FROM MATRICES TO GRAPHS

W. T. TUTTE

1. Introduction. All the matrices considered in this paper have their elements in the field of residues mod 2.

Two non-singular matrices are *equivalent* if each row of either matrix is a linear combination of rows of the other. The matrices then have equal numbers of rows and equal numbers of columns.

A *nodal* matrix is a non-singular matrix in which no column has more than two 1's. A *graphic* matrix is a non-singular matrix equivalent to a nodal matrix.

In this paper we present an algorithm for determining whether a given non-singular matrix is graphic, and if so for finding an equivalent nodal matrix.

Algorithms of this sort are of interest to electrical engineers, for whom the graphic matrices are the cut-set matrices of graphs. Those so far suggested have been based largely on graph-theoretical concepts (**1**, chap. 5). In the present paper we adopt a purely algebraic point of view.

2. Operations on matrices. Let M be a non-singular matrix. Suppose M' to be an equivalent matrix. We say that the (i, j)th element of M' *corresponds* to the (i, j)th element of M. If S is a submatrix of M, then the *corresponding* submatrix of M' is made up of the elements corresponding to those of S. We say that S is *transformed* into its corresponding submatrix in M' by any operation which changes M into M'.

We define a *central* column of M as one having just one 1. We write $C(M)$ for the submatrix of M, possibly null, made up of all the central columns. If $C(M)$ has a 1 in each row of M we call M a *central* matrix and say that $C(M)$ is its *centre*.

Suppose M has a 1 in the pth row and qth column. Let us replace each other row of M having a 1 in the qth column by its sum with the pth row. We refer to this process as "clearing the qth column with the pth row." It evidently transforms M into an equivalent matrix M'.

The column-clearing operation transforms the qth column into a central column of M', having its 1 in the pth row. It leaves unchanged every column having a zero in the pth row. Hence if the pth row has no 1 in $C(M)$ the operation increases the number of distinct central columns of M. On the other hand any central column of M with its 1 in the pth row transforms into a column of M' equal to the qth column of M. If this column of M' is cleared with the pth row the original matrix M is restored. From these observations we deduce the following theorems.

Received January 7, 1963.

https://doi.org/10.4153/CJM-1964-011-0 Published online by Cambridge University Press

(2.1) *Any non-singular matrix can be transformed into an equivalent central matrix by a sequence of column-clearing operations, no two of which are performed with corresponding rows.*

(2.2) *When a central matrix M is subjected to a column-clearing operation, it is transformed into a central matrix M'. Moreover M' can be changed back into M by another column-clearing operation.*

If M is central we write $\Lambda(M)$ for the set of all matrices derivable from M by column-clearing operations.

Our algorithm is based mainly on a theory of column-clearing operations. But we conclude this section by mentioning some other operations on matrices which are helpful.

We may, for example, permute the rows of a non-singular matrix, thereby changing it into an equivalent one. We may also permute columns. The latter operation does not necessarily change a given matrix into an equivalent one. It does, however, transform nodal matrices into nodal matrices, and therefore graphic matrices into graphic ones.

We define a *row-submatrix* of M as a submatrix made up of one or more complete rows of M. The operation of replacing a given row-submatrix S by a matrix equivalent to S evidently converts M into an equivalent matrix.

For nodal matrices we have the following theorem.

(2.3) *Let M_i be obtained from a nodal matrix M by replacing the ith row by the sum of all the rows of M. Then M_i is a nodal matrix equivalent to M.*

Proof. M_i is clearly equivalent to M. But the ith row of M_i has a 1 in the jth column if and only if M has just one 1 in that column. Hence M_i is nodal.

3. Central matrices. Let M be a central matrix. We consider the row-vectors which are linear combinations of rows of M. We shall speak of the jth component of such a vector as being "in" the jth column of M.

(3.1) *Let K be a non-zero vector which is a linear combination of rows of M. Then K has a* 1 *in some central column of M. Moreover K is the sum of those rows J of M such that J and K have* 1*'s in the same central column.*

Proof. Since K is non-zero it is the sum of a non-null set U of rows of M. Each $J \in U$ has a 1 in $C(M)$, since M is central, and each 1 of J in $C(M)$ gives rise to a 1 of K in the same column. On the other hand, if K has a 1 in a column X of $C(M)$ then U must include the row of M having a 1 in X. These results determine U uniquely, and establish the theorem.

(3.2) *Any row-submatrix S of a central matrix M is central.*

Proof. Each row of S has a 1 in a central column X of M, and the intersection of X with S is a central column of S.

(3.3) *Suppose M is graphic, with an equivalent nodal matrix N. Let S be any row-submatrix of M. Then S is graphic. Moreover S has an equivalent nodal matrix N_S such that every common row of S and N is a row of N_S.*

Proof. If M has only one row, the only row-submatrix of M is M itself. Thus the theorem is trivially true in this case.

Assume the theorem true whenever M has less than k rows, where k is an integer >1, and consider the case in which M has just k rows.

Let S be any row-submatrix of M. If $S = M$, the theorem is satisfied. We may therefore assume that S is contained in a row-submatrix T of M with exactly $k - 1$ rows. Let J be the row of M not in T and let X be a central column of M having its 1 in J.

Let N be a nodal matrix equivalent to M. Let Y be its column corresponding to X. By permuting rows of N we can arrange that the row J' corresponding to J has a 1 in Y. If some other row of N has a 1 in Y we replace it by the sum of all the rows. We thus transform N into a nodal matrix N' equivalent to N, by (2.3). In the remaining case we write $N' = N$. In each case we write X' for the column of N' corresponding to X, and T' for the submatrix of N' corresponding to T. Clearly X' is a central column of N' with its 1 in J', and each common row of N and S is a row of T'.

Any row of T is a linear combination of rows of N'. Since it has a 0 in X it is therefore a linear combination of rows of T'. Similarly each row of T' is a linear combination of rows of T. But T' is nodal, and therefore T is graphic. It follows, by the inductive hypothesis, that S is graphic, having an equivalent nodal matrix N_S which includes all the common rows of S and T', and therefore all the common rows of S and N.

The theorem now follows in general by induction.

4. Connection. Two distinct rows of an arbitrary matrix U are *linked* if there is a column of U having a 1 in each.

A *path* in U is a sequence of one or more distinct rows of U such that any two consecutive members of the sequence are linked. If A and B are its first and last terms respectively we call it a path *from A to B*. It is a *geodesic* path if no two non-consecutive terms are linked.

