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RESEARCH ARTICLE

Level correspondence of the K-theoretic I-function in Grassmann duality

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Abstract
In this paper, we prove a series of identities of the quasi-map K-theoretical I-functions with level structure between the Grassmannian and its dual Grassmannian. Those identities prove the quantum K-theory version mutation conjecture stated in [13]. Here we find an interval of levels within which two I-functions are the same, and on the boundary of that interval, two I-functions intertwine. We call this phenomenon the level correspondence in Grassmann duality.

Contents

1. Introduction

The quantum K-theory was introduced by Givental [6] and Lee [9] decades ago. Recently, Givental shows that q-hypergeometric solutions represent K-theoretic Gromov-Witten invariants in the toric case [5] and Ruan-Zhang [14] introduce the level structures in quantum K-theory. There is a serendipitous discovery that some special toric spaces with certain level structures result in Mock theta functions. Nevertheless, beyond the toric case, much less is known.

The recent explosion of study of the quantum K-theory was from a fundamental relation between 3d supersymmetric gauge theories and quantum K-theory of the so-called Higgs branch discovered by the works of Nekrasov [12], and Nekrasov and Shatashvili [10] [11], amongst many others. For the concrete case of massless theories with a nontrivial UV-IR flow, Jockers and Mayr [7] show a 3d gauge theory/quantum K-theory correspondence, connecting the BPS partition functions of specific $\mathcal{N}=2$ supersymmetric gauge theories to Givental's permutation equivariant K-theory. In addition, Jockers et al. [8] and Ueda-Yoshida [18] establish the correspondence between 3d gauge theory and the quantum K-theory of $Gr(r,n)$ independently. Now it is well-understood that the level structures introduced by Ruan-Zhang [14] are the key new feature for the so-called 3d $\mathcal{N}=2$ theory (Chern-Simons term).

One of the key features of gauge theory is Seiberg-duality, which has been studied in 2d by Bonelli et al. [1] and the first author. The 2d Seiberg-duality has a mathematical version known as mutation conjecture [13]. As far as the authors know very little is known in the 3d $\mathcal{N}=2$ case. The results of this article hopefully will contribute some clarity. The simplest example of the mutation conjecture is the Grassmannian $Gr(r,V)$ versus dual Grassmannian $Gr(n-r,V^*)$. However, it is unknown how to match the level structure. Without misunderstanding, we will use $Gr(r,n)$ and $Gr(n-r,n)$ to denote the Grassmannian and its dual, respectively. They are geometrically isomorphic. However, they encode very different combinatorial data. A long-standing problem is matching their combinatorial data directly. For example, the presentations of K-theoretic I-functions depend on their gauge theory representation/combinatorial data, and it is tough to see why the I-function of the Grassmannian equals the I-function of the dual Grassmannian. In this paper, we give the explicit formula of K-theoretic I-function of the Grassmannian with level structure by using abelian/nonabelian correspondence [20] as follows:

$$I_{T,d}^{Gr(r,n),E_r,l}=\sum_{d_1+d_2+\cdots+d_r=d}Q^d\prod_{i,j=1}^{r}\frac{\prod_{k=-\infty}^{d_i-d_j}(1-q^kL_iL_j^{-1})}{\prod_{k=-\infty}^{0}(1-q^kL_iL_j^{-1})}\prod_{i=1}^{r}\frac{(L_i^{d_i}q^{\frac{d_i(d_i-1)}{2}})^l}{\prod_{k=1}^{d_i}\prod_{m=1}^{n}(1-q^kL_i\Lambda_m^{-1})},$$

and

$$I_{T,d}^{Gr(n-r,n),E_{n-r},l}=\sum_{d_1+d_2+\cdots+d_{n-r}=d}Q^d\prod_{i,j=1}^{n-r}\frac{\prod_{k=-\infty}^{d_i-d_j}(1-q^k\tilde{L}_i\tilde{L}_j^{-1})}{\prod_{k=-\infty}^{0}(1-q^k\tilde{L}_i\tilde{L}_j^{-1})}\prod_{i=1}^{n-r}\frac{(\tilde{L}_i^{d_i}q^{\frac{d_i(d_i-1)}{2}})^l}{\prod_{k=1}^{d_i}\prod_{m=1}^{n}(1-q^k\tilde{L}_i\Lambda_m)}.$$

We want to remark here that the isomorphism between the Grassmannian and its dual would imply the equivalence of J-function when level l is 0. In fact, the I-function is known to be different from the J-function with negative levels.

In this paper, we use Theorem 1.2 to show the relations of the equivariant I-function between the Grassmannian $Gr(r,n)$ and that of the dual Grassmannian $Gr(n-r,n)$ with level structures; here we find an interval of levels within which two I-functions with levels are the same. On the boundary of that interval, two I-functions with levels are intertwining with each other. We call this phenomenon the level correspondence in Grassmann duality. The existence of a specific interval of level is very mysterious to us. We hope that our result will give some hints on formulating Seiberg-duality for a general target.

