)$

q \ n	1	2	3	4	5	6	7	8	9	10
1			1							
2				8	6	1				
3					25	44	32	9	1	
4						50	154	212	158	62
5							83	376	784	987
6								124	750	2133
7									173	1316
8										230

$D(q, n)$

q \ n	1	2	3	4	5	6	7	8	9	10
1				1						
2					12	18	8	1		
3						50	154	212	158	62
4							120	584	1396	2038
5								230	1526	5154
6									388	3276
7										602

$E(q, n)$

q \ n	1	2	3	4	5	6	7	8	9	10
1					1					
2						16	38	32	10	1
3							83	376	784	987
4								230	1526	5154
5									497	2668
6										932

7. The enumeration of free animals. General remarks. If $p \neq q$, there are five types of symmetry that an animal may possess (or lack). They are as follows:

(i) The animal may have no symmetry, that is, no rotation or reflection of the rectangle leaves the animal invariant. We shall use the symbol α when referring to free animals of this kind.

(ii) The animal may be invariant under reflection about the "horizontal" line (that is, parallel to the rows) which bisects the rectangle. We shall use the symbol B when referring to *fixed* animals having *at least* this type of symmetry,* and the symbol β when referring to *free* animals having *only* this type of symmetry.

(iii) The animal may be invariant under reflection about the "vertical" line (parallel to the columns) which bisects the rectangle. We shall use the symbol Γ when referring to *fixed* animals having *at least* this type of symmetry, and the symbol γ when referring to *free* animals having *only* this type of symmetry.

(iv) The animal may be invariant under rotation through 180° about the centre of the rectangle. We shall use the symbol Δ when referring to *fixed* animals having *at least* this type of symmetry, and the symbol δ when referring to *free* animals having *only* this type of symmetry.

(v) The animal may be invariant under each of the reflections given in (ii) and (iii). It is then also invariant under the rotation given in (iv). We shall use the symbol ϵ when referring to animals having this type of symmetry. The numbers of fixed and free animals of this kind are the same.

Examples of animals having these five types of symmetry are given by Figures 1, 4, 2, 7, and 8 respectively.

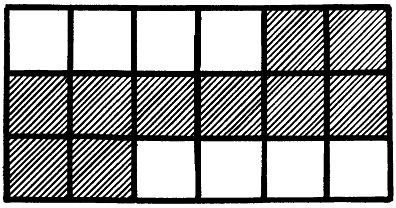


FIGURE 7

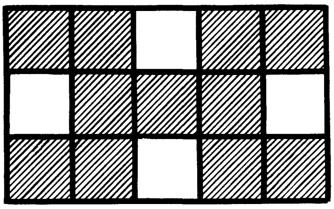


FIGURE 8

We note that if $p = q$ then other types of symmetry are possible. These will be considered later. Until then, any results obtained by putting $q = p$ in the general formulae are to be ignored.

For a given value of p let $Y(q, n)$ be the number of fixed proper $(p \times q)$-animals having n cells, and let $Y_{\mathrm{B}}(q, n)$, $Y_{\Gamma}(q, n)$ and $Y_{\Delta}(q, n)$ be the numbers of these that possess symmetry of types B, Γ, and Δ respectively. Let $y(q, n)$ be the number of free proper $(p \times q)$-animals having n cells, and let $y_\alpha(q, n)$, $y_\beta(q, n)$, $y_\gamma(q, n)$, $y_\delta(q, n)$, and $y_\epsilon(q, n)$ be the numbers of these possessing symmetry of types α, β, γ, δ, and ϵ respectively. We may now state a theorem which will be used several times in the following sections.

THEOREM 2. $4y(q, n) = Y(q, n) + Y_{\mathrm{B}}(q, n) + Y_{\Gamma}(q, n) + Y_{\Delta}(q, n)$.

