ON THE LOCI $|f(z)| = R$, $f(z)$ ENTIRE

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Introduction. The following result is found quite widely. Suppose $f(z)$ is a non-constant entire function such that $|f(z)| = 1$ along $|z| = 1$. Then, $f(z)$ has form cz^m, $|c| = 1$, $m \geqq 1$. See Ahlfors [**1**, p. 172, exercise 3], Dienes [**4**, p. 172, exercise 23], Hille [**6**, p. 317, exercise 2]. It is natural to inquire about a generalization of this result.

In particular, let $f(z)$ and $g(z)$ be non-constant entire functions. Suppose that C is a path in the finite plane along which $|f(z)| = |g(z)| = 1$. What then is the relation between $f(z)$ and $g(z)$?

Several results in this area are known. We thus have the following three theorems.

THEOREM 1. (Valiron [**12**], Cartwright [**2, 3**].) *Let C be a simple closed curve in the finite plane and let $f(z)$ and $g(z)$ be non-constant entire functions such that $|f(z)| = |g(z)| = 1$ along C. Then either there is some $\alpha > 0$ such that $|f(z)| \equiv |g(z)|^\alpha$ or else there exists an entire function $a(z)$ and some γ with $0 < |\gamma| < 1$, such that both $|f(z)|$ and $|g(z)|$ have the form*

$$|a(z)|^m \left| \frac{a(z)-\gamma}{1-\bar{\gamma}a(z)} \right|^n,$$

where m and n are non-negative integers.

THEOREM 2. (Cartwright [**3**].) *Let $f(z)$ and $g(z)$ be non-constant entire functions and let D be a simply-connected domain in the finite plane such that $|f(z)| > 1$, $|g(z)| > 1$ $(z \in D)$, and $|f(z)| = |g(z)| = 1$ $(z \in \partial D)$. Finally, suppose that $f(z)$ and $g(z)$ have finite order. Then there exists $\alpha > 0$ such that*

$$|f(z)| \equiv |g(z)|^\alpha.$$

THEOREM 3. (Cartwright [**3**].) *Let D be a domain in the finite plane such that ∂D is a simple path extending to ∞ in both directions. Let $f(z)$ and $g(z)$ be non-constant entire functions such that $|g(z)| > 1$, $|f(z)| < 1$ $(z \in D)$, $|f(z)| = |g(z)| = 1$ $(z \in \partial D)$, and such that $g(z)$ does not omit 0. Finally, suppose that $f(z)$ has finite order. Then there exist positive integers D and N, an entire function $a(z)$, and some γ with $|\gamma| > 1$, such that*

$$|f(z)| \equiv \left| \frac{a(z)-\gamma}{1-\bar{\gamma}a(z)} \right|^N, \qquad |g(z)| \equiv |a(z)|^D.$$

In this paper we will present a further development of Theorems 1, 2, and 3.

Preliminaries. For our development certain special preliminary results are necessary. For the sake of completeness and easy reference we list these explicitly at this point.

Result 1. Suppose that $u(z)$ is non-constant and harmonic over $|z| < R$. Let $u(0) = c$. Then there exists some δ, $0 < \delta < R$, such that for $|z| < \delta$ the locus $u(z) = c$ can be completely

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described as $\{z \mid z = re^{i\theta},\ -\delta < r < \delta,\ \theta = \theta_1(r), \ldots, \theta_m(r)\}$, where m is a suitable positive integer, where $\theta_j(r)$ is holomorphic over $|r| < \delta$ for each j, and where, for each r, all the $\theta_j(r)$ are distinct. We may assume that $\theta_{j+1}(0) - \theta_j(0) = \pi/m$ for $1 \leqq j \leqq m-1$. Thus, the tangents to the paths $\theta = \theta_j(r)$ at the origin are equally spaced.

Proof. We refer to Osgood [**10**, pp. 224–225] and use the implicit function theorem for analytic functions of several complex variables.

Remark. Let $f(z)$ be non-constant and entire. Application of Picard's theorem to $f(z)$ and of Result 1 (trivially modified) to $u(z) = \ln|f(z)|$ shows that the locus $|f(z)| = 1$ is non-empty and is locally analytic (to say the least).