Theorem 1.1 (Level correspondence). *For the Grassmannian $Gr(r,n)$ and its dual Grassmannian $Gr(n-r,n)$ with standard $T=(\mathbb{C}^*)^n$ torus action, let E_r, E_{n-r} be the standard representation of* $\mathrm{GL}(r,\mathbb{C})$ *and* $\mathrm{GL}(n-r,\mathbb{C})$*, respectively. Consider the following equivariant I-function:*

$$\begin{aligned}I_T^{Gr(r,n),E_r,l}&=1+\sum_{d=1}^{\infty}I_{T,d}^{Gr(r,n),E_r,l}Q^d,\\ I_T^{Gr(n-r,n),E_{n-r}^\vee,-l}&=1+\sum_{d=1}^{\infty}I_{T,d}^{Gr(n-r,n),E_{n-r}^\vee,-l}Q^d.\end{aligned}$$

Then we have the following relations between $I_{T,d}^{Gr(r,n),E_r,l}$ and $I_T^{Gr(n-r,n),E_{n-r}^\vee,-l}$ in $K_T^{loc}(Gr(r,n))\otimes\mathbb{C}(q)\cong K_T^{loc}(Gr(n-r,n))\otimes\mathbb{C}(q)$:

- *For $1-r \le l \le n-r-1$, we have*

$$I_{T,d}^{Gr(r,n),E_r,l} = I_{T,d}^{Gr(n-r,n),E_{n-r}^\vee,-l}.$$

- *For $l = n-r$, we have*

$$I_{T,d}^{Gr(r,n),E_r,l} = \sum_{s=0}^{d} C_s(n-r,d) I_{T,d-s}^{Gr(n-r,n),E_{n-r}^\vee,-l},$$

where $C_s(k,d)$ is defined as

$$C_s(k,d) = \frac{(-1)^{ks}}{(q;q)_s q^{s(d-s+k)} \left(\bigwedge^{top} \mathcal{S}_{n-r}\right)^s},$$

and $\mathcal{S}_{n-r}$ is the tautological bundle of $Gr(n-r,n)$.

- *For $l = -r$, we have*

$$I_{T,d}^{Gr(n-r,n),E_{n-r}^\vee,-l} = \sum_{s=0}^{d} D_s(r,d) I_{T,d-s}^{Gr(r,n),E_r,l},$$

where

$$D_s(r,d) = \frac{(-1)^{rs}}{(q;q)_s q^{s(d-s)} \left(\bigwedge^{top} \mathcal{S}_r\right)^s},$$

and $\mathcal{S}_r$ is the tautological bundle of $Gr(r,n)$.

Here we use q-Pochhammer symbol notation:

$$(a;q)_d := \begin{cases} (1-a)(1-qa)\cdots(1-q^{d-1}a) & d>0 \\ 1 & d=0 \\ \frac{1}{(1-q^{-1}a)\cdots(1-q^{-d}a)} & d<0 \end{cases}.$$

A key step in our proof is the following series of nontrivial q-Pochhammer symbol identities, which are of independent interest.

Theorem 1.2. *Denoted by $[n]$, the set of elements $\{1,\dots,n\}$, let $\emptyset \neq I \subsetneq [n]$ be a subset of $[n]$, $|I|$ be its cardinality and denoted by $I^{\complement}$, the complementary set of I in $[n]$. For constant positive integers d, n and l such that $1-|I| \le l \le n-|I|-1$, let $A_d(\vec{x},I,l)$ and $B_d(\vec{x},I,l)$ be two rational functions in $\vec{x}$ and q with an extra data l*

$$A_d(\vec{x},I,l) = \sum_{|d_I|=d} \frac{\left(\prod_{i\in I} x_i^{d_i} q^{\frac{d_i(d_i-1)}{2}}\right)^l}{\prod_{i,j\in I}\left(q^{d_{ij}+1}x_{ij};q\right)_{dj} \prod_{i\in I}\prod_{j\in I^{\complement}}(qx_{ij};q)_{d_i}},$$

$$B_d(\vec{x},I,l) = \sum_{|\vec{d}_I|=d} \frac{\left(\prod_{i\in I} x_i^{-d_i} q^{\frac{d_i(d_i+1)}{2}}\right)^l}{\prod_{i,j\in I}\left(q^{d_{ij}+1}x_{ji};q\right)_{dj} \prod_{i\in I}\prod_{j\in I^{\complement}}(qx_{ji};q)_{d_i}},$$

where $\vec{d}_I$ is $|I|$-tuple of non negative integers and $|\vec{d}_I| := \sum_{i\in I} d_i$. $x_i, i=1,\dots,n$ are parameters. For convenience, we use the notation $x_{ij} := x_i/x_j$ and $d_{ij} := d_i - d_j$. Then we have

$$A_d(\vec{x},I,l) = B_d\left(\vec{x},I^{\complement},-l\right).$$

Plan of the paper

This paper is arranged as follows. In Subsection 2.1, we prove Theorem 1.2 by constructing the rational function in equation (2.3) and then using the iterated residue method, which is useful in Nekrasov partition function [4]. In the following Subsection 2.2, we provide two explicit examples to explain the proof and also provide a nontrivial identity by using Theorem 1.2. In Subsection 2.3, we expand the restriction to the boundary: that is, $l = -|I|$ and $l = n - |I|$. In Section 3, we first revisit the K-theoretic quasi-map theory in which we review some basic definitions and theorems, especially the formula of equivariant I-function of the Grassmannian $Gr(r, n)$. Finally, we apply Theorem 1.2 to obtain the level correspondence of the I-function in Grassmann duality.