Proof. The number of fixed animals corresponding to a given free animal will depend on its symmetry type, and is 4 for α-symmetry, 2 for β-, γ-, and δ-symmetry, and 1 for ϵ-symmetry. Hence

$$Y(q, n) = 4y_\alpha(q, n) + 2y_\beta(q, n) + 2y_\gamma(q, n) + 2y_\delta(q, n) + y_\epsilon(q, n). \tag{7.1}$$

But

$$y(q, n) = y_\alpha(q, n) + y_\beta(q, n) + y_\gamma(q, n) + y_\delta(q, n) + y_\epsilon(q, n). \tag{7.2}$$

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Eliminating $y_\alpha(q, n)$ we find that

(7.3) $$4y(q, n) = Y(q, n) + 2y_\beta(q, n) + 2y_\gamma(q, n) + 2y_\delta(q, n) + 3y_\epsilon(q, n).$$

If we consider the set of B-symmetric fixed proper animals we see that it will include every β-symmetric free proper animal twice, and every ϵ-symmetric free proper animal once. Hence

(7.4) $$Y_B(q, n) = 2y_\beta(q, n) + y_\epsilon(q, n).$$

Similarly

(7.5) $$Y_\Gamma(q, n) = 2y_\gamma(q, n) + y_\epsilon(q, n)$$

and

(7.6) $$Y_\Delta(q, n) = 2y_\delta(q, n) + y_\epsilon(q, n).$$

Theorem 2 then follows at once from (7.3), (7.4), (7.5), and (7.6).

In the following sections Y will be replaced by B, C, D, or E according as $p = 2$, 3, 4, or 5. This statement will serve to define all functions (for example, $C_\Delta(q, n)$) thus obtained.

The method of determining $Y_B(q, n)$ will vary according to the value of p. To determine $Y_\Gamma(q, n)$ we note that if q is even then so is n, and a Γ-symmetric animal will be determined by the "subanimal" formed by its first $\frac{1}{2}q$ columns. The complete animal is proper if and only if the subanimal is proper. If q is odd, then the centre column has odd or even content according as n is odd or even, and the animal is determined by the subanimal consisting of the first $\frac{1}{2}(q + 1)$ columns.

To determine $Y_\Delta(q, n)$ we note that if q is even then so is n, and a Δ-symmetric animal is determined by the subanimal formed by the first $\frac{1}{2}q$ columns. This subanimal may or may not be proper. If q is odd then the centre column must be B-symmetric, its content is odd or even according as n is odd or even, and the animal is determined by the subanimal consisting of the first $\frac{1}{2}(q + 1)$ columns. This animal may or may not be proper.

In this way the numbers of animals having the various kinds of symmetry are made to depend on the numbers of certain animals of smaller size. This procedure will be realized for $p = 2$, 3, 4, and 5 in the following four sections.

8. Free proper $(2 \times q)$-animals. Clearly the number of free proper $(1 \times q)$-animals is 1 if $n = q$ and 0 otherwise.

A proper $(2 \times q)$-animal has B-symmetry if, and only if, it has $2q$ cells. Hence

(8.1) $$\left.\begin{array}{ll} B_B(q, n) = 1 & \text{if } n = 2q. \\ \text{and} \quad\quad\; = 0 & \text{if } n \neq 2q \end{array}\right\}.$$

Let us consider proper $(2 \times q)$-animals having Γ-symmetry. If q is even, the subanimal is a proper $(2 \times \frac{1}{2}q)$-animal having $\frac{1}{2}n$ cells. Hence, for q even,

$$B_\Gamma(q, n) = B(\tfrac{1}{2}q, \tfrac{1}{2}n) \quad \text{if } n \text{ is even} \quad \text{and} \quad = 0 \quad \text{if } n \text{ is odd}. \tag{8.2}$$

If q is odd, the subanimal is a proper $(2 \times \frac{1}{2}(q + 1))$-animal which will end in $\mathfrak{B}_3$, if n is even, and in $\mathfrak{B}_1$ or $\mathfrak{B}_2$ of n is odd and $> q$. Hence, for q odd

$$B_\Gamma(q, n) = B_3(\tfrac{1}{2}(q + 1), \tfrac{1}{2}n + 1) \quad \text{if } n \text{ is even} \quad \text{and} \quad = 2B_1(\tfrac{1}{2}(q + 1), \tfrac{1}{2}(n + 1)) \quad \text{if } n \text{ is odd}. \tag{8.3}$$

If $n = q$ then $B_\Gamma(q, n) = 0$, clearly.

We now consider proper $(2 \times q)$-animals having Δ-symmetry. These must have an even number of cells. If q is even the subanimal is a proper $(2 \times \frac{1}{2}q)$-animal ending in $\mathfrak{B}_3$. If q is odd it is a proper $(2 \times \frac{1}{2}(q + 1))$-animal, also ending in $\mathfrak{B}_3$. Hence

$$B_\Delta(q, n) = B_3(\tfrac{1}{2}q, \tfrac{1}{2}n) \quad \text{if } q \text{ is even} \quad \text{and} \quad = B_3(\tfrac{1}{2}q, \tfrac{1}{2}n + 1) \quad \text{if } q \text{ is odd}. \tag{8.4}$$

Theorem 2 then enables us to calculate the number $b(q, n)$ of free proper $(2 \times q)$-animals having n cells.

