INVOLUTORY AUTOMORPHISMS OF GROUPS OF ODD ORDER

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1. Introduction

Let G be a finite group of odd order with an automorphism ω of order 2. The Feit-Thompson theorem implies that G is soluble and this is assumed throughout the paper. Let G_ω denote the subgroup of G consisting of those elements fixed by ω. If $F(G)$ denotes the Fitting subgroup of G then the upper Fitting series of G is defined by $F_1(G) = F(G)$ and $F_{r+1}(G) =$ the inverse image in G of $F(G/F_r(G))$. $G^{(r)}$ denotes the rth derived group of G. The principal result of this paper may now be stated as follows:

THEOREM 1. *Let G be a group of odd order with an automorphism ω of order 2. Suppose that G_ω is nilpotent, and that $G_\omega^{(r)} = 1$. Then $G^{(r)}$ is nilpotent and $G = F_3(G)$.*

Examples given in [7] show that there exist groups G satisfying the hypothesis of theorem 1 for which $G \neq F_2(G)$. If H is any nilpotent group of odd order and derived length r, we can construct a group G satisfying the hypothesis of the theorem such that $G_\omega \cong H$ and $G^{(r-1)}$ is not nilpotent. Indeed let q be an odd prime not dividing the order of H and construct the group algebra A of K, the direct product of H and the cyclic group of order 2, over $GF(q)$, the Galois field with q elements. The mappings

$$x \to ax+b$$

of A into itself, where a runs over K and b runs over A, form a group Γ. Γ has a subgroup G of odd order and index 2. $G/F(G) \cong H$ and an inner automorphism of Γ of order 2 induces an automorphism ω of G with $G_\omega \cong H$.

L. Kovacs and G. E. Wall have constructed in [7] p groups of arbitrarily high derived length, each with an automorphism ω of order 2 such that the fixed point group of ω is cyclic. Taking K to be the splitting extension of a suitable one of these groups by its automorphism and applying the above construction we can show that given any integer n there exists a group G of odd

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order with an automorphism ω of order 2 such that G_ω is metabelian and $G^{(n)}$ is not nilpotent. Thus the assumption that G_ω is nilpotent in theorem 1 is essential.

If the group G has several automorphisms of order 2 satisfying the condition that each of the fixed point groups is nilpotent, then stronger assertions can be made. We have

THEOREM 2. *Let G be a group of odd order with a group of automorphisms A of order 4 and exponent 2 such that for each $\omega \in A$, $\omega \neq 1$, G_ω is nilpotent. Then G' is nilpotent.*

Under these conditions G need not be nilpotent but with even stronger hypotheses the nilpotence of G can be asserted:

THEOREM 3. *Let G be a group of odd order with a group of automorphisms A of order 8 and exponent 2 such that for each $\omega \in A$, $\omega \neq 1$, G_ω is nilpotent. Then G is nilpotent.*

A very much more elementary result is

THEOREM 4. *Let G be a group of odd order with an automorphism ω of order 2. If G_ω is a Hall-subgroup of G then there exists a normal abelian complement of G_ω in G.*

For further discussion of theorems of this kind we refer to [7].

I wish to express my thanks to G. E. Wall for his guidance in this work.

Notation. The notation is standard and agrees with that mentioned in [7]. By a proper subgroup is means a subgroup not equal to the whole group. A non-trivial subgroup is one containing more than one element. If G is a group, $|G|$ denotes the order of G, $Z(G)$ the centre of G and $\Phi(G)$ the Frattini subgroup of G. For subgroups H and K of G, $|G : H|$ is the index of H in G, $C_H(K)$ the centralizer of K in H and $N_H(K)$ the normalizer of K in H.

$\mathscr{F}$ always denotes the algebraic closure of $GF(p)$, the Galois field with p-elements. If $\mathscr{L}$ is a field $\mathscr{L}(G)$ denotes the group algebra of G over $\mathscr{L}$. If V is an $\mathscr{L}(G)$-module, we write scalars as left operators on V and elements of $\mathscr{L}(G)$ as right operators on V.

If p is a prime, a p' group is a group of order prime to p. A Hall p' subgroup of a group is a Hall subgroup, whose index is a power of p.

A frequently used property of a soluble group G is that $C_G(F(G)) \leqq F(G)$ ([1], p. 646).

2. Preliminary lemmas

LEMMA 1. *Let P be a p-group and H a proper subgroup of P. Then* $|P : H| > |P' : H \cap P'|$.

Proof. Since $P' \leqq \Phi(P)$, $|P : P'| > |P'H : P'|$. The result follows.

Lemma 2. *Let G be a soluble group operated on by a group A of automorphisms. Suppose that for some pair of integers* (m, n), $(n > 0)$, $G^{(m)} \nleqq F_n G)$ *but if H/K is any A-section*[2] *of G*, $H/K \neq G/1$, *then* $(H/K)^{(m)} \leqq F_n(H/K)$. *Then if H is a non trivial normal A-subgroup of G*, $F(G) \leqq H$. $F(G)$ *is an elementary abelian p-group for some prime p.*

Proof. It follows from the hypothesis that if $1 \neq H, K \neq 1$ are normal A-subgroups of G then $1 \neq H \cap K$. Thus G has a unique minimal normal A-subgroup M. Since G is soluble, M is an elementary abelian p-group. Now from [1] p. 647, $F(G/\Phi(G)) = F(G)/\Phi(G)$ so $F_n(G/\Phi(G)) = F_n(G)/\Phi(G)$. Thus $\Phi(G) = 1$. Since $\Phi(N) \leqq \Phi(G)$ if $N \lhd G$ ([3], p. 162), $F(G)$ is an elementary abelian p-group.

Write $H/M = F(G/M)$. Then $F(G)$ is the Sylow p-subgroup of H. Since $(G/M)^{(m)} \leqq F_n(G/M)$ whilst $G^{(m)} \leqq F_n(G)$, H properly contains $F(G)$.