2. The class of q-Pochhammer symbol identities

2.1. The proof of identities

Now we prove Theorem 1.2 for one case $I = \{1, \cdots, r\}$ by constructing the following symmetric complex rational function $f(w_1, \cdots, w_d)$ with parameters q and $x_1, \cdots, x_n$. We made the following assumptions for parameters:

$$\begin{aligned} &|q| < 1, \\ x_i x_j^{-1} \neq q^k, \qquad &\forall i \neq j \in [n], \forall k \in \mathbb{Z}. \end{aligned} \tag{2.1}$$

Furthermore, there exists some $\rho > 0$ such that

$$\max_{i \in [n]} |x_i| < \rho < \min_{i \in [n]} |q|^{-1} |x_i|, \tag{2.2}$$

where $[n] := \{1, \cdots, n\}$ and general situations follow from analytic continuation. Let $f(w_1, \cdots, w_d)$ be as follows:

$$f(w_1, \cdots, w_d) = \frac{1}{(1-q)^d d!} \prod_{i \neq j}^{d} \frac{w_i - w_j}{w_i - q w_j} \prod_{i=1}^{d} \frac{w_i^{l-1}}{\prod_{j=1}^{r} (1 - x_j / w_i) \prod_{j=r+1}^{n} (1 - q w_i / x_j)} \tag{2.3}$$

$$= g(w_1, \cdots, w_d) \prod_{i=1}^{d} \left(\prod_{u \in U} \frac{w_i - q^{-1} u}{w_i - u} \prod_{i<j} \frac{(w_i - w_j)^2}{(w_i - q w_j)(q w_i - w_j)} \right), \tag{2.4}$$

where U is a set of complex numbers all contained in open disk $|w| < \rho$, at the moment $U = \{x_1, \cdots, x_r\}$, and g is a symmetric function of the form

$$g(\vec{w}) = \frac{1}{(1-q)^d d!} \prod_{i=1}^{d} \frac{w_i^{l+r-1}}{\prod_{j=1}^{r} (w_i - q^{-1} x_j) \prod_{j=r+1}^{n} (1 - q w_i / x_j)}.$$

From the condition in inequality (2.2) and the restriction of l, we know $g(\vec{w})$ is analytical in the polydiscs $\{(w_1, \cdots, w_n) : |w_i| \leq \rho, \forall i \in [n]\}$ and g can only have possible zeros for some $w_j = 0$.

We consider the following integration

$$E_d := \int_{C_\rho} \frac{dw_d}{2\pi\sqrt{-1}} \cdots \int_{C_\rho} \frac{dw_1}{2\pi\sqrt{-1}} f(\hat{w}_1, \cdots, \hat{w}_d), \tag{2.5}$$

where $(\hat{w}_1, \cdots, \hat{w}_d)$ is any arrangement of $\{w_1, \cdots, w_d\}$ and the integration contour C_ρ for each variable w_i is the circle centred at origin with radius ρ and takes a counterclockwise direction. The condition in inequality (2.2) ensures that there isn't a pole on the integration contour. By Fubini's theorem, we could permute the order of integration variables; and since $f(w_1,\cdots,w_d)$ is a symmetric function, we can change $(w_1,\cdots,w_d)$ to another order, such as $(\hat{w}_1,\cdots,\hat{w}_d)$.

Suppose we have the following evaluating sequence for some $S_1 \le d$ by induction,

$$\hat{w}_1 = q\hat{w}_2, \hat{w}_2 = q\hat{w}_3, \cdots, \hat{w}_{S_1-1} = q\hat{w}_{S_1},$$

which are all simple poles inside $|w| < \rho$. Then we have

$$\begin{aligned}
&\operatorname*{Res}_{\hat{w}_{S_1-1}=q\hat{w}_{S_1}} \cdots \operatorname*{Res}_{\hat{w}_2=q\hat{w}_3} \operatorname*{Res}_{\hat{w}_1=q\hat{w}_2} f \\
&= \prod_{i=S_1+1}^{d} \left(\prod_{u\in U} \frac{\hat{w}_i - q^{-1}u}{\hat{w}_i - u} \prod_{i<j} \frac{(\hat{w}_i-\hat{w}_j)^2}{(\hat{w}_i - q\hat{w}_j)(q\hat{w}_i - \hat{w}_j)} \right) \\
&\quad \times \hat{w}_{S_1}^{S_1-1} \prod_{k=0}^{S_1-1} \prod_{u\in U} \frac{q^k\hat{w}_{S_1} - q^{-1}u}{q^k\hat{w}_{S_1} - u} \cdot \prod_{S_1<j} \frac{(\hat{w}_{S_1}-\hat{w}_j)(q^{S_1-1}\hat{w}_{S_1}-\hat{w}_j)}{(\hat{w}_{S_1}-q\hat{w}_j)(q^{S_1}\hat{w}_{S_1}-\hat{w}_j)} \\
&\quad \times \frac{(q-1)^{S_1}}{q^{S_1}-1} q^{-(S_1-1)(d-S_1)} g(q^{S_1-1}\hat{w}_{S_1}, q^{S_1-2}\hat{w}_{S_1}, \cdots, \hat{w}_{S_1}, \hat{w}_{S_1+1}, \cdots, \hat{w}_d).
\end{aligned} \tag{2.6}$$