In § 3 we were able to obtain generating functions for $B_1^*(q, n)$, $B_3^*(q, n)$ and $B^*(q, n)$, from which the generating functions for $B_1(q, n)$, $B_3(q, n)$ and $B(q, n)$ follow at once. The results of the present section then enable us to derive a generating function for $b(q, n)$. The derivation is quite straightforward, and we shall not follow it through in detail, but merely quote the final result, which is that

$$\sum_q \sum_n b(q, n)x^q y^n = \frac{1}{4}\left\{\frac{1 + xy}{1 - xy - xy^2 - x^2y^3} - \frac{3 + xy}{1 - xy} + \frac{xy^2}{1 - xy^2} + \frac{2(1 + xy + xy^2 + x^3y^4)}{1 - x^2y^2 - x^2y^4 - x^4y^6}\right\}. \tag{8.5}$$

9. Free proper $(3 \times q)$-animals. To find the number of B-symmetric $(3 \times q)$-animals we use Theorem 1, as in § 4, but we consider only those columns which are themselves B-symmetric, viz. $\mathfrak{C}_2$, $\mathfrak{C}_5$, $\mathfrak{C}_6$, and $\mathfrak{C}_8$. The transition matrix is

$$\mathbf{\Phi} = \begin{bmatrix} y & \cdot & \cdot & y \\ \cdot & y^2 & \cdot & y^2 \\ \cdot & \cdot & y^2 & \cdot \\ y^3 & y^3 & y^3 & y^3 \end{bmatrix}$$

and by (2.1) we find that, for $n > q$, $C_B(q, n)$ is given by

$$\sum C_B(q, n)y^n = [1, 1, 0, 1]\mathbf{\Phi}^{q-1} \begin{bmatrix} y \\ 0 \\ y^2 \\ y^3 \end{bmatrix} \tag{9.1}$$

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since if $n > q$ the animals thus counted are automatically proper. Clearly $C_B(q, q) = 0$.

For Γ-symmetric $(3 \times q)$-animals the method already used will suffice. If q is even then

$$(9.2) \qquad \begin{aligned} C_\Gamma(q, n) &= C(\tfrac{1}{2}q, \tfrac{1}{2}n) && \text{if } n \text{ is even} \\ \text{and} \qquad &= 0 && \text{if } n \text{ is odd} \end{aligned}$$

as before. If q is odd and n is even then the subanimal is a proper $(3 \times \frac{1}{2}(q+1))$-animal ending in $\mathfrak{C}_4$, $\mathfrak{C}_5$, or $\mathfrak{C}_7$. If q and n are both odd, the subanimal ends in $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, or $\mathfrak{C}_8$. Hence, for q odd we obtain

$$(9.3) \qquad \begin{aligned} C_\Gamma(q, n) &= 2C_4(\tfrac{1}{2}(q+1), \tfrac{1}{2}n+1) + C_5(\tfrac{1}{2}(q+1), \tfrac{1}{2}n+1) \\ & \qquad\qquad \text{if } n \text{ is even} \\ \text{and} \qquad &= 2C_1(\tfrac{1}{2}(q+1), \tfrac{1}{2}(n+1)) + C_2(\tfrac{1}{2}(q+1), \tfrac{1}{2}(n+1)) \\ & \qquad + C_8(\tfrac{1}{2}(q+1), \tfrac{1}{2}(n+3)) \text{ if } n \text{ is odd.} \end{aligned}$$

For Δ-symmetric animals we proceed similarly. If q is even so is n, and the subanimal is a $(3 \times \frac{1}{2}q)$-animal. If this subanimal is proper, then it must not end in $\mathfrak{C}_1$ or $\mathfrak{C}_3$. If it is improper then there is one proper $(3 \times q)$-animal with Δ-symmetry corresponding to every fixed proper $(2 \times \frac{1}{2}q)$-animal ending in $\mathfrak{B}_1$ or $\mathfrak{B}_2$ (see, for example, Figure 7); and there are two proper $(3 \times q)$-animals corresponding to every fixed proper $(2 \times \frac{1}{2}q)$-animal ending in $\mathfrak{B}_3$ (as shown in Figures 9 and 10).