As $F(G)$ is an elementary abelian normal Sylow p-subgroup of H, $F(G)$ is a completely reducible $H/F(G)$ module. Thus $F(G) = M \times N$ where $N \lhd H$. Since H/M is nilpotent and $F(G)$ is abelian, $N \leqq Z(H)$. Suppose $Z(H) > 1$. Then $Z(H)$ is characteristic in the normal A-subgroup H of G so $Z(H)$ is a normal A-subgroup of G. Hence $Z(H) \geqq F(G)$ so $H \leqq C_G(F(G)) = F(G)$, a contradiction. Thus $N \leqq Z(H) = 1$ and $F(G) = M$, proving the lemma.

We apply lemma 2 in the following way. Each of theorems 1, 2 and 3 is to be proved by induction on the order of G and by way of contradiction. For theorem 1 take A to be the group $\{1, \omega\}$. Let G be a group of minimal order not satisfying the hypothesis of the theorem in question. For theorem 1 we take $(m, n) = (0, 3)$ or $(r, 1)$; for theorem 2, $(m, n) = (1, 1)$ and for theorem 3, $(m, n) = (0, 1)$. Now if $H/K \neq G/1$ is an A-section of G, either A is represented faithfully as a group of automorphisms of H/K in which case by induction $(H/K)^{(m)} \leqq F_n(H/K)$ or for some automorphism $\omega \in A$ $(H/K)_\omega = H/K$ so H/K is nilpotent being isomorphic to a section of G_ω. Thus in either case since $|G|$ is odd the hypothesis of the lemma is satisfied and we conclude that $F(G)$ is the unique minimal normal A-subgroup of G.

The following lemma and its corollaries are stated for convenience. The method of proof is well known, see for example [7].

Lemma 3. *Let G be a group of odd order with an automorphism* ω *of order* 2. *Then there exists precisely one element of G which is inverted by* ω *in each left (right) coset of* G_ω.

[2] An A-section of G is a factor group H/K where $K \lhd H$ and H and K are A-subgroups of G.

Corollary 1. *Let G be a group of odd order with an automorphism ω of order 2. Every element of G may be expressed as the product of an element fixed by ω and an element inverted by ω.*

Corollary 2. *Let G be a group of odd order with an automorphism ω of order 2. Let H be a subgroup of G containing G_ω. Then $H^\omega = H$.*

Corollary 3. *Let G be an abelian group of odd order with an automorphism ω of order 2. Then if N is the set of elements of G which are inverted by ω, N is a subgroup of G and $G = N \times G_\omega$.*

Theorem 4 follows from lemma 3:

The Hall-subgroups of G which complement G_ω form a characteristic system of subgroups. Since G is of odd order one of these is fixed by ω; this subgroup consists of those elements of G inverted by ω and so is normalized by G_ω. Thus it is a normal abelian complement of G_ω in G.

3. Proof of theorem 1

The theorem is proved by induction on $|G|$ and by way of contradiction. Suppose therefore that G is a group of minimal order satisfying the hypothesis of the theorem but not the conclusion. It follows from [7] that the theorem is true for $r = 1$ so we may assume $r > 1$. Since $|G|$ is odd, G is soluble. We have already proved.

Lemma 1. *$F(G)$ is the unique minimal normal ω-subgroup of G. Therefore $F(G)$ is an elementary abelian p-group for some prime p.*

Notation. For each positive integer n, set $F_n = F_n(G)$. Let Γ denote the splitting extension of G by ω and p the unique prime dividing $|F_1|$.

Lemma 2. (i) $(G/F_1)_\omega \neq G/F_1$,

(ii) *F_1 is a faithful irreducible Γ/F_1-module,*

(iii) *$(F_1)_\omega > 1$. Therefore $p \mid |G_\omega|$.*

Proof. (i) If $(G/F_1)_\omega = G/F_1$, then G/F_1 is isomorphic to a section of G_ω and therefore is nilpotent of derived length less than or equal to r. It follows that G satisfies the conclusion of the theorem. Hence $(G/F_1)_\omega \neq G/F_1$.

(ii) Lemma 1 implies that F_1 is an irreducible Γ/F_1-module. To prove that F_1 is a faithful Γ/F_1-module we need to prove that $C_\Gamma(F_1) = F_1$. Since F_1 is the Fitting subgroup of G, and since G is soluble, $C_G(F_1) = F_1$. Hence if $C_\Gamma(F_1) > F_1$, $|C_\Gamma(F_1) : F_1| = 2$. In this case Γ/F_1 has a normal Sylow 2-subgroup so that $\Gamma/F_1 = G/F_1 \times gp\{\omega F_1\}$ from which it follows that $(G/F_1)_\omega = G/F_1$, contradicting (i). This proves (ii).

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(iii) $(F_1)_\omega > 1$ for if $(F_1)_\omega = 1$, ω must invert all the elements of F_1. Then, since Γ/F_1 is faithfully represented by its action on F_1, ωF_1 lies in the centre of Γ/F_1. But this again implies that $(G/F_1)_\omega = G/F_1$, contradicting (i).

LEMMA 3. *F_2/F_1 is a p'-group. G/F_1 has no non-trivial normal p-subgroups.*

PROOF. Suppose that P/F_1 is the Sylow p-subgroup of F_2/F_1. Then as F_1 is a p-group, P is a normal p-subgroup of G. Hence $P \leqq F_1$. The second statement follows from the first.