Now, integrating variable $\hat{w}_{S_1}$, we pick up the residue as $\hat{w}_{S_1} = q^{-k_1}u_1$ for some $0 \le k_1 < S_1$ and $u_1 \in U = \{x_1,\cdots,x_r\}$ because the condition in equation (2.2), $|\hat{w}_{S_1}| < \rho$, implies that $k_1 = 0$. Evaluating $\hat{w}_{S_1} = u_1$, we obtain

$$\begin{aligned}
&\operatorname*{Res}_{\hat{w}_{S_1}=u_1} \operatorname*{Res}_{\hat{w}_{S_1-1}=q\hat{w}_{S_1}} \cdots \operatorname*{Res}_{\hat{w}_2=q\hat{w}_3} \operatorname*{Res}_{\hat{w}_1=q\hat{w}_2} f \\
&= \prod_{i=S_1+1}^{d} \left(\frac{\hat{w}_i - q^{S_1-1}u_1}{\hat{w}_i - q^{S_1}u_1} \prod_{u\in U\setminus\{u_1\}} \frac{\hat{w}_i - q^{-1}u}{\hat{w}_i - u} \prod_{i<j} \frac{(\hat{w}_i-\hat{w}_j)^2}{(\hat{w}_i - q\hat{w}_j)(q\hat{w}_i - \hat{w}_j)} \right) \\
&\quad \times u_1^{S_1} \prod_{k=0}^{S_1-1} \prod_{u\in U\setminus\{u_1\}} \frac{q^k u_1 - q^{-1}u}{q^k u_1 - u} \cdot (q-1)^{S_1} q^{S_1(S_1-1-d)} \\
&\quad \times g(q^{S_1-1}u_1, q^{S_1-2}u_1, \cdots, qu_1, u_1, \hat{w}_{S_1+1}, \cdots, \hat{w}_d)
\end{aligned} \tag{2.7}$$

$$= \tilde{g}(\hat{w}_{S_1+1},\cdots,\hat{w}_d) \prod_{i=S_1+1}^{d} \left(\prod_{u\in\tilde{U}} \frac{\hat{w}_i - q^{-1}u}{\hat{w}_i - u} \prod_{i<j} \frac{(\hat{w}_i-\hat{w}_j)^2}{(\hat{w}_i - q\hat{w}_j)(q\hat{w}_i - \hat{w}_j)} \right), \tag{2.8}$$

where

$$\tilde{U} = U\setminus\{u_1\} \cup \{q^{S_1}u_1\}. \tag{2.9}$$

All elements of $\tilde{U}$ are still in the open disk $|w| < \rho$, and

$$\begin{aligned}
\tilde{g}(\hat{w}_{S_1+1},\cdots,\hat{w}_d) = u_1^{S_1} \prod_{k=0}^{S_1-1} \prod_{u\in U\setminus\{u_1\}} \frac{q^k u_1 - q^{-1}u}{q^k u_1 - u} \cdot (q-1)^{S_1} q^{S_1(S_1-1-d)} \\
\times g(q^{S_1-1}u_1, q^{S_1-2}u_1, \cdots, qu_1, u_1, \hat{w}_{S_1+1}, \cdots, \hat{w}_d).
\end{aligned} \tag{2.10}$$

So we just write $\tilde{f} := \operatorname*{Res}_{\hat{w}_{S_1}=u} \operatorname*{Res}_{\hat{w}_{S_1-1}=q\hat{w}_{S_1}} \cdots \operatorname*{Res}_{\hat{w}_2=q\hat{w}_3} \operatorname*{Res}_{\hat{w}_1=q\hat{w}_2} f$ in the same pattern as in the original form in equation (2.4). One could check that setting $S_1 = 1$ in equation (2.7) is valid.

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If one takes the following evaluation sequence of simple poles by induction

$$\hat{w}_1 = u_1, \hat{w}_2 = qu_1, \cdots, \hat{w}_{S_1-1} = q^{S_1-2}u_1, \hat{w}_{S_1} = q^{S_1-1}u_1, \tag{2.11}$$

we get

$$\begin{aligned}
&\operatorname*{Res}_{\hat{w}_{S_1}=q^{S_1-1}u_1} \cdots \operatorname*{Res}_{\hat{w}_2=qu_1} \operatorname*{Res}_{\hat{w}_1=u_1} f = \prod_{S_1<i}^{d}\left(\frac{\hat{w}_i - q^{S_1-1}u_1}{\hat{w}_i - q^{S_1}u_1} \prod_{u\in U\setminus\{u_1\}} \frac{\hat{w}_i - q^{-1}u}{\hat{w}_i - u} \prod_{i<j} \frac{(\hat{w}_i - \hat{w}_j)^2}{(\hat{w}_i - q\hat{w}_j)(q\hat{w}_i - \hat{w}_j)}\right)\\
&\times u_1^{S_1} \prod_{k=0}^{S_1-1} \prod_{u\in U\setminus\{u_1\}} \frac{q^k u_1 - q^{-1}u}{q^k u_1 - u} \cdot (q-1)^{S_1} q^{S_1(S_1-1-d)} g(u_1, qu_1, \cdots, q^{S_1-1}u_1, \hat{w}_{S_1+1}, \cdots \hat{w}_d),
\end{aligned} \tag{2.12}$$