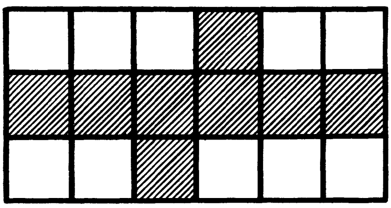

FIGURE 9

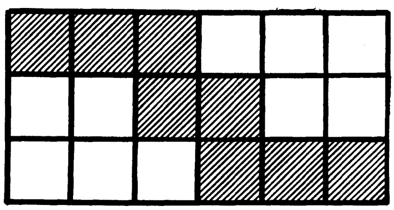

FIGURE 10

Hence if q is even

$$(9.4) \qquad \begin{aligned} C_\Delta(q, n) &= C(\tfrac{1}{2}q, \tfrac{1}{2}n) - 2C_1(\tfrac{1}{2}q, \tfrac{1}{2}n) \\ & \qquad + 2B_1(\tfrac{1}{2}q, \tfrac{1}{2}n) + 2B_3(\tfrac{1}{2}q, \tfrac{1}{2}n) \quad \text{if } n \text{ is even} \\ \text{and} \qquad &= 0 \quad \text{if } n \text{ is odd.} \end{aligned}$$

If q is odd and n is even the centre column must be $\mathfrak{C}_5$, and the subanimal is a proper $(3 \times \frac{1}{2}(q+1))$-animal ending in $\mathfrak{C}_5$. If q and n are both odd, the centre column is either $\mathfrak{C}_2$ or $\mathfrak{C}_5$. If it is $\mathfrak{C}_2$, then the subanimal may be proper, in which case it ends in $\mathfrak{C}_2$; or it may be improper in which case there are two possibilities for every $(2 \times \frac{1}{2}(q+1))$-animal ending in $\mathfrak{B}_1$. If the centre column is $\mathfrak{C}_8$ the subanimal is proper and ends in $\mathfrak{C}_8$. Hence, for q odd,

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$$C_\Delta(q, n) = C_5(\tfrac{1}{2}(q+1), \tfrac{1}{2}n + 1) \qquad \text{if } n \text{ is even}$$
(9.5)
$$\text{and} \quad = C_2(\tfrac{1}{2}(q+1), \tfrac{1}{2}(n+1)) + 2B_1(\tfrac{1}{2}(q+1), \tfrac{1}{2}(n+1)) + C_8(\tfrac{1}{2}(q+1), \tfrac{1}{2}(n+3)) \qquad \text{if } n \text{ is odd.}$$

The number $c(q, n)$ of free proper $(3 \times q)$-animals having n cells then follows from Theorem 2.

10. Free proper $(4 \times q)$**-animals.** A proper $(4 \times q)$-animal having B-symmetry will be determined by its first two rows. These will form a proper $(2 \times q)$-animal. Hence we have

(10.1)
$$\left.\begin{aligned} D_B(q, n) &= B(q, \tfrac{1}{2}n) && \text{if } n \text{ is even} \\ \text{and} \quad &= 0 && \text{if } n \text{ is odd} \end{aligned}\right\}.$$

The formulae for $D_\Gamma(q, n)$ and $D_\Delta(q, n)$ are formed in much the same way as before. We shall merely quote the final results which are as follows: For q even,

(10.2)
$$\left.\begin{aligned} D_\Gamma(q, n) &= D(\tfrac{1}{2}q, \tfrac{1}{2}n) && \text{if } n \text{ is even} \\ \text{and} \quad &= 0 && \text{if } n \text{ is odd} \end{aligned}\right\}.$$

For q odd,

(10.3)
$$\begin{aligned} D_\Gamma(q, n) &= 2D_3(\tfrac{1}{2}(q+1), \tfrac{1}{2}n+1) + 2D_4(\tfrac{1}{2}(q+1), \tfrac{1}{2}n+1) \\ &\quad + D_9(\tfrac{1}{2}(q+1), \tfrac{1}{2}n+1) + D_{12}(\tfrac{1}{2}(q+1), \tfrac{1}{2}n+2 \\ &\qquad\qquad \text{if } n \text{ is even} \\ \text{and} \quad &= 2D_1(\tfrac{1}{2}(q+1), \tfrac{1}{2}(n+1)) + 2D_2(\tfrac{1}{2}(q+1), \tfrac{1}{2}(n+1)) \\ &\quad + 2D_6(\tfrac{1}{2}(q+1), \tfrac{1}{2}(n+3)) + 2D_8(\tfrac{1}{2}(q+1), \tfrac{1}{2}(n+3)) \\ &\qquad\qquad \text{if } n \text{ is odd.} \end{aligned}$$