Two rows A and B of U are *connected* in U if there is a path from A to B in U. Connection in U is evidently an equivalence relation. It partitions U into disjoint non-null row-submatrices $U_1, U_2, \ldots, U_k$ such that two rows are connected in U if and only if they belong to the same submatrix U_i. We refer to the submatrices U_i as the *layers* of U. If $k = 1$, then U is *connected.*

From the above definitions and results we deduce the following theorems.

(4.1) *Any connected submatrix of U is a submatrix of some layer of U.*

(4.2) *Let S be a row-submatrix of U such that no column of U has both a* 1 *in S and a* 1 *outside S. Then S is a union of layers of U.*

We can determine the layers of U as follows. We assign the number 1 to an arbitrarily chosen row A. Then we assign numbers 2, 3, 4, . . . according to the following rule. When the number $n \geqslant 1$ has been assigned, the number $n + 1$ is attached to every as yet unnumbered row of U which is linked to a row numbered n. We continue this process until it terminates. The numbered rows then make up a connected submatrix $U(A)$ of U. This is the layer of U containing A, by (4.1) and (4.2).

If, in the above numbering process, we assign the number n to a row B we can evidently trace backwards from B a geodesic path from A to B in which there are just n terms, the jth term having been assigned the number j. We thus have an algorithm for constructing a geodesic path in U from a given row A to any other row in the same layer.

5. Connection in central matrices.

(5.1) *If a connected central matrix M is equivalent to a matrix M', then M' is connected.*

Proof. Suppose M' is not connected. Let U be one of its layers. Let S' be the submatrix of M' consisting of the columns having 1's in U, and let T' be the submatrix consisting of all other columns. Since M' is non-singular, it has 1's in both S' and T'. (See Fig. 1.)

In Figure 1 and some similar figures we show matrices partitioned by horizontal and vertical lines into rectangular submatrices. Each such submatrix is either left blank to indicate that it has zero elements only, blacked in to show that it consists entirely of 1's, or shaded diagonally to indicate that it may have elements of either kind.

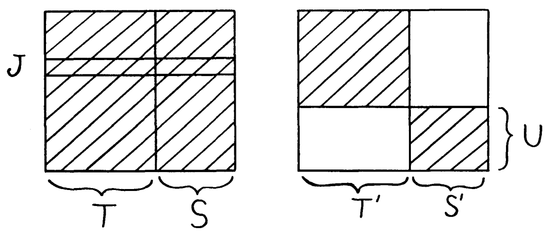

FIGURE 1

Let S and T be the submatrices of M corresponding to S' and T' respectively.

Let J be any row of M. Let J_S and J_T be the vectors derived from it by replacing its 1's in S and T respectively by zeros.

Now J is a linear combination of rows of M'. Hence both J_S and J_T are linear combinations of rows of M', and therefore of rows of M. Hence, by (3.1), if either J_S or J_T is non-zero it is equal to J.

We deduce that no row of M has 1's in both S and T. Since M is connected,

https://doi.org/10.4153/CJM-1964-011-0 Published online by Cambridge University Press

it follows that either S or T is a zero matrix. But then the corresponding matrix S' or T' must be zero, which is a contradiction.

(5.2) *Let M be any central matrix. Then if M is connected every member of $\Lambda(M)$ is connected.*

This follows from (5.1).

(5.3) *Let M be any central matrix. Then M is graphic if and only if its layers are all graphic.*

Proof. If the layers are all graphic we can replace each one by an equivalent nodal matrix, and so obtain a nodal matrix equivalent to M. The converse result follows from (3.3).

(5.4) *Let $P = (V_1, V_2, \ldots, V_k)$ be a geodesic path in a submatrix N of a central matrix M. Let J_i be the row of M containing V_i $(1 \leqslant i \leqslant k)$. Then by a sequence of $k - 1$ column-clearing operations on columns of M meeting N we can transform M into $M' \in \Lambda(M)$ such that $J_1 + J_2 + \ldots + J_k$ is a row of M'.*

Proof. Let X_i denote a column of M having a 1 in V_i and V_{i+1}, but no 1 in any other member of P. By the definition of a geodesic path such a column can be found whenever $1 \leqslant i < k$. Let Q denote the sequence $(J_1, J_2, \ldots, J_k)$, and let K_i $(1 \leqslant i \leqslant k)$ be the sum of the first i members of Q.

Suppose we have found a matrix $M_i \in \Lambda(M)$, where $1 \leqslant i < k$, such that K_i and the last $k - i$ members of Q are rows of M_i. The column of M_i corresponding to X_i has 1's in K_i and J_{i+1}, but no 1 in any member of P succeeding V_{i+1}. Clearing this column with K_i, we obtain a matrix $M_{i+1} \in \Lambda(M)$ having $K_{i+1} = K_i + J_{i+1}$ and the last $K - (i + 1)$ members of Q as rows. If $i + 1 < k$ we repeat the procedure with $i + 1$ replacing i, and so on.

Starting with M as M_1 and applying the above operation for each successive value of i up to $k - 1$ we obtain the required matrix $M' = M_k$.

6. J-Layers. Let M be a connected central matrix and let J be a row of M.

Consider the row-submatrix of M made up of all rows other than J. We partition it into two submatrices $M_0(J)$ and $M_1(J)$ as follows. An element belongs to $M_0(J)$ or $M_1(J)$ according as it is contained in a column of M having a 0 or a 1 respectively in J.

(6.1) *$M_0(J)$ is central.*

Proof. Each row K of M other than J has a 1 in a central column X of M. This column evidently contains a central column of $M_0(J)$. Thus each row of $M_0(J)$ has a 1 in a central column of $M_0(J)$. It follows that $M_0(J)$ is non-singular and central.

We enumerate the layers of $M_0(J)$ as $B_1, B_2, \ldots, B_k$. With each B_i we associate a *J-layer* L_i of M, defined as the row-submatrix of M having 1's in

https://doi.org/10.4153/CJM-1964-011-0 Published online by Cambridge University Press

B_i. Thus each row of M other than J belongs to just one J-layer, and J itself belongs to no J-layer of M.

We partition the part of any J-layer L in $M_1(J)$ into submatrices which we call the (J, L)*-fragments* of M. Each (J, L)-fragment is made up of one or more columns of L in $M_1(J)$, and two columns belong to the same (J, L)-fragment if and only if they are equal vectors.

(6.2) *For each J-layer L of M there are at least two distinct (J, L)-fragments of M, one of these being a matrix of zeros only.*

Proof. There is a column of $C(M)$ having its 1 in J. This column contains a zero column of L. Hence a zero (J, L)-fragment of M exists.

Suppose there is no other (J, L)-fragment of M. Then no column of M has a 1 in L and a 1 outside L. Hence M is not connected, by (4.2), which is contrary to its definition.