Result 2 (Factorization). Take $f(z) \not\equiv 0$ to be holomorphic and uniformly bounded over $\mathscr{R} = \{z \mid \operatorname{Re}(z) > 0\}$. Let $\{a_n\}$ be the zeros of $f(z)$ in $\mathscr{R}$, listed by multiplicity. Then

$$\sum_n \frac{\operatorname{Re}(a_n)}{1+|a_n|^2} < \infty,$$

and $f(z) = E(z)B(z)$, where

$$B(z) = \prod_{|a_j|<1} \frac{z-a_j}{z+\bar{a}_j} \cdot \prod_{|a_k|\geqq 1} \frac{a_k - z}{\bar{a}_k + z} \cdot \frac{\bar{a}_k}{a_k}$$

and

$$E(z) = \exp\left\{-\int_{-\infty}^{\infty} \frac{d\alpha(t)}{z-it} + c\right\},$$

where c = constant, and $\alpha(t)$ is monotonic increasing on $(-\infty, +\infty)$. Both $B(z)$ and $E(z)$ are holomorphic and uniformly bounded on $\mathscr{R}$. $B(z)$ is called a Blaschke product.

Proof. The result is stated in Hille [**6**, pp. 457–458]. A proof can be given by transforming the proof in Hayman [**5**, p. 179, Theorem 6.13] over from the unit disk to the right half-plane. See also R. Nevanlinna [**9**, p. 201] or Hoffman [**7**, pp. 132–133].

Result 3 (Blaschke products). Let $\{a_n\}$ be any sequence of points in $\mathscr{R}$. Let

$$\sum_n \frac{\operatorname{Re}(a_n)}{1+|a_n|^2} < \infty.$$

Then the products

$$\prod_{|a_j|<1} \frac{z-a_j}{z+\bar{a}_j} \quad \text{and} \quad \prod_{|a_k|\geqq 1} \frac{a_k - z}{\bar{a}_k + z} \cdot \frac{\bar{a}_k}{a_k}$$

converge uniformly and absolutely on any compact subset of the finite plane which lies at a positive distance from the set $\{-\bar{a}_n\}$. Finally, if

$$B(z) = \prod_{|a_j|<1} \frac{z-a_j}{z+\bar{a}_j} \cdot \prod_{|a_k|\geqq 1} \frac{a_k - z}{\bar{a}_k + z} \cdot \frac{\bar{a}_k}{a_k},$$

then $B(z)$ is holomorphic over $\mathscr{R}$, $|B(z)| < 1$ over $\mathscr{R}$, and, for almost all y, $\lim_{x \to 0+} |B(x+iy)| = 1$.

Proof (partial).

$$\frac{z-a_j}{z+\bar{a}_j} = 1 - \frac{2\mathrm{Re}(a_j)}{z+\bar{a}_j}$$

and

$$\frac{a_k-z}{\bar{a}_k+z}\cdot\frac{\bar{a}_k}{a_k} = 1 - \frac{2z\,\mathrm{Re}(a_k)}{|a_k|^2+za_k},$$

along with the usual methods for infinite products yield the first part. For the second part we can refer to Hille [**6**, p. 457], or transforming from the unit disk to the right half-plane, R. Nevanlinna [**9**, p. 207] or Hayman [**5**, p. 182].

Result 4. Let $u(z)$ be non-negative and harmonic on $\mathcal{R}$. Let $u(z)$ be continuous on $\{z \mid \mathrm{Re}(z) \geqq 0\}$ with $u(iy) \equiv 0$ (y real). Then $u(z) \equiv c\,\mathrm{Re}(z)$ for some $c \geqq 0$.

Proof. See Tsuji [**11**, pp. 149–151] or Hoffman [**7**, p. 134, exercise 6]. B. Ja. Levin [**8**, p. 230] has an elementary proof.

Notation. In what follows we write X for the finite plane, X^* for the extended plane, U for the unit disk and $\mathcal{R}$ for the right half-plane.

Development of the Theorems.