LEMMA 4. *If G_ω is a p-group then $G = F_2 G_\omega$ and $(G_\omega)^{(r-1)}$ is not contained in F_1. F_2/F_1 is abelian.*

PROOF. We know, by lemma 2, that F_1 is a p-group and, by lemma 3, that F_2/F_1 is a p'-group. Since G/F_1 is soluble and F_2/F_1 is a normal subgroup of G/F_1, F_2/F_1 is contained in every Hall p'-subgroup of G/F_1. Now the Hall p'-subgroups of G/F_1 are all conjugate and the order of G is odd so the number of Hall p'-subgroups is odd. Clearly the automorphism ω permutes these Hall p'-subgroups and since the number of them is odd, at least one is fixed by ω. Thus we can choose a Hall p'-subgroup H/F_1 such that $H^\omega = H$. Now $H_\omega = H \cap G_\omega$ is a p-group so that $H_\omega \leqq F_1$. Thus ω acts as a regular automorphism on H/F_1 so that H/F_1 is abelian. Since G/F_1 is a soluble group, $C_{G/F_1}(F_2/F_1) \leqq F_2/F_1$. But $F_2/F_1 \leqq H/F_1$ so that as H/F_1 is abelian, $H/F_1 \leqq C_{G/F_1}(F_2/F_1) \leqq F_2/F_1 \leqq H/F_1$. Thus F_2/F_1 is the unique Hall p'subgroup of G/F_1. It follows that G/F_2 is a p-group. Therefore $G = F_3$.

Since $G = F_3$ and G does not satisfy the conclusion of the theorem, $G^{(r)}$ is not nilpotent.

Suppose by way of contradiction that $G_\omega^{(r-1)} \leqq F_1$. Then $(G/F_1)_\omega$ has derived length at most $r-1$, so by the minimality of G, $(G/F_1)^{(r-1)}$ is nilpotent. Thus $G^{(r-1)} \leqq F_2$ and since, as we have already seen, F_2/F_1 is abelian, $G^{(r)} \leqq F_1$. This contradiction proves that $G_\omega^{(r-1)}$ is not contained in F_1.

Finally we show that if $G_\omega F_2 < G$, $G_\omega^{(r-1)}$ is contained in F_1. It then follows from the conclusion of the previous paragraph that $G_\omega F_2 = G$. Suppose then that $G_\omega F_2 < G$, and let K be a maximal subgroup of G containing $G_\omega F_2$. Since K is a maximal subgroup of G containing F_2 and since G/F_2 is nilpotent, K is a normal subgroup of G. By § 2, lemma 3 corollary 2, as $G_\omega \leqq G_\omega F_2 \leqq K$, K is a ω-subgroup of G. Therefore, by the minimality of G, $K^{(r)}$ is nilpotent. But $K^{(r)}$ is a characteristic subgroup of K, a normal subgroup of G, and therefore $K^{(r)}$ is a normal subgroup of G. Hence $K^{(r)} \leqq F_1$ so that $K^{(r-1)} \leqq F_2$. Now $G_\omega^{(r-1)} \leqq K^{(r-1)} \leqq F_2$. But G_ω is

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a p-group and F_1 is the Sylow p-subgroup of F_2 so $G_\omega^{(r-1)} \leqq F_1$. This completes the proof of the lemma.

We have shown that F_1 is the unique minimal normal ω-subgroup of G. Since G is a normal subgroup of Γ, $F(G) \leqq F(\Gamma)$. If $F(\Gamma) \neq F(G)$ then $|F(\Gamma) : F(G)| = 2$ so that $\omega \in F(\Gamma)$. But in this case, since $F(\Gamma)$ is nilpotent, $(F_1)_\omega = F_1$ contradicting lemma 2(*ii*). Thus $F_1 = F(\Gamma)$ is the unique minimal normal subgroup of Γ. $|\Gamma : G| = 2$ so the solubility of Γ follows from that of G. Therefore ([1], p. 651) there exists a complement N of F_1 in Γ. By Sylow's theorem we can suppose, by taking a suitable conjugate of N if necessary, that $\omega \in N$. Let $M = G \cap N$. Then M is a complement of F_1 in G.

Since the elements of N form a complete set of coset representatives of F_1 in Γ, we may consider F_1 as a $GF(p)(N)$-module. We now summarize the results obtained so far in module notation.

(1) F_1 is a faithful irreducible N-module over $GF(p)$.

(2) $(F_1)_\omega > 0$.

(3) If $f \in (F_1)_\omega$ and $x_i \in (M_\omega)^{(i)}$ $(i = 0, \cdots, r-1)$ then $f(1-x_0)(1-x_1)\cdots(1-x_{r-1}) = 0$.

(4) If $f \in (F_1)_\omega$ and $x \in M_\omega$ is of order prime to p, then since G_ω is nilpotent, $fx = f$.

It also follows from lemma 2 that $M_\omega \neq M$.

If we extend the field of scalars from the prime field $GF(p)$ to its algebraic closure $\mathscr{F}$, F_1 splits into a direct sum of absolutely irreducible $\mathscr{F}(N)$-modules, which are algebraically conjugate. ([2], section 70). Taking V as one of these irreducible $\mathscr{F}(N)$-modules, we obtain an $\mathscr{F}(N)$ module with the following properties:

(1) V_1 is a faithful irreducible N-module over $\mathscr{F}$,

(2) $V_\omega = \{v \in V | v\omega = v\} > 0$,

(3) If $v \in V_\omega$ and $x_i \in (M_\omega)^{(i)}$ $(i = 0, 1, \cdots, r-1)$ then $v(1-x_0)(1-x_1)\cdots(1-x_{r+1}) = 0$,

(4) If $v \in V_\omega$ and $x \in M_\omega$ is of order prime to p, then $vx = v$.

Notation. $Q = F(M)$.