It is convenient to permute the columns of M so that those with 1's in J appear first. It is also convenient to permute the rows so that J is the first row, the rows of L_1 appear next, then those of L_2, and so on. As a further refinement we can permute the columns with 0's in J so that those with 1's in L_1 appear first, then those with 1's in L_2, and so on.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
J	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
	0	1	0	0	1	1	0	1	0	0	0	0	0	0	0
L_1	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0
	0	1	1	1	1	0	0	0	1	0	0	0	1	0	0
	0	1	0	0	0	0	0	0	0	1	0	0	1	1	0
	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
L_2	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1

FIGURE 2

Figure 2 shows a connected central matrix written in this way. Here the zero (J, L_1)-fragment is a single column of L_1. The third and fourth columns of L_1 constitute a second (J, L_1)-fragment, and the second and fifth columns make up a third. There is a (J, L_2)-fragment occupying the third, fourth, and fifth columns of L_2, and the other two (J, L_2)-fragments have one column each.

(6.3) *Let R be a row-submatrix of M made up of J and one or more J-layers of M. Then R is a connected central matrix. Its J-layers are the J-layers of M contained in R, and any such J-layer L has the same (J, L)-fragments in R as in M.*

Proof. Since each of the submatrices B_i of M is connected, each J-layer of M is connected, by (4.1). By (6.2) each J-layer of M contained in R belongs to the same layer of R as does J. Hence R is connected. It is central by (3.2).

By (4.1) and (4.2) the layers of $R_0(J)$ are the layers of $M_0(J)$ associated with the J-layers of M contained in R. Hence these J-layers of M are the J-layers of R. Since the (J, L)-fragments, for a given J-layer L, are determined by J and L alone, the theorem follows.

We now consider those column-clearing operations on M which leave J unchanged. There are just two kinds, those in which some column is cleared with J and those in which the column being cleared has a 0 in J. We refer to these as *J-preserving operations* of the *first* and *second* kind respectively.

(6.4) *Let M be transformed into $R \in \Lambda(M)$ by a J-preserving operation. Then each J-layer L of M is transformed into a J-layer L' of R. Moreover each (J, L)-fragment of M is transformed into a (J, L')-fragment of R.*

Proof. We observe that R can be transformed into M by a column-clearing operation, by (2.2), and that this is necessarily J-preserving.

A J-preserving operation of the first kind on M leaves $M_0(J)$ unaltered and an operation of the second kind changes it in one J-layer only. In either case a column of M having all its 1's in a J-layer L is transformed into a column of R having all its 1's in the submatrix L' of R corresponding to L. Hence, by (4.2), any J-layer of M transforms into a union of J-layers of R. But a similar result must hold for the operation transforming R into M. Hence J-layers of M transform into J-layers of R.

It is easily verified that a J-preserving operation of either kind on M transforms equal columns of a J-layer L of M into equal columns of the corresponding J-layer L' of R. It therefore transforms each (J, L)-fragment into a submatrix of a (J, L')-fragment. An analogous result holds for the transformation of R into M. The theorem follows.

7. J-Layers and K-layers.

Let M be a connected central matrix and let J and K be distinct rows of M. We proceed to relate the J-layers of M to the K-layers.

Suppose S and T are submatrices of M. We say that S *covers* T if each column of M containing a column of T contains also a column of S. We say also that S *covers* those elements of M which are in columns meeting S.

(7.1) *Let E and F be distinct J-layers of M. Then no non-zero (J, F)-fragment covers the zero (J, E)-fragment.*

Proof. There is a central column X of M having its 1 in J, and X meets the zero (J, L)-fragment for each J-layer L of M.

(7.2) *Let L be a J-layer of M which does not contain K. If any non-zero (J, L)-fragment of M covers a zero of K, then L is contained in the K-layer H of M having J as a row. Otherwise L is a K-layer of M.*

Proof. The part D of L in $M_0(J)$ is connected, by the definition of a J-layer. It follows that the submatrix B of D, consisting of all the non-zero columns, is connected. But B is a submatrix of $M_0(K)$, since K is not a row of L. Hence L is contained in a K-layer L' of M. (See Fig. 3.)

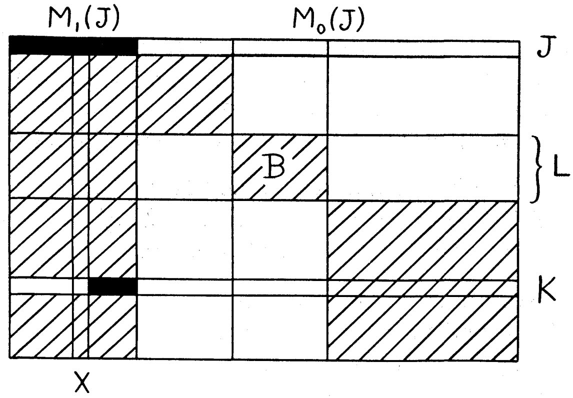

FIGURE 3

Suppose some column X of $M_0(K)$ has a 1 in L and a 1 not in L. By the definition of L, X must have a 1 in J. Hence J is a row of L'. In the remaining case L contains a K-layer of M, by (4.2), and therefore $L = L'$.

(7.3) *Let L be a K-layer of the J-layer G of M containing K. If L has a* 1 *which is in $M_1(J)$ but not $M_1(K)$, then L is part of the K-layer H of M containing J. Otherwise L is a K-layer of M.*

Proof. The part of L in $M_0(K)$ is connected. Hence there is a K-layer L' of M containing L.

Suppose some column X of $M_0(K)$ has a 1 in L and a 1 not in L. Since the second 1 is not in G, X must have a 1 in J. Hence J is a row of L'. In the remaining case, L contains a K-layer of M, by (4.2), and therefore $L = L'$. (See Fig. 4.)

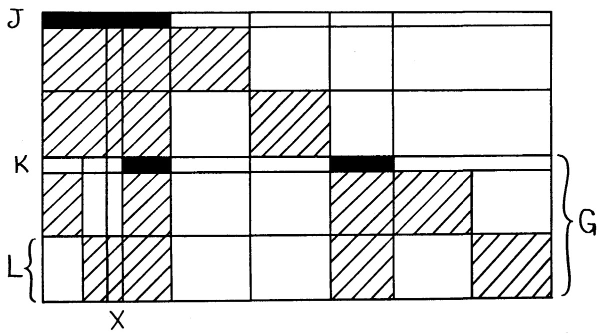

FIGURE 4

https://doi.org/10.4153/CJM-1964-011-0 Published online by Cambridge University Press

We can now recognize three kinds of K-layers, with respect to J. The K-layers of the *first kind* are those which are also J-layers of M. There is only one K-layer of the *second kind*; it is the one containing J. The K-layers of the *third kind* are those which are also K-layers of the J-layer G containing K. Theorems (7.2) and (7.3) show that each K-layer of M is of just one of these kinds.