DEFINITION. *$D \in \mathcal{A}$ if and only if* (i) *D is a domain in X;* (ii) *the image of ∂D on the Riemann sphere is a simple closed curve passing through the north pole.*

THEOREM 4. *Let $f(z)$, $g(z)$ be non-constant and entire. Let $|f(z)| > 1$, $|g(z)| > 1$, for $z \in D$, $D \in \mathcal{A}$, and let $|f(z)| = |g(z)| = 1$ for $z \in \partial D \cap X$. Then, for some real $c > 0$,*

$$|g(z)| \equiv |f(z)|^c \qquad (z \in X).$$

Proof. Let $\xi = \psi(z)$ be a one-to-one conformal mapping of D onto $\mathcal{R}$, which extends continuously to a one-to-one mapping of $\bar{D}$ onto $\bar{\mathcal{R}}$ [*closures in* X^*] in such a way that $\xi = \infty$ corresponds to $z = \infty$. (An auxiliary map of D onto U may be useful.) Let the inverse map be $z = \mu(\xi)$. Consider $\ln|f[\mu(\xi)]|$ and $\ln|g[\mu(\xi)]|$ for $\xi \in \mathcal{R}$. These are positive harmonic functions with continuous boundary value 0 along the imaginary axis. Therefore, by Result 4,

$$\ln|f[\mu(\xi)]| \equiv c_f\,\mathrm{Re}(\xi) \qquad \text{and} \qquad \ln|g[\mu(\xi)]| \equiv c_g\,\mathrm{Re}(\xi),$$

where $\xi \in \mathcal{R}$, $c_f > 0$, $c_g > 0$. By harmonic continuation and with $c = c_g/c_f$ we get our result.

Remark. c need not be rational in Theorem 4. For instance, $f(z) = e^z$ and $g(z) = e^{cz}$, $c > 0$. However, if $g(z) = 0$ at least once, comparing orders of zeros of $f(z)$ and $g(z)$ shows c to be rational.

DEFINITION. *Let $h(z)$ be a non-constant entire function. If $h(z) \neq 0$ for $z \in X$, set $\phi(h) = 0$. Otherwise, let $\{z_n\}$ be the countable set of distinct zeros of $h(z)$, and let $z = z_n$ have multiplicity m_n. Set $\phi(h) =$* g.c.d. $\{m_1, m_2, \ldots\}$.

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THEOREM 5. *Let $D \in \mathscr{A}$. Let $f(z), g(z)$ be non-constant and entire. Let $|f(z)|<1$ for $z \in D$, and $|f(z)|=1$ for $z \in \partial D \cap X$. Let $|g(z)|>1$ for $z \in D$, and $|g(z)|=1$ for $z \in \partial D \cap X$. Finally let $\phi(g)=1$. Then, for a suitable positive integer N and for a suitable γ with $|\gamma|>1$,*

$$|f(z)| \equiv \left|\frac{g(z)-\gamma}{1-\bar{\gamma} g(z)}\right|^N \quad \text{for} \quad z \in X.$$

Proof. Suppose first of all that $f(z) \neq 0$ in D. Then, $1/f(z)$ is holomorphic on D. Also, $|1/f(z)|>1$, $|g(z)|>1$ for $z \in D$. As in the proof of Theorem 4, we have $\xi=\psi(z)$, $z=\mu(\xi)$. Looking at positive harmonic functions $-\ln|f[\mu(\xi)]|$ and $\ln|g[\mu(\xi)]|$ for $\xi \in \mathscr{R}$, as in the proof of Theorem 4, shows at once that $|f(z)| \equiv|g(z)|^c$, where $c<0$ for $z \in X$. $\phi(g)=1$ yields an immediate contradiction. Hence, $f(z)$ has at least one zero in D.

Now with $\xi=\psi(z)$, $z=\mu(\xi)$ as above, form $f[\mu(\xi)]$ and $g[\mu(\xi)]$. Apply Result 4 to $\ln|g[\mu(\xi)]|$. Thus, let $\ln|g[\mu(\xi)]| \equiv c \operatorname{Re}(\xi)$, where $c>0$ for $\xi \in \mathscr{R}$. Now, $\xi=\psi(z)$ implies $\ln|g(z)| \equiv c \operatorname{Re} \psi(z)$ for $z \in D$. Select a single-valued branch of $\log g(z)$ for $z \in D$—call it $\operatorname{Log} g(z)$. Hence, $\operatorname{Log} g(z) \equiv c \psi(z)+i d$ (d real, $z \in D$). Certainly $c \psi(z)+i d$ has the same mapping properties as does $\xi=\psi(z)$. Without loss of generality, therefore, we can take $\operatorname{Log} g(z) \equiv \psi(z)$ for $z \in D$. Next, $|f[\mu(\xi)]|<1$, $\xi \in \mathscr{R}$. Apply Result 2. We obtain