LEMMA 5. *V is an irreducible $\mathscr{F}(M)$-module.*

PROOF. By way of contradiction suppose that there exists an irreducible $\mathscr{F}(M)$-submodule W of V such that $0 < W < V$. Since $W\omega$ is also an irreducible $\mathscr{F}(M)$-submodule of V and since $W + W\omega$ is an $\mathscr{F}(N)$-module we have $V = W + W\omega$ and so as an $\mathscr{F}(M)$-module

$$V = W + W\omega.$$

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Suppose that G_ω is not a p-group. Then there exist an element $x \neq 1$ in M_ω of order prime to p. Let $w \in W$ be arbitrary. Then $w+w\omega \in V_\omega$ so by property (4) of V, $(w+w\omega)x = w+w\omega$. Equating the W and $W\omega$ components of both sides we deduce that x acts trivially on both W and $W\omega$ and so on V. But this contradicts property (1) of V. Hence we may assume that G_ω is a p-group.

Let $x \in Q$ and suppose that for all $w \in W$, $wx = w$. Since $x \in Q$ and G_ω is a p-group, $x^\omega = x^{-1}$. Thus if $w \in W$, $w\omega x = wx^{-1}\omega = w\omega$ so x also acts trivially on $W\omega$. Hence x acts trivially on $W+W\omega = V$ so $x = 1$. Therefore W is a faithful Q-module. Since $Q = F(M)$ and M is soluble, any normal subgroup of M has non-trivial intersection with Q. Hence if W were not a faithful M-module, W would not be a faithful Q-module. It follows that W is a faithful M-module.

Let $w \in W$. Then $w+w\omega \in V_\omega$ so if $x_i \in (M_\omega)^{(i)}$ $(i = 0, 1, \cdots, r-1)$ it follows from property (3) of V that

$$(w+w\omega)(1-x_0)(1-x_1)\cdots(1-x_{r-1}) = 0$$

and hence equating the W-components we have

$$w(1-x_0)(1-x_1)\cdots(1-x_{r-1}) = 0.$$

Consider W as an $\mathscr{F}(Q)$-module. Since W is an irreducible $\mathscr{F}(M)$-module it follows that

$$W = W_1 + \cdots W_n$$

where $W_1, W_2, \cdots, W_n$ are the homogeneous components of W as an $\mathscr{F}(Q)$-module. ([2], section 49). Since M_ω is a p-group, Q is an abelian p'-group by lemma 4. Thus as $\mathscr{F}$ is of characteristic p and algebraically closed, the irreducible $\mathscr{F}(Q)$-submodules of W are one-dimensional. Thus the action of $x \in Q$ on $w \in W_i$ may be described by

$$wx = \chi_i(x)w.$$

M/Q is a transitive permutation group on the W_i. Since, by lemma 4, M_ω is a complement of Q in M, we may consider M_ω as a transitive permutation group on the W_i. Set $H_i = \{x \in M_\omega | W_i x = W_i\}$.

We now prove that if $K \leqq H_1$ and $K \lhd M_\omega$ then $K = 1$. For all the H_i are conjugate in M_ω so $K \leqq H_i$ for all i. Now let $y \in K$, $x \in Q$. Then if $w \in W_i$, $wy^{-1} \in W_i$ so

$$w(y^{-1}xy) = ((wy^{-1})x)y = \chi_i(x)wy^{-1}y = wx$$

Hence $w(y^{-1}xy) = wx$ for all $w \in W_i$ and $x \in Q$. But since i was arbitrary and W is a direct sum of the W_i, $y^{-1}xy$ acts on W in the same way as x. But W is a faithful M-module so $y^{-1}xy = x$, or as x was arbitrary in Q, $y \in C_M(Q)$.

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But as $Q = F(M)$ and M is soluble, $C_M(Q) \leqq Q$ so $y \in Q \cap M_\omega = 1$. This proves the statement made at the beginning of the paragraph.

Let $x \in (M_\omega)^{(r)}$ and $w \in W_1$. Since M_ω is a p-group and H_1 is a subgroup of M_ω containing no non-trivial normal subgroup of M_ω, it follows from § 2, lemma 1, that

$$(*) \qquad |(M_\omega)^{(i)} : M_\omega^{(i)} \cap H_1| > |(M_\omega)^{(i+1)} : (M_\omega)^{(i+1)} \cap H_1|$$

for all i such that $(M_\omega)^{(i)} \neq 1$. By lemma 4, $(M_\omega)^{(r-1)} \neq 1$, so that (*) is true for $0 \leqq i \leqq r-1$. But $|(M_\omega)^{(i)} : (M_\omega)^{(i)} \cap H_1|$ is the number of W_i in the same system of transitivity as W_1 under $(M_\omega)^{(i)}$. Hence for each $(0 \leqq i \leqq r-1)$ we can choose $x_i \in (M_\omega)^{(i)}$ such that $W_1 x_i$ is not in the same system of transitivity as W_1 under $(M_\omega)^{(i+1)}$. Now

$$w(1-x_0)(1-x_1) \cdots (1-x_{r-1})(1-x) = 0.$$

Since $W_1 x_0$ is not in the same system of transitivity as W_1 under $(M_\omega)'$ we can conclude that

$$w(1-x_1) \cdots (1-x_{r-1})(1-x) = 0$$

and finally $w(1-x) = 0$. Hence $x \in H_1$ since w was arbitrary in W_1. But x was arbitrary in $(M_\omega)^{(r)}$ so that $(M_\omega)^{(r)} \leqq H_1$. But $(M_\omega)^{(r)} \lhd M_\omega$ so $(M_\omega)^{(r)} = 1$ contradicting lemma 4. This completes the proof of lemma 5.

LEMMA 6. *If $L \neq 1$ is a normal ω-subgroup of M, then L_ω is a non-trivial proper subgroup of L.*

PROOF. Since M is soluble and every soluble group contains a characteristic subgroup which is abelian, it is sufficient to prove the lemma for abelian L. Therefore L is supposed to be a normal abelian subgroup of M. Now L is contained in $F(M) = Q$. It follows from lemma 3 that L is a p'-group. Write

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

where V is considered as an $\mathscr{F}(L)$-module and the V_i are the homogeneous components. Since L is an abelian p'-group whilst $\mathscr{F}$ is algebraically closed of characteristic p, the action of $x \in L$ on $v \in V_i$ may be described by

$$vx = \chi_i(x)v.$$

The characters χ_i are all conjugate and the number, s, of homogeneous components divides the order of M ([2], section 49). Thus none of the characters χ_i $(i = 1, 2, \cdots, s)$ is the trivial character since V is a faithful module. Also s is odd.