which agrees with equation (2.7), since $g(\vec{w})$ is a symmetric function. That is to say, we get the same results from two different evaluation sequences

$$\operatorname*{Res}_{\hat{w}_{S_1}=q^{S_1-1}u_1} \cdots \operatorname*{Res}_{\hat{w}_2=qu_1} \operatorname*{Res}_{\hat{w}_1=u_1} f = \operatorname*{Res}_{\hat{w}_{S_1}=u_1} \operatorname*{Res}_{\hat{w}_{S_1-1}=q\hat{w}_{S_1}} \cdots \operatorname*{Res}_{\hat{w}_2=q\hat{w}_3} \operatorname*{Res}_{\hat{w}_1=q\hat{w}_2} f. \tag{2.13}$$

As with the evaluation process for the sequence in equation (2.6), we now pick up the residues of $\tilde{f}$ in the following sequence:

$$\hat{w}_{S_1+1} = q\hat{w}_{S_1+2} \quad \hat{w}_{S_1+2} = q\hat{w}_{S_1+3} \quad \cdots \quad \hat{w}_{S_1+S_2-1} = q\hat{w}_{S_1+S_2}. \tag{2.14}$$

Suppose $\hat{w}_{S_1+S_2} = u_2$. We have two cases here: $u_2 \neq q^{S_1}u_1$ or $u_2 = q^{S_1}u_1$. With a little computation, we obtain the following.

Case 1: $u_2 \neq q^{S_1}u_1$,

$$\begin{aligned}
&\operatorname*{Res}_{\hat{w}_{S_1+S_2}=u_2} \operatorname*{Res}_{\hat{w}_{S_1+S_2-1}=q\hat{w}_{S_1+S_2}} \cdots \operatorname*{Res}_{\hat{w}_{S_1+2}=q\hat{w}_{S_1+3}} \operatorname*{Res}_{\hat{w}_{S_1+1}=q\hat{w}_{S_1+2}} \operatorname*{Res}_{\hat{w}_{S_1}=u_1} \operatorname*{Res}_{\hat{w}_{S_1-1}=q\hat{w}_{S_1}} \cdots \operatorname*{Res}_{\hat{w}_2=q\hat{w}_3} \operatorname*{Res}_{\hat{w}_1=q\hat{w}_2} f\\
&= \operatorname*{Res}_{\hat{w}_{S_1+S_2}=u_1} \operatorname*{Res}_{\hat{w}_{S_1+S_2-1}=q\hat{w}_{S_1+S_2}} \cdots \operatorname*{Res}_{\hat{w}_{S_2+2}=q\hat{w}_{S_2+3}} \operatorname*{Res}_{\hat{w}_{S_2+1}=q\hat{w}_{S_2+2}} \operatorname*{Res}_{\hat{w}_{S_2}=u_2} \operatorname*{Res}_{\hat{w}_{S_2-1}=q\hat{w}_{S_2}} \cdots \operatorname*{Res}_{\hat{w}_2=q\hat{w}_3} \operatorname*{Res}_{\hat{w}_1=q\hat{w}_2} f.
\end{aligned} \tag{2.15}$$

Case 2: $u_2 = q^{S_1}u_1$,

$$\begin{aligned}
&\operatorname*{Res}_{\hat{w}_{S_1+S_2}=q^{S_1}u_1} \operatorname*{Res}_{\hat{w}_{S_1+S_2-1}=q\hat{w}_{S_1+S_2}} \cdots \operatorname*{Res}_{\hat{w}_{S_1+2}=q\hat{w}_{S_1+3}} \operatorname*{Res}_{\hat{w}_{S_1+1}=q\hat{w}_{S_1+2}} \operatorname*{Res}_{\hat{w}_{S_1}=u_1} \operatorname*{Res}_{\hat{w}_{S_1-1}=q\hat{w}_{S_1}} \cdots \operatorname*{Res}_{\hat{w}_2=q\hat{w}_3} \operatorname*{Res}_{\hat{w}_1=q\hat{w}_2} f\\
&= \operatorname*{Res}_{\hat{w}_{S_1+S_2}=q^{S_1+S_2-1}u_1} \operatorname*{Res}_{\hat{w}_{S_1+S_2-1}=q^{S_1+S_2-2}u_1} \cdots \operatorname*{Res}_{\hat{w}_{S_1+2}=q^{S_1+1}u_1} \operatorname*{Res}_{\hat{w}_{S_1+1}=q^{S_1}u_1} \operatorname*{Res}_{\hat{w}_{S_1}=q^{S_1-1}u} \cdots \operatorname*{Res}_{\hat{w}_2=qu_1} \operatorname*{Res}_{\hat{w}_1=u_1} f.
\end{aligned} \tag{2.16}$$

To summarise all of the above, together with equations (2.13), (2.15) and (2.16), we know that the iterated residue does not depend on the order of the poles we pick but depends on the final set of poles we choose. So we can integrate all variables for the integrand of the form as in equation (2.4) with one less variable each time.