(10.4)
$$\begin{aligned} D_\Delta(q, n) &= 2D_6(\tfrac{1}{2}q, \tfrac{1}{2}n) + D_8(\tfrac{1}{2}q, \tfrac{1}{2}n) + D_9(\tfrac{1}{2}q, \tfrac{1}{2}n) \\ &\quad + D_{10}(\tfrac{1}{2}q, \tfrac{1}{2}n. + D_{12}(\tfrac{1}{2}q, \tfrac{1}{2}n) + 2C_4(\tfrac{1}{2}q, \tfrac{1}{2}n) \\ &\quad + 2C_8(\tfrac{1}{2}q, \tfrac{1}{2}n) \qquad \text{if } q \text{ and } n \text{ are even.} \end{aligned}$$

(10.5)
$$\begin{aligned} D_\Delta(q, n) &= D_9(\tfrac{1}{2}(q+1), \tfrac{1}{2}n+1) + D_{10}(\tfrac{1}{2}(q+1), \tfrac{1}{2}n+1) \\ &\quad + D_{12}(\tfrac{1}{2}(q+1), \tfrac{1}{2}n+2) + 2C_4(\tfrac{1}{2}(q+1), \tfrac{1}{2}n+1 \\ &\qquad\qquad \text{if } q \text{ is odd and } n \text{ is even} \end{aligned}$$

while $D_\Delta(q, n) = 0$ if n is odd.

The number $d(q, n)$ of proper free $(4 \times q)$-animals is then given by Theorem 2.

11. Free proper $(5 \times q)$**-animals.** The calculation of $E_B(q, n)$ can be carried out using Theorem 1, as was done for $C_B(q, n)$. The calculation of $E_\Gamma(q, n)$ and $E_\Delta(q, n)$ can be performed by the same methods as used hitherto. We shall not consider these calculations in any detail.

We shall need, later on, the value of $e(6, 10)$ and this we shall find from first principles. A B-symmetric (5×6)-animal having 10 cells is determined by its top three rows, and the animal formed by them will have 6, 7, or 8 cells according as the third row has 2, 4, or 6 cells. But if the animal is proper the first two possibilities are ruled out. Thus the centre row of the animal must have 6 cells, and hence the remaining 4 cells are all in one column. Since we are counting fixed animals we have $E_B(6, 10) = 6$.

For a Γ-symmetric animal, the subanimal would be a proper (5×3)-animal having 5 cells. This is clearly impossible, so that $E_\Gamma(6, 10) = 0$.

For a Δ-symmetric animal, the subanimal is a (5×3)-animal with 5 cells, which cannot be proper. The third column of this subanimal cannot have 5 or 4 cells. If it has 3 they must be the top 3 or the bottom 3; if it has 2, one cell must be in the centre, the other adjacent to it; if it has one only, this cell must be in the centre. The numbers of subanimals in these three cases are found empirically to be 6, 6, and 10. Thus $E_\Delta(6, 10) = 22$.

From Theorem 2 we then have

$$\begin{aligned} e(6,10) &= \tfrac{1}{4}\{932 + 6 + 0 + 22\} \\ &= 240. \end{aligned}$$

12. Numerical results. The methods of the preceding sections yield the following results.

$b(q, n)$

q \ n	1	2	3	4	5	6	7	8	9	10
1		1								
2			1	1						
3				3	2	1				
4					3	6	2	1		
5						5	11	10	3	1
6							5	19	22	15
7								7	28	52
8									7	40
9										9

$c(q, n)$

q \ n	1	2	3	4	5	6	7	8	9	10
1			1							
2				3	2	1				
3					9	12	12	4	1	
4						15	39	59	42	21
5							25	96	210	255
6								35	188	550
7									49	332
8										63

$d(q, n)$

q \ n	1	2	3	4	5	6	7	8	9	10
1				1						
2					3	6	2	1		
3						15	39	59	42	21
4							30	148	349	518
5								61	383	1304
6									97	822
7										155

Note. The values for $p = q$ have been included for completeness, although they are not significant for our purpose.