We complete the proof of the lemma by showing that if $L_\omega = 1$ or $L_\omega = L$ then we can choose an i such that χ_i is the trivial character.

Since ω has order 2 and V is an $\mathscr{F}(N)$-module, for each i $(i = 1, 2, \cdots, s)$

there exists j such that $V_i\omega = V_j$ and $V_j\omega = V_i$. Since s is odd there exists at least one i for which $V_i\omega = V_i$. Suppose $v \in V_i$ and $x \in L$. Then $\chi_i(x)v_i\omega = v_i\omega x = v_i x^\omega \omega = \chi_i(x^\omega)v_i\omega$ so that $\chi_i(x) = \chi_i(x^\omega)$, for all $x \in L$. Now if $L_\omega = 1$, then for all $x \in L$, $x^\omega = x^{-1}$ so that $\chi_i(x) = \chi_i(x^{-1})$ or $\chi_i(x^2) = 1$. Since L has odd order, it follows that χ_i is the trivial character. Thus $L_\omega > 1$.

Now suppose that $L_\omega = L$. By the second property of V, $V_\omega > 0$ so that there exists $0 \neq v = v\omega \in V$. Since $L = L_\omega$ is a p'-group, it follows from property (4) of V that $v = vx$ for all $x \in L$. Thus $\{kv | k \in \mathscr{F}\}$ is a trivial $\mathscr{F}(L)$-submodule of V and therefore is contained in some V_j. For this V_j, $\chi_j = 1$ clearly. This contradiction proves the lemma.

Remark. In lemma 6, L_ω cannot be a normal subgroup of M, for if it were we would obtain a contradiction by applying lemma 6 to L_ω. But $(Z(M))_\omega$ is a normal subgroup of M so $Z(M) = 1$. Therefore we can now assume that $Q = F(M)$ is a proper subgroup of M.

LEMMA 7. *Q is abelian.*

PROOF. We consider V as a $\mathscr{F}(Q)$-module and write

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

where the V_i are the homogeneous components of V. Let Q_i be the kernel of the representation of Q obtained on V_i for each $i = 1, \ldots, s$. Then the Q_i are all conjugate, ([2], section 49), so that if $Q' \leqq Q_i$ for some i then $Q' \leqq Q_i$ for all i. Therefore in this case Q' is contained in the kernel of $V_1 + V_2 + \cdots + V_s = V$. But V is a faithful M-module so that this implies that $Q' = 1$, and proves the lemma.

Now suppose that Q_ω is contained in one of the groups Q_i $(i = 1, \ldots, s)$, say Q_j. Then by §2, lemma 3, corollary 2, $Q_j^\omega = Q_j$. Therefore ω induces a regular automorphism on Q/Q_j so that Q/Q_j is abelian. Consequently $Q' \leqq Q_j$. Thus it is sufficient to prove that for some i, Q_ω is contained in Q_i.

Suppose that there exists an i such that $V_i\omega \neq V_i$. Let $v \in V_i$. Then $v + v\omega \in V_\omega$ so that as Q_ω is a p'-group if $x \in Q_\omega$, by the fourth property of V, $(v + v\omega)x = v + v\omega$. Equating the V_i components of both sides, we see that $vx = v$ so that Q_ω is contained in Q_i.

Finally suppose by way of contradiction that $V_i\omega = V_i$ for all i and fix i. Considering V_i as a $Z(Q)$-module, we may write

$$V_i = W_{i1} \oplus W_{i2} \oplus \cdots \oplus W_{iu}$$

where for each j, W_{ij} is a homogeneous component of V_i. Since $V_i\omega = V_i$ we find, as we have done previously in similar circumstances, that there exists a j such that

$$W_{ij}\omega = W_{ij}.$$

Since $Z(Q)$ is an abelian p'-group, the elements of $Z(Q)$ act as scalar multipliers on the W_{ij}. Suppose that if $x \in Z(Q)$ and $w \in W_{ij}$, $wx = \chi_{ij}(x)w$. Then $\chi_{ij}(x)w\omega = w\omega x = wx^{\omega}\omega = \chi_{ij}(x^{\omega})w\omega$ so that $\chi_{ij}(x) = \chi_{ij}(x^{\omega})$. Since $Z(Q)$ is a non-trivial normal abelian subgroup of M, $(Z(Q))_{\omega} < Z(Q)$ by lemma 6. Therefore the set H of elements of $Z(Q)$ inverted by ω forms a non-trivial subgroup of $Z(Q)$ (§2, lemma 3, corollary 3). Since H is a subgroup of $Z(Q)$, H is normal in Q. Now if $x \in H$, $\chi_{ij}(x) = \chi_{ij}(x^{\omega}) = \chi_{ij}(x^{-1})$. Since H is of odd order, for all $x \in H$, $\chi_{ij}(x) = 1$. Thus H is contained in the kernel of the representation of $Z(Q)$ given by W_{ij}. Since for $k \neq j$, the kernel of W_{ik} is conjugate to that of W_{ij} in Q and since H is a normal subgroup of Q, H is contained in the kernel of W_{ik} for all k. Thus H is contained in the kernel of $W_{i1}+ \cdots +W_{iu} = V_i$. But this is true for all i so that H is contained in the kernel of $V_1+V_2+ \cdots +V_s = V$. Since V is a faithful M-module, this implies that $H = 1$ and this contradiction, to the fact that H is a non-trivial subgroup of $Z(Q)$, completes the proof of the lemma.