When there is only one variable left

$$f(w) = g(w) \prod_{u\in U} \frac{w - q^{-1}u}{w - u}, \tag{2.17}$$

we still update the set U to $U\setminus\{u\} \cup \{qu\}$ after choosing a pole at $\hat{w} = u \in U$. Using the same argument to get equations (2.15) and (2.16) after picking up poles for all $w_i, i \in [d]$, the result only depends on the final set U, which is of the form

$$\{q^{d_1}x_1, \cdots, q^{d_r}x_r\}, \tag{2.18}$$

where $d_1 + \cdots + d_r = d$, which means for each sequence, the final result can be indexed by a r-tuple partition of d.

Suppose there is a sequence with final set $\{q^{d_1}x_1, \cdots, q^{d_r}x_r\}$. Then we can compute the result with the following sequence:

$$(\hat{w}_1, \cdots, \hat{w}_d) = (x_1, qx_1 \cdots, q^{d_1-1}x_1, x_2, \cdots, q^{d_2-1}x_2, x_r, \cdots, q^{d_r-1}x_r). \tag{2.19}$$

Note that we can do permutations on the left side, so for each partition $|\vec{d}| = d$, we have $d!$ possible evaluation sequences.

In all, we obtain the following lemma to compute E_d.

Lemma 2.1. *We can write E as*

$$E_d = \sum_{|\vec{d}|=d} d! E_{\vec{d}}, \tag{2.20}$$

where

$$E_{\vec{d}} = \lim_{w_d \to \hat{w}_d} \cdots \lim_{w_1 \to \hat{w}_1} \left(\prod_{i=1}^{n} (w_i - \hat{w}_i) f(\vec{w}) \right), \tag{2.21}$$

here

$$(\hat{w}_1, \ldots, \hat{w}_d) = (x_1, qx_1, \ldots, q^{d_1-1}x_1, x_2, qx_2, \ldots, q^{d_2-1}x_2, \ldots, x_r, \ldots, q^{d_r-1}x_r),$$

and the order in which to take the limit is from w_1 to w_d.

We now evaluate one specific configuration of these simple pole residues for given $\vec{d}$ by changing variables:

$$w_{i,n_i} = x_i q^{n_i-1} z_{i,n_i}, \qquad i = 1, \ldots, r \quad n_i = 1, \ldots, d_i.$$

Notations: From now on, we frequently use the following notations:

$$x_{ij} := x_i/x_j \quad n_{ij} := n_i - n_j. \tag{2.22}$$

Then

$$\begin{aligned} f(\vec{w}) =& \frac{1}{(1-q)^d d! \prod_{i,n_i} z_{i,n_i}} \cdot \prod_{i=1}^{r} \prod_{n_i \neq n_j}^{d_i} \frac{1 - q^{n_{ij}} z_{i,n_i}/z_{i,n_j}}{1 - q^{n_{ij}+1} z_{i,n_i}/z_{i,n_j}} \\ &\times \prod_{i,j=1|i\neq j}^{r} \prod_{n_i=1}^{d_i} \prod_{n_j=1}^{d_j} \frac{1 - q^{n_{ij}} z_{i,n_i}/z_{j,n_j} x_{ij}}{1 - q^{n_{ij}+1} z_{i,n_i}/z_{j,n_j} x_{ij}} \\ &\times \frac{\prod_i^r \prod_{n_i=1}^{d_i} (x_i q^{n_i-1} z_{i,n_i})^l}{\prod_{i,j=1|i\neq j}^{r} \prod_{n_i=1}^{d_i} (1 - x_{ji} q^{1-n_i}/z_{i,n_i})} \\ &\times \frac{1}{\prod_{i=1}^{r} \prod_{n_i=1}^{d_i} (1 - q^{1-n_i}/z_{i,n_i})} \cdot \frac{1}{\prod_{i=1}^{r} \prod_{j=r+1}^{n} \prod_{n_i=1}^{d_i} (1 - x_{ij} q^{n_i} z_{i,n_i})}. \end{aligned}$$

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Now we pick up the simple pole terms and evaluate the function with $z_{i,j} = 1$. Note that

$$\lim_{z_{i,d_i}\to 1} \cdots \lim_{z_{i,1}\to 1}\left(\prod_{n_i=1}^{d_i}(z_{i,n_i}-1)\cdot\frac{1}{(1-(z_{i,1})^{-1})(1-z_{i,1}/z_2^i)\dots(1-z_{i,d_i-1}/z_{i,d_i})z_{i,1}\cdots z_{i,d_i}}\right) = 1,$$

where the order in which to take the limit is from $z_{i,1}$ to z_{i,d_i}. So this specific configuration of residues is

$$\frac{1}{(1-q)^d d!}\cdot\prod_{i=1}^{r}\left(\prod_{n_i\neq n_j|n_{ij}\neq -1}^{d_i}\frac{1-q^{n_{ij}}}{1-q^{n_{ij}+1}}\cdot\prod_{n_i=2}^{d_i}\frac{1-q^{-1}}{1-q^{1-n_i}}\right)$$
$$\times\prod_{i,j=1|i\neq j}^{r}\prod_{n_i=1}^{d_i}\prod_{n_j=1}^{d_j}\frac{1-q^{n_{ij}}x_{ij}}{1-q^{n_{ij}+1}x_{ij}}$$
$$\times\frac{\prod_i^r\prod_{n_i=1}^{d_i}(x_iq^{n_i-1})^l}{\prod_{i,j=1|i\neq j}^{r}\prod_{n_i=1}^{d_i}(1-x_{ji}q^{1-n_i})}\cdot\frac{1}{\prod_{i=1}^{r}\prod_{j=r+1}^{n}\prod_{n_i=1}^{d_i}(1-x_{ij}q^{n_i})},$$