$$f[\mu(\xi)]=E(\xi) B(\xi) \qquad (\xi \in \mathscr{R}).$$

Each set $\{\xi \mid |\xi|<r\} \cap \mathscr{R}$ contains only finitely many zeros of $f[\mu(\xi)]$, so that we may assume without loss of generality that $B(\xi)$ has the form

$$B(\xi)=\prod_k \frac{\xi_k-\xi}{\bar{\xi}_k+\xi} \frac{\bar{\xi}_k}{\xi_k}.$$

Now, $E(\xi)$ will be bounded on $\mathscr{R}$. In fact, $|E(\xi)| \leqq 1$ for $\xi \in \mathscr{R}$. For, consider boundary values along the imaginary axis. $|f[\mu(\xi)]|=1$ for ξ purely imaginary. $|B(\xi)|=1$ almost everywhere for purely imaginary ξ. This implies that $|E(\xi)|=1$ almost everywhere for purely imaginary ξ. By means of Poisson integral representations (for the half-plane) [cf. Result 2 and Hille [6, p. 445]] we get $|E(\xi)| \leqq 1$ for $\xi \in \mathscr{R}$.

We apply Result 4 to $-\ln|E(\xi)|$. Result 3 and the remark about $\{\xi \mid |\xi|<r\} \cap \mathscr{R}$ above show $|B(\xi)|$ to be continuous on $\overline{\mathscr{R}} \cap X$, whence $-\ln|E(\xi)|$ is continuous on $\overline{\mathscr{R}} \cap X$. Hence, for $a \geqq 0$,

$$\ln|E(\xi)| \equiv-a \operatorname{Re}(\xi) \qquad \text{for} \qquad \xi \in \mathscr{R}.$$

And

$$\begin{aligned} |E(\xi)| &\equiv|g[\mu(\xi)]|^{-a} && (\xi \in \mathscr{R}), \\ |E[\psi(z)]| &\equiv|g(z)|^{-a} && (z \in D). \end{aligned}$$

It follows that for $z \in D$

$$|f(z)| \equiv|g(z)|^{-a} \cdot\left|\prod_k \frac{\xi_k-\operatorname{Log} g(z)}{\bar{\xi}_k+\operatorname{Log} g(z)} \frac{\bar{\xi}_k}{\xi_k}\right|. \tag{*}$$

We must now study continuations of (*) outside of D. Note that we use Result 3 for analytic continuation of the Blaschke product. First choose any $z_0 \notin D$, $g(z_0)=0$. Select

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$z_1 \in D$ so that $g(z) \neq 0$ along $\overrightarrow{z_1 z_0}$ except for $z = z_0$. This is possible since the zeros of $g(z)$ are countable. Continue $\operatorname{Log} g(z)$ along $\overrightarrow{z_1 z_0}$. Let $\operatorname{Log}^* g(z)$ thus be defined. Continue both sides of (*) along $\overrightarrow{z_1 z_0}$. It is apparent that unless $|f(z)| = \infty$ somewhere along $\overrightarrow{z_1 z_0}$, $\operatorname{Log}^* g(z) \neq -\bar{\xi}_k$ along $\overrightarrow{z_1 z_0}$. But then, as $z \to z_0$ along $\overrightarrow{z_1 z_0}$, $|B[\operatorname{Log}^* g(z)]| \geqq 1$ and $|g(z)|^{-a} \to \infty$ unless $a = 0$. Thus $a > 0$ implies $|f(z_0)| = \infty$, which is a contradiction. Hence $a = 0$ and $|E(\xi)| \equiv 1$.

$$|f(z)| \equiv \left| \prod_k \frac{\xi_k - \operatorname{Log} g(z)}{\bar{\xi}_k + \operatorname{Log} g(z)} \cdot \frac{\bar{\xi}_k}{\xi_k} \right| \quad \text{for} \quad z \in D. \tag{**}$$

Lemma 1. *For $g(z)$ with $\phi(g) = 1$, the general analytic function* $\log g(z)$ *is monogenic.*