13. Animals contained in a square. We have already remarked that when $p = q$ the animal may possess types of symmetry other than those given in § 7. Thus the results already obtained will not hold for animals contained in a square. We could extend the scope of Theorem 2—essentially a variation on the principal of inclusion and exclusion—to cover the new problem, but we would meet a new difficulty, namely, that whereas, for the types of symmetry so far considered, the corresponding subanimals were fixed animals contained in a rectangle, for the new types of symmetry the subanimals would be contained in a triangular arrangement of unit squares. For example, animals contained in a 7×7 square and possessing all possible symmetries (corresponding to the dihedral group of order 8) would be determined by subanimals contained in the arrangement shown in Figure 11.

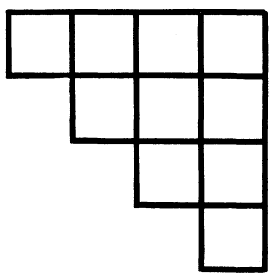

FIGURE 11

Moreover, these subanimals would have to be chosen in such a way that the resulting animal would satisfy the requirement of connectedness. This is a different problem altogether from those we have been considering, and we shall make no attempt to tackle it in the general case. Our treatment of the cell growth problem thus remains incomplete in at least this respect.

We shall consider four special cases, however, namely, those for $p = q = 4$ and 5, and $n = 9$ and 10. Fortunately, for these low values, the animals can be enumerated *in extenso* by a process of enlightened trial and error, using the information derivable from the requirements of connectedness, properness, and symmetry (if any). We shall give the results that have been found in this way.

If $p = 4$ and $n = 9$, the only symmetry possible is about a single diagonal. There are 13 free animals with this kind of symmetry, each giving rise to 4 fixed animals. All the other free animals give rise to 8 fixed animals each. Hence if $z_p(n)$ denotes the number of free proper $(p \times p)$-animals having n cells, we have

$$8\{z_4(9) - 13\} + 52 = D(4, 9) = 1396$$

Hence

$$z_4(9) = 181.$$

If $p = 4$ and $n = 10$, there are 6 β- (or γ-) symmetric free animals; 3 δ-symmetric free animals; and 12 free animals symmetrical about one diagonal. These each give rise to 4 fixed animals. There is also one free animal with symmetry about both diagonals, and this gives rise to 2 fixed animals. Thus

$$8\{z_4(10) - 22\} + 86 = D(4, 10) = 2038$$

Hence

$$z_4(10) = 266.$$

If $p = 5$, $n = 9$, there are 20 free animals possessing β-symmetry, δ-symmetry, or symmetry about one diagonal, thus giving rise to 4 fixed animals each; and there is one animal with all possible symmetries, to which corresponds one fixed animal only. Thus

$$8\{z_5(9) - 21\} + 81 = E(5, 9) = 497$$

Hence

$$z_5(9) = 73.$$

If $p = 5$, $n = 10$, there are 3 free animals with β-symmetry, and 10 which are symmetrical about one diagonal. These each give rise to 4 fixed animals. Thus

$$8\{z_5(10) - 12\} + 52 = E(5, 10) = 2668$$

Hence

$$z_5(10) = 340.$$

14. The cell-growth problem for n = 9, 10. We are now in a position to find the numbers of free animals of unrestricted shape having 9 and 10

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cells. This is simply a matter of abstracting the relevant data from the previous sections. We may exhibit the results in tabular form, as follows.

$n = 9$

Size of rectangle	No. of animals
1×9	1
2×8	7
2×7	28
2×6	22
2×5	3
3×7	49
3×6	188
3×5	210
3×4	42
3×3	1
4×6	97
4×5	383
4×4	181
5×5	73
Total No. of animals	1285

$n = 10$

Size of rectangle	No. of animals
1×10	1
2×9	9
2×8	40
2×7	52
2×6	15
2×5	1
3×8	63
3×7	332
3×6	550
3×5	255
3×4	21
4×7	155
4×6	822
4×5	1304
4×4	266
5×6	240
5×5	340
Total No. of animals	4466

These numbers will include animals that are multiply-connected. By an empirical process (that of adding cells in all possible ways to the animal of Figure 3, and to the unique multiply-connected (3×3)-animals having 7 and 8 cells) I find that there are 37 multiply-connected animals having 9 cells, and 195 which have 10 cells. Thus, as a final result, we may augment Table I in the following manner.

TABLE II

Area n	9	10
No. of simply-connected animals	1248	4271
No. of multiply-connected animals	37	195

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