and the factor with only x_{ij} for $i = 1, \dots, r$ and $j = r+1, \dots, n$ is

$$A := \frac{1}{\prod_{i=1}^{r}\prod_{j=r+1}^{n}\prod_{n_i=1}^{d_i}(1-x_{ij}q^{n_i})} = \frac{1}{\prod_{i=1}^{r}\prod_{j=r+1}^{n}(qx_{ij};q)_{d_i}}.$$

The factor that does not involve any x_{ij} is

$$B := \frac{1}{(1-q)^d}\cdot\prod_{i=1}^{r}\left(\prod_{n_i\neq n_j|n_{ij}\neq -1}^{d_i}\frac{1-q^{n_{ij}}}{1-q^{n_{ij}+1}}\cdot\prod_{n_i=2}^{d_i}\frac{1-q^{-1}}{1-q^{1-n_i}}\right).$$

And define P_d as

$$P_d := \begin{cases}\displaystyle\prod_{i\neq j|i-j\neq -1}^{d}\frac{1-q^{i-j}}{1-q^{i-j+1}}\cdot\prod_{i=2}^{d_i}\frac{1-q^{-1}}{1-q^{1-i}} & d>1\\ 1 & d=0,1\end{cases}.$$

By simple induction, it is easy to show that

$$\frac{P_d}{(1-q)^d} = \frac{1}{(q;q)_d}, \qquad d\geq 0,$$

and

$$B = \prod_{i=1}^{r}\frac{P_{d_i}}{(1-q)^{d_i}} = \prod_{i=1}^{r}\frac{1}{(q;q)_{d_i}}.$$

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The factor left is

$$\begin{aligned}
C :=& \left(\prod_{i,j=1|i\neq j}^{r} \prod_{n_i=1}^{d_i} \prod_{n_j=1}^{d_j} \frac{1-q^{n_{ij}}x_{ij}}{1-q^{n_{ij}+1}x_{ij}} \right) \frac{\prod_{i=1}^{r}\prod_{n_i=1}^{d_i}(x_i q^{n_i-1})^l}{\prod_{i,j=1|i\neq j}^{r}\prod_{n_i=1}^{d_i}(1-x_{ji}q^{1-n_i})} \\
=& \prod_{i\neq j}^{r} \prod_{n_j=1}^{d_j} \left(\left(\prod_{n_i=1}^{d_i} \frac{1-q^{n_{ij}}x_{ij}}{1-q^{n_{ij}+1}x_{ij}} \right) \cdot \frac{1}{1-x_{ij}q^{1-n_j}} \right) \cdot \prod_{i=1}^{r} x_i^{ld_i} q^{\frac{ld_i(d_i-1)}{2}} \\
=& \prod_{i\neq j}^{r} \prod_{n_j=1}^{d_j} \frac{1}{1-q^{d_i-n_j+1}x_{ij}} \cdot \prod_{i=1}^{r} x_i^{ld_i} q^{\frac{ld_i(d_i-1)}{2}} \\
=& \prod_{i=1}^{r} x_i^{ld_i} q^{\frac{ld_i(d_i-1)}{2}} \cdot \prod_{i\neq j}^{r} \frac{1}{(q^{d_{ij}+1}x_{ij};q)_{d_j}}.
\end{aligned}$$

The above equations prove that the summand in equation (2.20) corresponding to a given $\vec{d}$ equals to one summand in $A_d(\vec{x}, I, l)$. Thus we have

$$A_d(\vec{x}, I, l) = \sum_{|\vec{d}|=d} d! E_{\vec{d}} = E_d.$$

Now calculate the integration in a clockwise direction:

$$E'_d := \int_{C'_\rho} \frac{dw_{i_d}}{2\pi\sqrt{-1}} \cdots \int_{C'_\rho} \frac{dw_{i_1}}{2\pi\sqrt{-1}} f(w_{i_1}, \cdots, w_{i_d}). \tag{2.23}$$

The assumption with l ensures that when integrating in any order, for each variable w, the residue at infinity is 0. By definition, we can calculate this integration by taking the sum of residues outside the circle $|w_i| = \rho$.

The iterated residues, in this case, are similar to the previous counterclockwise direction. Arguments similar to those in equation (2.1) show

$$E'_d = \sum_{|\vec{d}'|=d} d! E'_{\vec{d}'},$$

where

$$E'_{\vec{d}'} = \lim_{w_d \to \hat{w}_d} \cdots \lim_{w_1 \to \hat{w}_1} \left(\prod_{i=1}^{n} (w_i - \hat{w}_i) f(\vec{w}) \right),$$

here

$$\{\hat{w}_1, \ldots, \hat{w}_d\} = \left\{ x_{r+1}q^{-1}, x_{r+1}q^{-2}, \ldots, x_{r+1}q^{-d_{r+1}}, \ldots, x_n q^{-1}, x_n q^{-2}, \ldots, x_n q^{-d_n} \right\},$$

and the order in which to take the limit is from w_1 to w_d.