Proof (informal). The terminology is that of Osgood [**10**, pp. 174–175]. We must show that if $w' = \log g(z')$ and $w'' = \log g(z'')$, then there exists a path Γ in the finite plane going from z' to z'' such that $w = \log g(z)$ can be continued along Γ from (z', w') to (z'', w''). $1 = \text{g.c.d.}\,\{m_1, m_2, \ldots\}$. Hence, for some large s, $1 = \text{g.c.d.}\,\{m_1, m_2, \ldots, m_s\}$. By the usual algebraic considerations, there are integers $e_1, \ldots, e_s$ such that $1 = e_1 m_1 + \ldots + e_s m_s$. Let m_j correspond to the zero z_j of $g(z)$. Draw any path Γ_0 in X from z' to z'' passing very near to each z_j $(1 \leqq j \leqq m)$, and along which $g(z) \neq 0$. Deform Γ_0 slightly so that the new path Γ circulates around each $z = z_j$, $1 \leqq j \leqq m$, ke_j times counterclockwise, where $k =$ a suitable integer. It is now easily verified that (z'', w'') is accessible from (z', w') along Γ (for suitable choice of k).

Now choose any $z_0 \in D$. Choose any path Γ in X starting at $z = z_0$ along which $g(z) \neq 0$. Of course, (**) holds for $z \in D$ so that

$$|f(z)| \equiv \left| \prod_k \frac{\xi_k - \operatorname{Log} g(z)}{\bar{\xi}_k + \operatorname{Log} g(z)} \frac{\bar{\xi}_k}{\xi_k} \right|.$$

We can certainly continue the left-side of (**) along Γ. Hence the same holds for the right-hand side. Since $|f(z)| < \infty$ for $z \in X$, the continuation of $\operatorname{Log} g(z)$ must always avoid the values $\{-\bar{\xi}_k\}$. But $\log g(z)$ is monogenic. Varying Γ suitably shows easily that $g(z) \neq e^{-\bar{\xi}_k}$ for all k. Thus, the monogeneity of $\log g(z)$ implies that

$$|f(z)| \equiv \left| \prod_k \frac{\xi_k - \log g(z)}{\bar{\xi}_k + \log g(z)} \frac{\bar{\xi}_k}{\xi_k} \right|, \tag{***}$$

for $z \in X$ and any branch of $\log g(z)$. The usual techniques inform us that

$$B(\xi + 2\pi i) \equiv B(\xi).$$

Looking at the zeros of $B(\xi)$ tells us that $B(\xi_k + 2h\pi i) = 0$, where h is integral for any ξ_k. We apply Picard's theorem to $g(z) \neq e^{-\bar{\xi}_k}$; $g(z)$ can omit at most one finite value. From these considerations, it follows at once that we can take $\xi_k = \xi_0 + 2k\pi i$ $(-\infty < k < \infty)$, where

$\xi_0 \in \mathscr{R}$ is suitably chosen, *each with multiplicity N*. Therefore $z \in D$ implies that

$$|f(z)| \equiv \left| \prod_{k=-\infty}^{+\infty} \frac{\xi_0 + 2k\pi i - \operatorname{Log} g(z)}{\bar{\xi}_0 - 2k\pi i + \operatorname{Log} g(z)} \cdot \frac{\bar{\xi}_0 - 2k\pi i}{\xi_0 + 2k\pi i} \right|^N .$$

LEMMA 2. *Let* $\xi_k = A + 2k\pi i$ $(-\infty < k < \infty,\ A \in \mathscr{R})$. *Then*

$$\left| \frac{e^z - e^A}{1 - e^{\bar{A}} e^z} \right| \equiv \left| \prod_{k=-\infty}^{+\infty} \frac{\xi_k - z}{\bar{\xi}_k + z} \frac{\bar{\xi}_k}{\xi_k} \right| \quad \textit{for} \quad z \in X.$$

Proof. Let $f(z) = e^z + e^{-\bar{A}}$ and

$$g(z) = \frac{f(z) - e^A}{1 - e^{\bar{A}} f(z)}.$$

It is easily seen that $f(z), g(z)$ are non-constant and entire with $|f(z)| < 1$ if and only if $|g(z)| > 1$. A simple calculation shows that $|f(z)| = 1$ divides X into two disjoint simply-connected class $\mathscr{A}$ regions. Repeat the above proof on $f(z)$ and $g(z)$, $D = \{z \mid |g(z)| < 1\}$. We find easily that

$$|g(z)| \equiv \left| \prod_{k=-\infty}^{+\infty} \frac{\xi_k - \operatorname{Log} f(z)}{\bar{\xi}_k + \operatorname{Log} f(z)} \frac{\bar{\xi}_k}{\xi_k} \right| \qquad (N = 1)$$

for $z \in D$. But $\operatorname{Log} f(z)$ is an open mapping on D, so that

$$\left| \frac{e^z - e^A}{1 - e^{\bar{A}} e^z} \right| \equiv \left| \prod_{k=-\infty}^{+\infty} \frac{\xi_k - z}{\bar{\xi}_k + z} \frac{\bar{\xi}_k}{\xi_k} \right| \quad \text{for} \quad z \in X.$$