(7.4) *Let L be a K-layer of M which is also a J-layer. Then the non-zero (K, L)-fragments of M are identical with the non-zero (J, L)-fragments of M.*

Proof. By (7.2) any non-zero column of L which is covered by one of $M_1(J)$ and $M_1(K)$ is covered by both of them. The theorem follows.

8. Nodal sequences. Let M be a connected central matrix and let J be a row of M.

Let Q be a sequence of one or more J-layers of M. If G is any J-layer of M we define a set $A(Q, G)$ as follows. If $G \in Q$, then $A(Q, G)$ is the set of all members of Q preceding G, otherwise it is the set of all members of Q other than the last.

A (J, G)-fragment Y_G *enfolds* Q if $A(Q, G)$ is non-null and for each $E \in A(Q, G)$ the matrix Y_G covers all but one of the (J, E)-fragments of M. We note that a (J, G)-fragment cannot cover all the columns of $M_1(J)$ by (6.2), and therefore cannot cover all the (J, E)-fragments, for any J-layer E.

We call Q a *nodal sequence* of J in M if, for each J-layer G such that $A(Q, G)$ is non-null, there exists a (J, G)-fragment Y_G which enfolds Q.

Any set obtained by selecting one enfolding (J, G)-fragment for each J-layer G such that $A(Q, G)$ is non-null will then be called an *enfolding set* of Q.

The enfolding set may not be uniquely determined by J and Q. An ambiguity for Y_G arises when, for each $E \in \{G\} \cup A(Q, G)$, the set $A(Q, G)$ being non-null, there are just two (J, E)-fragments, and these determine the same partition of the set of columns of $M_1(J)$ for each such E. However when, in what follows, we speak of a nodal sequence Q of J we assume that some one enfolding set $Y(Q)$ is specified. We then say that Q is *simple* if each member of $Y(Q)$ consists entirely of 1's. If I is a row of the first member of Q, if I has at least one 1 in $M_1(J)$, and if each member of $Y(Q)$ covers all the 1's of I in $M_1(J)$, then we say that Q is *pinned* by I.

A trivial example of a nodal sequence of J is obtained by selecting an arbitrary J-layer as the only term of Q. Then $Y(Q)$ is null and Q is simple. Moreover Q is pinned by any row of the chosen J-layer which has 1's in $M_1(J)$.

A nodal sequence (L_1, L_2, L_3, L_4) is shown diagrammatically in Figure 5. It is pinned by I. The (J, L_i)-fragments are shown. In the case illustrated each of these, except Y_3 and Y_4, consists of a single block of consecutive columns. The heart of the enfolding set in these four J-layers is $\{Y_2, Y_3, Y_4\}$.

Suppose M is transformed into M' by a column-clearing operation. Any sequence P of submatrices of M will be said to *transform* into the sequence P'

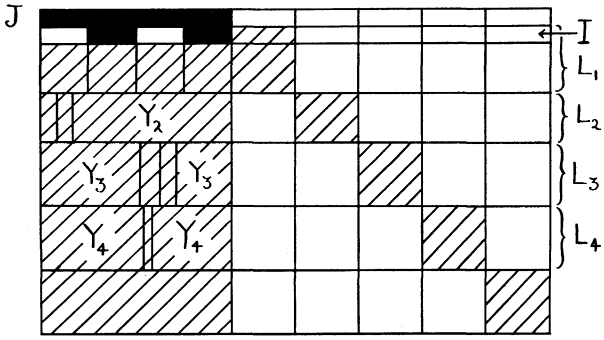

FIGURE 5

of submatrices of M' obtained by replacing each term of P by the corresponding submatrix of M'.

(8.1) *Let Q be a nodal sequence of J in M, pinned by a row I. Let M be transformed into $M' \in \Lambda(M)$ by a column-clearing operation which does not alter I or J. Then Q is transformed into a nodal sequence Q' of J in M', the members of $Y(Q)$ being transformed into the members of $Y(Q')$. Moreover Q' is pinned by I.*

This theorem is a consequence of (6.4).

(8.2) *Let Q be a nodal sequence of J. Let its last term be U. Let K be a row of a J-layer $H \notin Q$. If Q has more than one term, let K have a 1 in Y_H. Let U_1 be the K-layer of M containing U;* see (7.2). *Let Q_1 be derived from Q by replacing U by U_1. Then Q_1 is a nodal sequence of K in M.*

Moreover Q_1 is simple if Q is simple, and if Q is pinned by a row I having a 1 *in $M_1(K)$, then Q_1 is pinned by I.*

Proof. We may suppose Q to have more than one term since otherwise the theorem is trivially true.

Suppose $E \in A(Q, U)$. Then the zero (J, E)-fragment covers all the 0's of K in $M_1(J)$, by (7.1). Hence, by (7.2), E is a K-layer of M. Moreover the non-zero (J, E)-fragments are the non-zero (K, E)-fragments and they are covered by both $M_1(J)$ and $M_1(K)$; see (7.4).

Let L be any K-layer of M other than the first term of Q.

Case 1. *L is of the first kind* (*with respect to J*). This case arises, for example, whenever $L \in A(Q, U)$.

We define a (K, L)-fragment T_L as follows. If Y_L is non-zero, then $T_L = Y_L$; see (7.4). If Y_L is zero, then T_L is the zero (K, L)-fragment. In each case T_L covers all the columns of Y_L in $M_1(K)$, by (7.4).

It follows that if Q is pinned by I, then T_L covers all the 1's of I in $M_1(K)$. Moreover, if $F \in A(Q, L)$ any non-zero (K, F)-fragment which is covered by Y_L is covered also by T_L.