Lemma 8. $G = F_3(G)$.

Proof. Suppose by way of contradiction that $G > F_3(G)$. It follows from [8] that $G_{\omega} \leqq F_3(G)$. Therefore ω induces a regular automorphism on G/F_3 so that G/F_3 is abelian. If H is any subgroup of G containing F_3 then by §2 lemma 3, corollary 2, since $G_{\omega} < F_3 \leqq H$, $H^{\omega} = H$. Since G/F_3 is abelian, H is a normal subgroup of G. Suppose that $H \neq G$. Then H satisfies the hypothesis of the theorem and therefore, by the minimality of G, $H = F_3(H)$. Since H is normal in G, $F_3(H) \leqq F_3$ so $F_3 = F_3(H) = H$. It follows that G/F_3 is cyclic of prime order.

Since $G_{\omega} < F_3 < G$, $M_{\omega} \leqq F_2(M) < M$ and by §2, lemma 3, we can choose an element $x \in M$ such that $M = \{x, F_2(M)\}$ and $x^{\omega} = x^{-1}$. Now consider the ω-subgroup of G, $K = \{x, Q, F_1\}$. Since $x^{\omega} = x^{-1}$, whilst $F_2 = QF_1$ is a normal subgroup of K, $K_{\omega} \leqq (QF_1)_{\omega}$. But Q is an abelian p'-group, F_1 is an abelian p-group, and G_{ω} is nilpotent; therefore K_{ω} is abelian. Thus, as the theorem is true for $r = 1$, K' is nilpotent.

Write $K' = A \times B$ where A is a Sylow p-subgroup of K'. Then B is a normal p'-subgroup of K and since F_1 is a p-group, $B \cap F_1 = 1$. Since F_1 is also a normal subgroup of K and G is soluble,

$$B \leqq C_K(F_1) \leqq C_G(F_1) \leqq F_1.$$

Thus $B = 1$ and therefore $K' \leqq QF_1$ is a p-group. Therefore $K' \leqq F_1$. Let $L = \{x, Q\}$. Then L is a subgroup of M and $K = F_1L$. Now $L \cong K/F_1$ is abelian so that $x \in C_M(Q)$. But $Q = F(M)$ and M is soluble, so this implies that $x \in Q$. This contradiction to the choice of x proves the lemma.

Corollary. *Since $G = F_3(G)$ by lemma 8, whilst G does not satisfy the conclusion of theorem 1, it follows that $G^{(r)}$ is not nilpotent. Thus $M^{(r)} > 1$.*

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LEMMA 9. *There exists an ω-complement D of Q in M. Q is a q-group for some prime $q \neq p$ and M/Q is a q'-group.*

PROOF. To construct an ω-complement of Q in M we use properties of Sylow systems of a soluble group (see [5] and [6]).

Since M is soluble there exists a Sylow system of M. Since all such Sylow systems are conjugate in M and since the order of M is odd, there is an odd number of Sylow systems of M. The automorphism ω maps any given Sylow system of M onto another and since ω has order 2, at least one Sylow system of M is fixed by ω. Form the system normalizer D of this system. Clearly D is an ω-group and by the covering theorem, since $M = F_2(M)$, $M = DQ$. Suppose that $D \cap Q > 1$. Then since Q is a normal subgroup of M, $D \cap Q$ is a normal subgroup of D. Also Q is abelian, so that $D \cap Q$ is a normal subgroup of $DQ = M$. Let K be a minimal normal subgroup of M contained in $D \cap Q$. Then by the covering theorem, since $K \leqq D$, K is centralized by M. But by the remark at the end of the proof of lemma 6, $Z(M) = 1$. Hence $K = 1$ and therefore $D \cap Q = 1$. Thus D is an ω-complement of Q in M.

We next show that if K is a proper ω-subgroup of M, $K^{(r)} = 1$. For if K is a proper ω-subgroup of M, F_1K is a proper ω-subgroup of G. Since $(G_\omega)^{(r)} = 1$, $((F_1K)_\omega)^{(r)} = 1$, and therefore the minimality of G implies that $(F_1K)^{(r)}$ is nilpotent. Thus we may write $(F_1K)^{(r)} = A \times B$ where A is a p-group and B is a p'-group. By the minimality of G, $M^{(r)} \leqq Q$ so $(F_1K)^{(r)} \leqq F_1Q$. Thus $A \leqq F$. Also $B \lhd F_1K$, $F_1 \lhd F_1K$ and as their orders are relatively prime, $B \cap F_1 = 1$. Hence $B \leqq C_G(F_1) = F_1$ so $B = 1$. Now $(F_1K)^{(r)} = A \leqq F_1$ so $K^{(r)} \leqq F_1 \cap M = 1$ as required.

Now suppose that Q is not a q-group for any prime q. Then we may write $Q = Q_1Q_2$ where Q_1 and Q_2 are Hall subgroups of Q of relatively prime orders. Since $Q = F(M)$, the Q_i are normal ω-subgroups of M. Thus for each i, DQ_i is a proper ω-subgroup of M and so $(DQ_i)^{(r)} = 1$. Since Q_i is abelian, it follows that

$$(Q_i, D, D', \cdots, D^{(r-1)}) = 1 \qquad (i = 1, 2).$$

Also D is a proper ω-subgroup of G so that $D^{(r)} = 1$. Now

$$\begin{aligned} M^{(r)} &= (DQ_1Q_2)^{(r)} \\ &= D^{(r)}(Q_1, D, D', \cdots, D^{(r-1)})(Q_2, D, D', \cdots, D^{(r-1)}) \\ &= 1, \end{aligned}$$

using, in addition to the above results, the fact that $Q = Q_1Q_2$ is an abelian group. But this contradicts the corollary to lemma 8. Thus Q is a q-group for some prime $q \neq p$.