Returning to the proof of the theorem, we deduce from Lemma 2 that

$$|f(z)| \equiv \left| \frac{g(z) - e^{\xi_0}}{1 - e^{\bar{\xi}_0} g(z)} \right|^N$$

for $z \in X$, $\xi_0 \in \mathscr{R}$; $\gamma = e^{\bar{\xi}_0}$ so that $|\gamma| > 1$.

THEOREM 6. *Let* $D \in \mathscr{A}$. *Let* $f(z)$ *and* $g(z)$ *be non-constant and entire. Let* $|f(z)| < 1$ *for* $z \in D$ *and* $|f(z)| = 1$ *for* $z \in \partial D \cap X$. *Let* $|g(z)| > 1$ *for* $z \in D$ *and* $|g(z)| = 1$ *for* $z \in \partial D \cap X$. *Let* $\phi(g) = D \geqq 1$. *Then, for a suitable positive integer N and some* γ *with* $|\gamma| > 1$,

$$|f(z)| \equiv \left| \frac{[g(z)]^{1/D} - \gamma}{1 - \bar{\gamma}[g(z)]^{1/D}} \right|^N \qquad (z \in X)$$

for a suitable single-valued (entire) branch of $g(z)^{1/D}$.

Proof. $\phi(g) = D$ implies that we can find an entire function $h(z)$ with $h(z)^D = g(z)$, $\phi(h) = 1$. (Monodromy theorem.) Apply Theorem 5 to $f(z)$ and $h(z)$.

We now come to a number of important counterexamples. It is convenient to make the following definition.

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DEFINITION. *Let $A(z)$, $B(z)$ be non-constant and entire. $A(z)$ and $B(z)$ are said to be algebraically related if and only if there exists some non-trivial complex polynomial $P(z,w)$ in (z,w) such that $P[A(z),B(z)]\equiv 0$ for $z\in X$.*

Example 1. Let $A(z)=e^z+1$. Let $B(z)=\exp\{(A(z)+1)/(A(z)-1)\}$. Clearly $B(z)$ is entire. A simple calculation shows that $\{z \mid |A(z)|<1\}$ consists of infinitely many simply-connected class $\mathscr{A}$ components. Choose one of them—call it D. Clearly $|B(z)|=1$ if and only if $|A(z)|=1$ ($A(z)\neq 1$), and $|A(z)|<1$ if and only if $|B(z)|<1$. Let $C(z)=1/B(z)$. $C(z)$ is entire. Here $|A(z)|<1$ if and only if $|C(z)|>1$, etc. Note that neither A and B nor A and C are algebraically related, because of the exponential factor. [$\exp\{(z+1)/(z-1)\}$ is not an algebraic function.]

THEOREM 7. *In Theorem* 6, *if $\phi(g)=0$, then f and g need not even be algebraically related.*

Proof. Take $f(z)=A(z)$, $g(z)=C(z)$, and D as above.

THEOREM 8. *Let $D\in\mathscr{A}$. Let $f(z)$, $g(z)$ be non-constant and entire. Let $|f(z)|<1$ and $|g(z)|<1$ for $z\in D$, and $|f(z)|=|g(z)|=1$ for $z\in\partial D\cap X$. Then $f(z)$ and $g(z)$ need not be algebraically related.*

Proof. Take $f(z)=A(z)$, $g(z)=B(z)$, D as above.

Remark. A bit more work shows that, for example, there is no non-constant entire function $w(z)$ such that both $|A(z)|$ and $|B(z)|$ have the form

$$|w(z)|^{\alpha}\left|\frac{w(z)-\gamma}{1-\bar{\gamma}w(z)}\right|^{\beta}.$$

where $|\gamma|\neq 0,1$, $(\alpha,\beta)\neq(0,0)$.

Example 2. Take $f(z)=e^{2\cosh z}$, $g(z)=e^{e^z}$.