Suppose Y_L covers the zero (J, F)-fragment (Fig. 6). Then Y_L and T_L are zero matrices, by (7.1). Outside $M_1(J)$ T_L covers the same columns of M

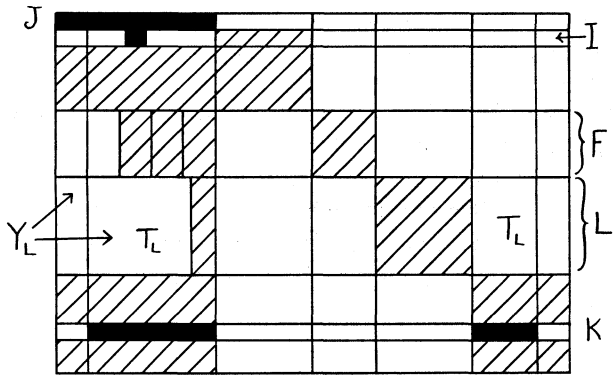

FIGURE 6

as does the zero (K, F)-fragment, namely those covered by $M_1(K)$. Inside $M_1(J) \cap M_1(K)$ the zero (J, F)-fragment and the zero (K, F)-fragment coincide. Hence T_L covers the zero (K, F)-fragment; see (7.4).

We deduce that T_L covers all but one of the (K, F)-fragments.

If Q is simple T_L consists of 1's only. For since Y_L is then non-zero we have $T_L = Y_L$.

Case 2. *L is of the second kind.*

Let C be the set of all J-layers E of M, other than H, such that some non-zero (J, E)-fragment covers a zero of K in $M_1(J)$. Let D be the set of all K-layers G of H such that G has a 1 which is in $M_1(J)$ but not $M_1(K)$. By (7.2) and (7.3) L is made up of J, the members of C, and the members of D.

Let S be the set of all columns X of L such that X is covered by $M_1(J)$, $M_1(K)$, Y_H, and each Y_E such that $E \in C$. Clearly any two members of S are equal.

Suppose S is non-null. Then there is a (K, L)-fragment T_L which contains every member of S. If Q is pinned by I, then by the definition of S, T_L covers all the 1's of I in $M_1(K)$. Moreover if Q is simple each member of S is a column of 1's only, and therefore T_L consists entirely of 1's.

Suppose $F \in A(Q, U)$. We know that each of $M_1(J)$ and $M_1(K)$ covers all the non-zero (J, F)-fragments. But these are also covered by Y_H, by (7.1), since Y_H is non-zero. They are also covered by each Y_E such that $E \in C$. For suppose this is not true for some $E \in C$. Then, since $E \notin A(Q, U)$, Y_E covers the zero (J, F)-fragment, which itself covers all the zeros of K in $M_1(J)$. This implies that Y_E is zero, by (7.1), and the definition of C is contradicted.

We deduce, since $A(Q, U)$ is non-null, that S is non-null. Hence T_L is defined and T_L covers all the non-zero (K, F)-fragments for each $F \in A(Q, U)$.

Case 3. *L is of the third kind.*

Since $A(Q, U)$ is non-null and Y_H covers all the non-zero (J, F)-fragments, for each $F \in A(Q, U)$, it follows that Y_H covers at least one column of $M_1(K)$.

Hence there is a uniquely defined (K, L)-fragment T_L covering all the columns of Y_H in $M_1(K)$. Accordingly T_L, like Y_H, covers all the non-zero (K, F)-fragments for each $F \in A(Q, U)$. Moreover, if Q is pinned by I, then T_L covers all the 1's of I in $M_1(K)$. If Q is simple, Y_H consists entirely of 1's and therefore T_L consists entirely of 1's.

The foregoing analysis shows that Q_1 is a nodal sequence of K in M, with the enfolding set $\{T_L\}$. If Q is simple we have seen that each matrix T_L consists entirely of 1's, so that Q_1 is also simple. If Q is pinned by a row I having a 1 in $M_1(K)$ we have seen that each matrix T_L covers all the 1's of I in $M_1(K)$, so that Q_1 is also pinned by I.

9. Nodal rows. Let M be a connected central matrix. A nodal sequence of a row J of M is *complete* if it includes all the J-layers of M. If J has a complete nodal sequence we call it a *nodal row* of M.

Consider a sequence Q of $q \geqslant 1$ submatrices of M. A second sequence Q_1 of q submatrices of M is an *improvement* of Q in M if it differs from Q only in the qth term, and if the qth term of Q_1 has more rows than the qth term of Q.

A sequence formed from Q by adjoining a new submatrix of M as a $(q + 1)$th term is an *extension* of Q in M.

Suppose M is transformed into $M' \in \Lambda(M)$ by a sequence of column-clearing operations. Then Q is transformed into a sequence Q' of q submatrices of M'. Any improvement or extension of Q' in M' will be called an *improvement* or *extension* of Q respectively in M'.

(9.1) *Let Q be a simple nodal sequence of a row J of M. Then if Q is not complete, we can find a simple nodal sequence Q_1 of a row K of M, so that Q_1 is either an improvement or an extension of Q in M.*

Proof. Let the last term of Q be U. Since Q is not complete, there are J-layers of M not in Q.

Suppose first that there is a row K of one such J-layer H such that one zero of K is covered by a non-zero (J, U)-fragment of M. By (7.2), there is a K-layer U_1 of M which contains both U and J. The sequence Q_1 obtained from Q by replacing U by U_1 is thus an improvement of Q.

If Q has more than one term, then Y_H is defined and consists entirely of 1's. Hence K has a 1 in Y_H. It follows, by (8.2), that Q_1 is a simple nodal sequence of K, whether Q has one term or more.

In the remaining case each J-layer H not in Q has a (J, H)-fragment T_H which covers all the non-zero (J, U)-fragments and consists entirely of 1's. We adjoin one such J-layer, H_1 say, to Q to form an extension Q_1 of Q in M.

If Q has more than one term, the matrices Y_H are defined for $H \notin Q$, and

https://doi.org/10.4153/CJM-1964-011-0 Published online by Cambridge University Press

$Y_H = T_H$ since Y_H consists entirely of 1's. Hence Q_1 is a simple nodal sequence of J in M, with $Y(Q_1) = Y(Q)$ or $Y(Q_1) = \{T_H\}$.

(9.2) *Suppose M has more than one row. Then if I is any row of M we can find a nodal row J of M other than I. Moreover we can choose J to have a nodal sequence which is both simple and complete.*

Proof. Choose a row K of M other than I and let U be the K-layer of M which includes I. Starting with the simple nodal sequence (U) of K we repeatedly construct improvements and extensions in M by the method of (9.1). Since each step increases the total number of rows appearing in the members of the nodal sequence the process must terminate. When it does we have a simple complete nodal sequence Q of some row J of M. But I is a row of the first member of Q and is therefore distinct from J.

We can use Figure 2 to provide some rather trivial exercises on the method of (9.1) and (9.2). Let us denote the ith row by J_i. To find a nodal row it seems reasonable to start with the nodal sequence (L_2) of J, since L_2 is the J-layer with the greatest number of rows. We observe that J_3 has a 0 in a column of $M_1(J)$ having a 1 in L_2. We can therefore improve L_2 by replacing it by the nodal sequence (L_3) of J_3, where L_3 is the J_3-layer of M containing J and L_2. We see from the second column that L_3 also contains J_2. Hence J_3 is a nodal row, with a complete nodal sequence (L_3).