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Since M/Q is nilpotent, Q is a q-group and $Q = F(M)$ it follows that $D \cong M/Q$ is a q'-group.

LEMMA 10. $D_\omega = D$.

PROOF. Suppose that $D_\omega < D$. Then, since $D \cong M/Q$ is nilpotent by lemma, 8, there exists a proper normal subgroup K of D containing D_ω. Form KQF_1, a proper normal subgroup of G. Since $G_\omega = (F_1)_\omega Q_\omega D_\omega$ is contained in KQF_1, KQF_1 is an ω-subgroup of G by § 2, lemma 3, corollary 2. Hence by the minimality of G, $(KQF_1)^{(r)} \leqq F_1$ and therefore $(KQF_1)^{(r-1)} \leqq F_2 = F_1Q$. Thus $D_\omega^{(r-1)} \leqq K^{(r-1)} \leqq D \cap F_1Q = 1$. Since $r > 1$, G_ω is nilpotent and D is a q'-group whilst Q is a q-group, $M_\omega = D_\omega Q_\omega$ has derived length at most $r-1$. Thus $M^{(r-1)} \leqq F(M) = Q$ and since Q is abelian $M^{(r)} = 1$. But this contradicts the corollary to lemma 8. Thus $D_\omega = D$.

Finally since Q is abelian, $Q \leqq N_M(Q_\omega)$ and since $Q_\omega = Q \cap M_\omega$ and $D = D_\omega \leqq M_\omega$, $D \leqq N_M(Q_\omega)$. Thus $Q_\omega \lhd DQ = M$ contradicting lemma 6. This contradiction completes the proof of the theorem.

4. Proof of theorem 2

Suppose that the theorem is false and choose a counterexample G of minimal order. Then $F(G)$ is the unique minimal normal A-subgroup of G. $F(G)$ is an elementary abelian p-group for some prime p.

Let Γ denote the splitting extension of G by A and write $F = F(G)$.

Suppose that $(G/F)_\omega = G/F$ for some $\omega \in A$, $\omega \neq 1$. Then since G_ω is nilpotent, G/F is nilpotent. It is now an easy consequence of the minimality of G that G/F is a q-group for some prime $q \neq p$. Therefore we can choose a Sylow q-subgroup Q to complement F in G. Since $N_\Gamma(Q)F = \Gamma$, by taking a suitable conjugate of Q if necessary, we may assume A normalizes Q. Since $(G/F)_\omega = G/F$, $Q_\omega = Q$. Now $Z(G) = 1$ for if $Z(G) > 1$. $Z(G) \geqq F$ which is false since G is soluble. Since G_ω is nilpotent and $Q = Q_\omega$ is a group of order prime to p, whilst F is an abelian p-group,

$$F_\omega = G_\omega \cap F \leqq Z(FQ) = Z(G) = 1.$$

Therefore $F_\omega = 1$. Now we may write $\omega = \omega_1\omega_2$ where ω_1 and ω_2 are non-trivial elements of A. Since $Q_{\omega_1\omega_2} = Q$, it follows that $Q_{\omega_1} = Q_{\omega_2}$. Now form F_{ω_1} and F_{ω_2}. Since $F_{\omega_1\omega_2} = 1$, it follows from § 2, lemma 3, that $F = F_{\omega_1}F_{\omega_2}$. Now G_{ω_1} and G_{ω_2} are nilpotent so, as before, $Q_{\omega_1} = Q_{\omega_2}$ is centralized by F_{ω_1} and F_{ω_2}. Therefore $Q_{\omega_1} \leqq C_G(F_{\omega_1}F_{\omega_2}) = C_G(F) = F$. Thus ω_1 induces a regular automorphism on Q, which implies that Q is abelian. Since $G = FQ$, we conclude that $G' \leqq F$ contrary to the definition of G. Therefore for no $\omega \in A$, $\omega \neq 1$, is $(G/F)_\omega = G/F$.

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If $F_\omega = 1$ or F for some $\omega \in A$, $\omega \neq 1$, then, as in § 3, lemma 2, ω either inverts or fixes all the elements of F so that $(G/F)_\omega = G/F$. Since we have already shown that this is false, we conclude that for each $\omega \in A, \omega \neq 1$, $F > F_\omega > 1$. Also, since $C_G(F) = F$, $C_\Gamma(F) = F$.

We have shown that $C_\Gamma(F) = F$ so it follows that $F = F(\Gamma)$ is the unique minimal normal subgroup of Γ. Therefore we may deduce (see [1]) that there exists a complement N of F in Γ. By Sylow's theorem we may suppose that $A \leqq N$. Let $M = G \cap N$ and $F(M) = Q$. The modular law implies that M is a complement of F in G.

For convenience we now summarize the properties of F which we have obtained.

(a) F is the unique minimal normal subgroup of Γ.

(b) $C_\Gamma(F) = F$.

(c) If $\omega \in A, \omega \neq 1$ then $Q_\omega \leqq C_G(F_\omega)$. This follows since F is a p-group, $Q \cong F_2(G)/F$ and G_ω is nilpotent.

(d) For each $\omega \in A$, $\omega \neq 1$, $F_\omega > 1$.

Properties (a) and (b) enable us to consider F as a faithful irreducible Γ/F-module over $GF(p)$. Applying the same method as in the proof of § 3, theorem 1, we may deduce the existence of an $\mathscr{F}(N)$-module V, where $\mathscr{F}$ denotes the algebraic closure of $GF(p)$, with the following properties:

(1) V is a faithful irreducible N-module over $\mathscr{F}$.

(2) For each $\omega \in A, \omega \neq 1, V\omega = \{v \in V | v\omega = v\} > 0$.

(3) For each $\omega \in A, \omega \neq 1$, if $v \in V_\omega$ and $x \in Q_\omega$ then $vx = v$.