Then $|f(z)|=1\Leftrightarrow \operatorname{Re}\{2\cosh z\}=0 \Leftrightarrow \operatorname{Re}[e^z+e^{-z}]=0 \Leftrightarrow e^x\cos y+e^{-x}\cos y=0 \Leftrightarrow \cos y=0 \Leftrightarrow y=(2k+1)\pi/2$ (k integral). Next, $|g(z)|=1 \Leftrightarrow \operatorname{Re}\{e^z\}=0\Leftrightarrow e^x\cos y=0 \Leftrightarrow \cos y=0 \Leftrightarrow y=(2k+1)\pi/2$. Clearly, $|f(z)|<1 \Leftrightarrow |g(z)|<1$, and similarly for >1. It is readily checked that $f(z)$ and $g(z)$ are not algebraically related. Thus, it is in general necessary for ∂D to be a simple closed curve on the sphere in order for our theorems to hold.

Example 3. Take $f(z)=e^{e^{e^z}}$, $g(z)=e^{e^{-e^z}}$.

Then $|f(z)|=1\Leftrightarrow \operatorname{Re}\{e^{e^z}\}=0\Leftrightarrow \operatorname{Re}\{e^{e^x\cos y+ie^x\sin y}\}=0\Leftrightarrow e^{e^x\cos y}\cos(e^x\sin y)=0\Leftrightarrow e^x\sin y=(2k+1)\pi/2$. $|g(z)|=1 \Leftrightarrow \operatorname{Re}\{e^{-e^z}\}=0 \Leftrightarrow \operatorname{Re}\{e^{-e^x\cos y-ie^x\sin y}\}=0 \Leftrightarrow e^x\sin y=(2k+1)\pi/2$ as above. And here we see that $f(z)$ and $g(z)$ are not algebraically related, and how complicated the locus $|f(z)|=|g(z)|=1$ can be.

THEOREM 9. *Let C be a closed path in the finite plane continuously differentiable with respect to arc length. Let S be an open subarc of C. Let $f(z)$, $g(z)$ be non-constant and entire such that $|f(z)|=1$ for $z\in C$, and $|g(z)|=1$, for $z\in S$. Then, the conclusion of Theorem* 1 *holds.*

Proof. Select $z_0 \in S$ so that $f'(z_0)$ and $g'(z_0)$ are nonzero. By hypothesis, and by Result 1, we can choose $z = z(s)$ continuously differentiable relative to arc length s such that

(i) $C = \{z \mid z = z(s), 0 \leqq s \leqq L\}$;

(ii) $z(0) = z(L) = z_0$;

(iii) As s increases, $z = z(s)$ never retraces itself.

Near $s = 0$ and $s = L$ it is readily verified that $|g[z(s)]| \equiv 1$. Let

$$A = \{s \mid 0 \leqq s \leqq L, \ |g[z(s)]| \neq 1\}.$$

Suppose that A is non-empty. Let $\eta = \inf A$. Clearly, $0 < \eta < L$. Application of Result 1 and the continuity of the tangent vector $z'(s)$ yields an immediate contradiction to the choice of the infimum. Hence A is empty, whence $|g[z(s)]| \equiv 1$ $(0 \leqq s \leqq L)$. Now apply Theorem 1 to some simple closed component oval of C.

COROLLARY. *Let $f(z)$ be a non-constant polynomial. Let $g(z)$ be non-constant and entire. Let $|g(z)| = 1$ along an open subarc of $|f(z)| = 1$. Then $g(z)$ is a polynomial and there exist positive integers m and n such that*

$$|f(z)|^m \equiv |g(z)|^n.$$

Proof. Let C be the component of $|f(z)| = 1$ containing the given open subarc. Use Result 1 here and apply Picard's theorem to the result of Theorem 1.

Two counterexamples related to Theorem 9 merit attention here.

Example 4. The choice

$$f(z) = e^z, \qquad g(z) = e^{z^3}$$

shows that a straightforward generalization of Theorem 9 to the " unbounded " case is not possible.

Example 5. The choice

$$f(z) = e^z + \tfrac{1}{2}, \qquad g(z) = f(z)\frac{f(z)-2}{1-2f(z)}$$

(with an easy calculation) shows that in Theorem 9 it is essential that (in terms of arc length) C be continuously differentiable. Note that in this example $\{z \mid |f(z)| = 1\}$ is a proper subset of $\{z \mid |g(z)| = 1\}$.

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