If the 0's in the second, third, fourth, and fifth columns of L_2 are replaced by 1's, then (L_1, L_2) becomes a simple and complete nodal sequence of J in the altered matrix.

(9.3) *Let Q be a nodal sequence, pinned by a row I, of some row J of M. Then if Q is not complete, we can find a nodal sequence Q_1, pinned by I, of a row K of some $M' \in \Lambda(M)$, such that Q_1 is either an improvement or an extension of Q in M'.*

Proof. Let the last term of Q be U. Since Q is not complete, there are J-layers of M not in Q. A row K of such a J-layer will be called a *weak* row of M if it has a 1 common to $M_1(I)$ and $M_1(J)$ and a 0 covered by a non-zero (J, U)-fragment.

Suppose K is a weak row of M in a J-layer H. By (7.2) there is a K-layer U_1 of M which contains both U and J. Replacing U by U_1 in Q we obtain an improvement Q_1 of Q in M.

If Q has more than one term, then Y_H is defined and covers all the 1's of I in $M_1(J)$. Hence K has a 1 in Y_H. Using (8.2) we deduce that Q_1 is a nodal sequence of K, pinned by I. The enfolding set $Y(Q_1)$ can be obtained from $Y(Q)$ by the definitions set out in the proof of (8.2).

We deduce that the theorem is true if M has a weak row.

In the remaining case if U is the only term of Q, then all the 1's of I in $M_1(J)$ are covered by a single (J,H)-fragment Y_H, for each $H \notin Q$. If Q has more than

https://doi.org/10.4153/CJM-1964-011-0 Published online by Cambridge University Press

one term we already know this to be true, with $Y_H \in Y(Q)$. In either case we continue the argument as follows.

Suppose first that there is a J-layer $H \notin Q$ such that Y_H leaves two distinct (J, U)-fragments of M, Z_1, and Z_2 say, uncovered. Let X_1 and X_2 be columns of M, with 1's in J, which meet Z_1 and Z_2 respectively, but do not meet Y_H (Fig. 7).

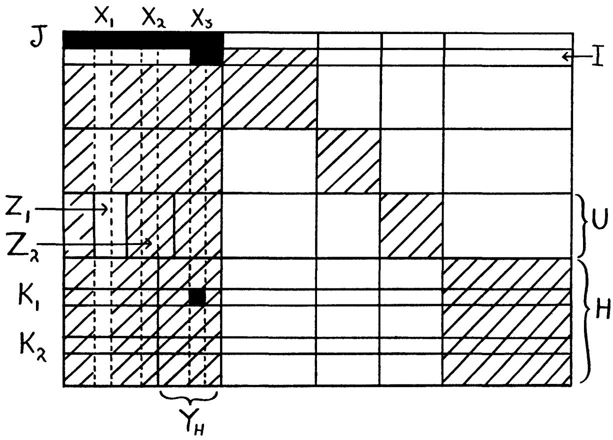

FIGURE 7

We clear X_1 with J. Since the 1's of I in $M_1(J)$ are all covered by Y_H this operation does not alter the rows I and J. By (8.1) it transforms Q into a nodal sequence Q', pinned by I, of J in the new matrix. The members of $Y(Q)$ transform into the members of $Y(Q')$. If Q has only one term we apply (6.4) to the matrices Y_H. We thus reduce our problem to the case in which Z_1 is a zero matrix.

We can now find a column X_3 of M, with 1's in both I and J, which meets Y_H. Since Y_H must now be non-zero we can find a row K_1 of H with a 1 in X_3. Since X_2 and X_3 are covered by different (J, H)-fragments we can find a row K_2 of H whose elements in X_2 and X_3 are different. But we may assume that no row of H can be chosen as both K_1 and K_2. Such a row would be a weak row of M and would lead to a verification of the theorem.

Let B be the layer of $M_0(J)$ corresponding to H. Let W_1 and W_2 be the parts of K_1 and K_2 respectively in B. Using the construction of §4 we form a geodesic path $P = (V_1, V_2, \ldots, V_k)$ from $W_1 = V_1$ to $W_2 = V_k$ in B. Let J_i be the row of M containing V_i $(1 \leqslant i \leqslant k)$. If we find that some J_t, where $1 < t < k$, has either a 1 in X_3 or unequal elements in X_2 and X_3 we can obviously make a new choice of K_1 and K_2 so as to obtain a shorter path P. We can therefore arrange that each J_i other than J_1 has a 0 in X_3 and that each J_i other than J_k has equal elements in X_2 and X_3.

By the procedure of (5.4) we can transform M into a matrix $M' \in \Lambda(M)$

having the sum $K = J_1 + J_2 + \ldots J_k$ as a row, effecting this transformation by clearing $k - 1$ columns with 0's in J and 1's in H. This sequence of operations preserves I and J. We thus reduce, by (8.1), to the case in which M has a weak row, for the sum K has a 1 in X_3 and 0's in both X_1 and X_2. We can now verify the theorem.

In the remaining case we find on inspection that, for each J-layer $H \notin Q$, Y_H covers all but one of the (J, U)-fragments. We then adjoin one such J-layer, H_1 say, to Q to form an extension Q_1 of Q in M. It is clear that Q_1 is a nodal sequence of J in M, pinned by I, with $Y(Q_1) = Y(Q)$ if Q has more than one term, and $Y(Q_1) = \{Y_H\}$ otherwise.

(9.4) *Suppose M has more than one row. Then if I is any row of M we can find a nodal row J of some $M' \in \Lambda(M)$ such that J has a complete nodal sequence pinned by I.*

Proof. Since M is connected we can find a row K of M having a 1 in $M_1(I)$, but distinct from I. Let U be the K-layer of M containing I. Starting with the nodal sequence (U) of K, which is pinned by I, we repeatedly construct improvements and extensions by the method of (9.3). When the process terminates we have a nodal row J of some $M' \in \Lambda(M)$, with a complete nodal sequence pinned by I.

Consider again the matrix of Figure 2. Let its ith row be denoted by J_i, as before. We construct a complete nodal sequence, pinned by J_3.

We begin with the nodal sequence (L_1) of J. The (J, L_2)-fragment, Y_2 say, occupying the third, fourth, and fifth columns covers all the 1's of J_3 in $M_1(J)$. But it leaves two (J, L_1)-fragments, represented by the first and second columns of L_1, uncovered. We can take the first, second, and third columns of M as the X_1, X_2, and X_3 respectively of (9.3).