We now show

(i) V is an irreducible $\mathscr{F}(M)$-module.

Suppose, by way of contradiction, that V is not an irreducible $\mathscr{F}(M)$-module. Let W be an irreducible $\mathscr{F}(M)$-submodule of V. Then for at least two elements $\omega_1, \omega_2 \in A$, $(\omega_1, \omega_2 \neq 1)$ we have $W\omega_1 \neq W$ and $W\omega_2 \neq W$. Let $w \in W$ so that $w + w\omega_i \in V_{\omega_i}$ $(i = 1, 2)$. Now if $y \in Q_{\omega_i}$ then by property (3)

$$(w + w\omega_i)y = w + w\omega_i.$$

Equating the W components of each side, we deduce that Q_{ω_i} acts trivially on W and so on V. But V is a faithful N-module over $\mathscr{F}$ so it follows that $Q_{\omega_i} = 1$ for $i = 1, 2$. By §2, lemma 3, ω_1 and ω_2 both invert all the elements of Q so that $\omega_1\omega_2$ fixes all the elements of Q. Now we have already shown that $F_{\omega_1\omega_2} > 1$ so since F is an abelian p-group, Q is of order prime to p and $G_{\omega_1\omega_2}$ is nilpotent, $F_{\omega_1\omega_2} \leqq Z(Q_{\omega_1\omega_2}F) = Z(QF) = Z(F_2(G))$. Therefore $Z(G_2(G)) > 1$. But $Z(F_2(G))$ is a normal A-subgroup of G so as F is the unique minimal normal A-subgroup of G, $F \leqq Z(F_2(G))$. This implies that

$F_2(G)$ is nilpotent, a contradiction since G is soluble and non-nilpotent. This contradiction proves (i).

In the same way as we proved § 3 lemma 6, we may now deduce

(ii) If $\omega \in A$, $\omega \neq 1$, and L is a non-trivial normal ω-subgroup of M, then $1 < L_\omega < L$.

It follows from (ii), as in the remark after the proof of § 3, lemma 6, that

(iii) $Z(M) = 1$.

This last result implies that $F_2(G)$ is a proper subgroup of G, so by the minimality of G, $Q \cong F_2(G)/F$ is abelian. We may also deduce from the minimality of G that M/Q is characteristically simple. Therefore M/Q is an elementary abelian r-group for some prime r.

Suppose that r divides the order of Q. Let R be a Sylow r-subgroup of M. Since Q is a normal subgroup of M and r divides the order of Q, $Z(R) \cap Q > 1$. But $Z(R) \cap Q \leqq Z(RQ) = Z(M)$ since Q is abelian. Thus $Z(M) > 1$ contradicting (iii). We conclude therefore that r does not divide the order of Q.

Let R be a Sylow r-subgroup of M. Since Q is of order prime to r, $R \cap Q = 1$. Clearly $RQ = M$. Now form $N_N(R)$. It is easily shown that $N_N(R)Q = N$ so, by taking a conjugate of R if necessary, we may suppose that A normalizes R.

If $\omega \in A$, $\omega \neq 1$ is such that $R_\omega = R$, then since $M_\omega \leqq G_\omega$ is nilpotent, r does not divide the order of Q and Q is abelian, $Q_\omega \leqq Z(RQ) = Z(M)$. Now on the one hand, (iii) implies that $Q_\omega = 1$ whilst on the other hand (ii) implies $Q_\omega > 1$, a contradiction. Thus for no $\omega \in A$, $\omega \neq 1$ is $R_\omega = R$.

It is an easy consequence of the minimality of G that the representation of A on R is irreducible. But an irreducible representation, over a field of characteristic not equal to two, of the non-cyclic group of order 4 is one-dimensional. Therefore for at least one $\omega \in A$, $\omega \neq 1$, is $R_\omega = R$ contradicting the conclusion of the last paragraph.

This contradiction completes the proof of theorem 2.

5. Proof of theorem 3

PROOF. Suppose that the theorem is false and choose a counterexample G of minimal order. Then $F = F(G)$ is the unique minimal normal A-subgroup of G.

Let L be a proper normal A-subgroup of G. Then L is nilpotent. It follows that F is the unique maximal normal A-subgroup of G and that G/F is an elementary abelian r-group for some prime r, since G is soluble. Thus G/F is an irreducible A-module over $GF(r)$. Now A is of exponent two, so any irreducible representation of A, over a field of characteristic not equal

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to two, is one-dimensional. Therefore the kernel of the representation of A on G/F must have order 4, at least. Let ω_1, ω_2 be two distinct non-unit elements of A in the kernel. Then

$$G/F = (G/F)_{\omega_1} = (G/F)_{\omega_2} = (G/F)_{\omega_1\omega_2}.$$

Suppose that $\omega \in A$, $\omega \neq 1$, and $(G/F)_\omega = G/F$. Since F is the unique minimal normal A-subgroup of the soluble group G, F is an elementary abelian p-group. By definition G is not nilpotent, so G/F is not a p-group. Therefore $r \neq p$. Now $(G/F)_\omega = G/F$ is isomorphic to a section of G_ω so the Sylow r-subgroup R of G_ω is a complement of F in G. Since G_ω is nilpotent and F is abelian, $F_\omega = F \cap G_\omega$ is centralized by $RF = G$. Now if $Z(G) > 1$, since F is the unique minimal normal A-subgroup of G, $Z(G) \geqq F = F(G)$, a contradiction since G is soluble. Therefore $F_\omega \geqq Z(G) = 1$. It follows that ω inverts all the elements of F.

Combining these results we have for $x \in F$,

$$x^{\omega_1} = x^{\omega_2} = x^{\omega_1\omega_2} = x^{-1}.$$

Thus $x^{-1} = x^{\omega_1\omega_2} = (x^{-1})^{\omega_2} = x$, a contradiction since the order of F is odd. This proves the theorem.

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