We must now take J_4 and J_5 as the K_1 and K_2 of (9.3). The sequence (J_4, J_5) corresponds to a geodesic path in the part of L_2 in $M_0(J)$. We therefore clear the thirteenth column with J_4, thus transforming M into the matrix M' of Figure 8.

The row $K = J_4 + J_5$ of M' has two K-layers, L_4 corresponding to the first four rows and L_5 corresponding to the last two. We find that (L_4, L_5) is a complete nodal sequence of K, pinned by J_3.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
J	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
J_2	0	1	0	0	1	1	0	1	0	0	0	0	0	0	0
J_3	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0
J_4	0	1	1	1	1	0	0	0	1	0	0	0	1	0	0
J_4+J_5	0	0	1	1	1	0	0	0	1	1	0	0	0	1	0
J_6	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
J_7	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1

FIGURE 8

https://doi.org/10.4153/CJM-1964-011-0 Published online by Cambridge University Press

10. Nodal rows in graphic matrices. Let M be a connected, central, graphic matrix, and let J be a row of M.

(10.1) *If M has not more than one J-layer, and if N is any nodal matrix equivalent to M, then J is either a row of N or the sum of all the rows of N.*

Proof. If J is the only row of M, then $M = N$ and the theorem is trivially true. We may now suppose that M has just one J-layer, L say. Let the part of L in $M_0(J)$ be L_0. Let N_0 denote the submatrix of N made up of those columns whose corresponding columns in M have 0's in J. Let R be a row-submatrix of N_0 defined by a maximal set of linearly independent rows. Since M and N are equivalent it follows that L_0 and R are equivalent.

Assume that the theorem fails. Then we can partition N into two non-null row-submatrices S and T such that S has at least two rows and the rows of S sum to J. Let the intersections with N_0 of S and T be S_0 and T_0 respectively. Since N is nodal and the rows of S sum to J, no column of N_0 can have a 1 in S_0 and a 1 in T_0.

But each row of N has a 1 in N_0, since it is a sum of rows of the central matrix M and is not J; see (3.1). Since the rows of N_0 are linear combinations of those of R it follows that the intersections with R of S_0 and T_0 are both non-null.

We deduce that R is not connected. But L_0 is central, by (3.2) and (6.1), and equivalent to R. Hence L_0 is not connected, by (5.1). But this is contrary to the definition of L as a J-layer. The theorem follows.

We call a row of M *nodular* if it is also a row of some nodal matrix N equivalent to M.

(10.2) *Let J be a nodal row of M with a complete nodal sequence Q. Then there exists a nodal matrix N which is equivalent to M and has J as a row. Moreover, if Q is pinned by a nodular row I of M, then N can be chosen to have I also as one of its rows.*

Proof. Let q be the number of terms of Q.

Suppose first that $q = 1$. We can choose a nodal matrix N equivalent to M, and having I as a row if I is defined. By (2.3) and (10.1) we can arrange that N has J as a row.

Assume the theorem true whenever q is less than some integer $s > 1$ and consider the case $q = s$. We write $Q = (L_1, L_2, \ldots, L_s)$ and denote the member of $Y(Q)$ in L_i by Y_i $(1 < i \leqslant s)$.

Let R_1 be the row-submatrix of M defined by J and all the J-layers L_i other than L_s. Let R_2 be the row-submatrix defined by J and L_s. Then R_1 and R_2 are connected, central, and graphic, by (3.3) and (6.3).

By (6.3) J is a nodal row of R_1 with a complete nodal sequence

$$Q_1 = (L_1, L_2, \ldots, L_{s-1}),$$

where $Y(Q_1) = Y(Q) - \{Y_s\}$. If I is defined, it pins Q_1, and it is a nodular

https://doi.org/10.4153/CJM-1964-011-0 Published online by Cambridge University Press

row of R_1 by (3.3). By the inductive hypothesis we can replace R_1 in M by an equivalent nodal matrix N_1 having J, and I too if it is defined, as a row. We thus transform M into an equivalent matrix M'.

Now the columns of R_1 which meet $M_1(J)$ and are not covered by Y_s in M are all equal, and if I is defined, they have 0's in I. Using (2.3) we can therefore adjust N_1 so that J is the only row of M' having 1's in the columns of N_1 not covered by Y_s.

By (2.3), (6.3), and (10.1) we can replace R_2 by an equivalent nodal matrix N_2 having J as a row. We thus transform M' into an equivalent matrix M''. But the columns of R_2 covered by Y_s are all equal. We can therefore adjust N_2, using (2.3), so that J is the only row of M'' having 1's in both N_1 and N_2. Since N_1 and N_2 are nodal it follows that M'' is nodal.

The above argument shows that the theorem is true for $q = s$. By induction it is true for all values of q.

(10.3) *Let I be a nodular row of M. Let J be a nodal row of M having a complete nodal sequence Q which is pinned by I. Then those columns of M which have* 1'*s in both I and J are all equal.*

Proof. By (10.2) there is a nodal matrix N, equivalent to M, which has both I and J as rows. Since N is nodal the columns of N having 1's in both I and J are all equal. The same result must hold for the equivalent matrix M.

(10.4) *Let I and J be as in* (10.3). *Let M be transformed into $M' \in \Lambda(M)$ by clearing with J a column X having* 1'*s in both I and J. Let R be the row-submatrix of M' made up of all rows other than J. Then $I + J$ is a nodular row of R.*

Proof. By (10.2) there is a nodal matrix N, equivalent to M, which has both I and J as rows. We replace I in N by $I + J$, thus obtaining an equivalent matrix N'. Let N_1 be the submatrix of N' made up of all rows other than J. Clearly N_1, though not necessarily N', is nodal.

Now the columns corresponding to X in M' and N' are central, and each has its 1 in J. It follows, by an argument like the one used in the proof of (3.3), that R and N_1 are equivalent.

But $I + J$ is a row both of R and of the equivalent nodal matrix N_1. That is, $I + J$ is a nodular row of R.

11. The graphic algorithm.

We are now in a position to explain the application of the algorithm to a given non-singular matrix M. We first replace M by an equivalent central matrix M_1, by the process described in § 2. Then we determine the layers of M_1, as in § 4.

Suppose first that M_1 is found to be connected. We may assume it to have more than two rows since otherwise it is already in nodal form. As a preliminary operation we apply the method of (9.1) and (9.2) to find a nodal row I_1 of M_1. We denote by N_1 the matrix having I_1 as its only row.

At the beginning of the ith stage of the algorithm we have two matrices M_i and N_i, and one row of M_i is distinguished by the symbol I_i. The following propositions hold.

(i) *N_i has i rows and M_i has $r - i + 1$, where r is the number of rows of M.*

(ii) *M_i is connected and central.*

(iii) *N_i is nodal.*

(iv) *If the* jth *column of N_i has two* 1's, *then the* jth *column of M_i has zeros only.*

(v) *I_i is the sum of the rows of N_i.*

(vi) *The matrix T_i formed by adjoining to N_i all the rows of M_i except I_i is equivalent to M.*

(vii) *If M is graphic, then M_i is graphic and I_i is a nodular row of M_i.*

In the case $i = 1$ the first six of these propositions are trivially true, and the seventh follows from (10.2).

If $i < r$ we carry out the ith stage as follows. We apply the process of (9.3) and (9.4) to M_i and obtain a matrix $M_i' \in \Lambda(M_i)$ in which there is a row J_i having a complete nodal sequence pinned by I_i. We then examine the columns of M_i' having 1's in both I_i and J_i. If these columns are not all equal, we terminate the algorithm and assert that M is not graphic. This is justified by (vii) and (10.3). If they are equal we clear one of them, and therefore all of them, with J_i. This operation transforms M_i' into $M_i'' \in \Lambda(M_i)$. It also replaces I_i by $I_i + J_i$, which we denote by I_{i+1}. We define M_{i+1} as the matrix obtained from M_i'' by deleting the row J_i and N_{i+1} as the matrix obtained from N_i by adjoining J_i as a new row.

We must now verify that Propositions (i) to (vii) continue to hold when i is replaced by $i + 1$. For (i) and (v) this is evident.

For (ii) we observe that M_i'', like M_i, is connected and central. Hence M_{i+1} is central by (3.2). Any J_i-layer L of M_i'' is connected, by definition. It has a 1 in a column X having another 1 in J_i, by (6.2). By the construction of M_i'' the non-central column X must also have a 1 in I_{i+1}. Since M_{i+1} is the union of the J_i-layers of M_i'' and has I_{i+1} as a row, it must be connected.

To prove that N_{i+1} is nodal we observe that by (iv) a column X of M_i'' with a 1 in J_i corresponds to a central or zero column Y of N_i, the first alternative arising, by (v), only when X has a 1 in I_i. Since some non-central column of M_i'' has a 1 in J_i we deduce that N_{i+1} is non-singular and has at most two 1's in each column. It is thus nodal. Moreover the columns with two 1's correspond to zero columns of M_i'' or to columns of M_i'' with 1's in both I_i and J_i, that is to zero columns of M_{i+1}. Thus (iii) and (iv) continue to hold at the $(i + 1)$th stage.

Using (v) it is easy to verify that each row of T_i is a linear combination of rows of T_{i+1}, and each row of T_{i+1} is a linear combination of rows of T_i. Hence, since T_{i+1} has exactly r rows, it is equivalent to T_i and M.

If M is graphic, then M_i'' is graphic and has I_{i+1} as a nodular row of the graphic row-submatrix M_{i+1}, by (10.4).

We have now verified our seven propositions with i replaced by $i + 1$.

It follows that we can start with $i = 1$ and carry out the algorithm stage by stage until either it terminates in the manner described above, M being non-graphic, or we complete the $(r - 1)$th stage and obtain the matrices M_r and N_r. Then M_r consists of the single row I_r by (i), and so $N_r = T_r$. Thus N_r is a nodal matrix equivalent to M, by (iii) and (vi).

There remains the case in which M_1 is not connected. We then apply the algorithm to each layer separately and use (5.3).

As an example we turn to Figure 8, derived from Figure 2 by operations belonging to the first stage of the algorithm. We change the notation to fit the present section, so that J_3 becomes I_1 and $J_4 + J_5$ becomes J_1. Clearing with J_1 the two equal columns with 1's in both I_1 and J_1 we arrive at the matrix of Figure 9. Here N_2 is shown above the horizontal division and M_2 below it.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
I_1	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0
J_1	0	0	1	1	1	0	0	0	1	1	0	0	0	1	0
I_2	0	0	0	0	1	0	1	1	1	1	0	0	0	1	0
	1	1	0	0	0	0	0	0	1	1	0	0	0	1	0
J_2	0	1	0	0	1	1	0	1	0	0	0	0	0	0	0
	0	1	0	0	0	0	0	0	0	1	0	0	1	1	0
	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1

FIGURE 9

Short-circuiting the recommended application of (9.4) we observe that the third row of M_2 is nodal, and acceptable as J_2. The columns having 1's in both I_2 and J_2 being equal, we carry out the next stage of the algorithm and obtain M_3 as the matrix of Figure 10.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
I_3	0	1	0	0	0	1	1	0	1	1	0	0	0	1	0
	1	1	0	0	0	0	0	0	1	1	0	0	0	1	0
	0	1	0	0	0	0	0	0	0	1	0	0	1	1	0
	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
J_3	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1

FIGURE 10

In this matrix the last row may be taken as J_3. Its complete nodal sequence, pinned by I_3, has two terms, the second consisting of the fourth row only. We now determine M_4 as the matrix of Figure 11.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
I_4	0	1	0	0	0	1	1	0	1	1	0	1	0	0	1
	1	1	0	0	0	0	0	0	1	1	0	1	0	0	1
	0	1	0	0	0	0	0	0	0	1	0	1	1	0	1
J_4	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1

FIGURE 11

The fourth row of this matrix can be taken as J_4. We then obtain M_5 as the matrix of Figure 12.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
I_5	0	1	0	0	0	1	1	0	1	1	1	1	0	0	0
	1	1	0	0	0	0	0	0	1	1	1	1	0	0	0
J_5	0	1	0	0	0	0	0	0	0	1	1	1	1	0	0

FIGURE 12

Here the third row is acceptable as J_5, and the sum of the second and third must be taken as J_6 in the last stage of the algorithm. We conclude that the matrices of Figures 2 and 8 are equivalent to the nodal matrix of Figure 13.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
I_1	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0
J_1	0	0	1	1	1	0	0	0	1	1	0	0	0	1	0
J_2	0	1	0	0	1	1	0	1	0	0	0	0	0	0	0
J_3	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1
J_4	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
J_5	0	1	0	0	0	0	0	0	0	1	1	1	1	0	0
J_6	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0

FIGURE 13

REFERENCE

1. S. Seshu and M. B. Reed, *Linear graphs and electrical networks* (Reading, Mass., 1961).

University of Waterloo,
Waterloo, Ontario