We now do the following, changing variables and calculating the residues:

$$w_{i,n_i} = x_i q^{-n_i} z_{i,n_i}, \quad i = r+1, \ldots, n \quad n_i = 1, \ldots, d_i.$$

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Similarly,

$$\begin{aligned}
f(\vec{w}) =& \frac{1}{(1-q)^d d! \prod_{i,n_i} z_{i,n_i}} \cdot \prod_{i=r+1}^{n} \prod_{n_i \neq n_j}^{d_i} \frac{1-q^{n_{ji}} z_{i,n_i}/z_{i,n_j}}{1-q^{n_{ji}+1} z_{i,n_i}/z_{i,n_j}} \\
&\times \prod_{i,j=r+1|i\neq j}^{n} \prod_{n_i=1}^{d_i} \prod_{n_j=1}^{d_j} \frac{1-q^{n_{ji}} z_{i,n_i}/z_{j,n_j} x_{ij}}{1-q^{n_{ji}+1} z_{i,n_i}/z_{j,n_j} x_{ij}} \\
&\times \frac{\prod_{i=r+1}^{n} \prod_{n_i=1}^{d_i} (x_i q^{-n_i} z_{i,n_i})^{l-1}}{\prod_{i,j=r+1|i\neq j}^{n} \prod_{n_i=1}^{d_i} (1 - x_{ij} q^{1-n_i} z_{i,n_i})} \\
&\times \frac{1}{\prod_{i=r+1}^{n} \prod_{n_i=1}^{d_i} (1-q^{1-n_i} z_{i,n_i})} \cdot \frac{1}{\prod_{i=r+1}^{n} \prod_{j=1}^{r} \prod_{n_i=1}^{d_i} (1 - x_{ji} q^{n_i}/z_{i,n_i})}.
\end{aligned}$$

Note that

$$\lim_{z_{i,d_i}\to 1} \cdots \lim_{z_{i,1}\to 1} \left(\prod_{n_i=1}^{d_i} (z_{i,n_i} - 1) \cdot \frac{1}{(1-z_{i,1})(1-z_{i,2}/z_{i,1}) \ldots (1 - z_{i,d_i}/z_{i,d_i-1}) z_{i,1} \cdots z_{i,d_i}} \right) = (-1)^{d_i},$$

where the order in which to take the limits is from $z_{i,1}$ to z_{i,d_i}. So the residues for one specific configuration of residues of type $\vec{d}'$ are

$$\begin{aligned}
&\frac{(-1)^d}{(1-q)^d d!} \cdot \prod_{i=r+1}^{n} \left(\prod_{n_i \neq n_j | n_{ji} \neq -1}^{d_i} \frac{1-q^{n_{ji}}}{1-q^{n_{ji}+1}} \cdot \prod_{n_i=2}^{d_i} \frac{1-q^{-1}}{1-q^{1-n_i}} \right) \\
&\times \prod_{i,j=r+1|i\neq j}^{n} \prod_{n_i=1}^{d_i} \prod_{n_j=1}^{d_j} \frac{1-q^{n_{ji}} x_{ij}}{1-q^{n_{ji}+1} x_{ij}} \\
&\times \frac{\prod_{i=r+1}^{n} \prod_{n_i=}^{d_i} (x_i q^{-n_i})^{l}}{\prod_{i,j=r+1|i\neq j}^{n} \prod_{n_i=1}^{d_i} (1 - x_{ij} q^{1-n_i})} \cdot \frac{1}{\prod_{i=r+1}^{n} \prod_{j=1}^{r} \prod_{n_i=1}^{d_i} (1 - x_{ji} q^{n_i})}.
\end{aligned}$$

After almost the same computation as for $E_{\vec{d}}$, we can simplify the above equation to

$$(-1)^d \frac{\prod_{i=r+1}^{n} x_i^{ld_i} q^{-\frac{ld_i(d_i+1)}{2}}}{\prod_{i=r+1}^{n} (q;q)_{d_i} \prod_{i\neq j|i,j=r+1}^{n} (q^{d_{ij}+1} x_{ji}; q)_{d_j} \prod_{i=r+1}^{n} \prod_{j=1}^{r} (q x_{ji}; q)_{d_i}},$$

which proves

$$E'_d = (-1)^d B_d\left(\vec{x}, I^{\complement}, -l\right).$$

Since the residue at infinity is zero, using the Cauchy Residue Theorem d times,

$$\begin{aligned}
&\int_{C_\rho} \cdots \int_{C_\rho} f(\vec{w}) \frac{dw_1}{2\pi\sqrt{-1}w_1} \cdots \frac{dw_d}{2\pi\sqrt{-1}w_d} \\
&= (-1)^d \int_{C'_\rho} \cdots \int_{C'_\rho} f(\vec{w}) \frac{dw_1}{2\pi\sqrt{-1}w_1} \cdots \frac{dw_d}{2\pi\sqrt{-1}w_d},
\end{aligned}$$

we arrive at equations (2.24), (2.25) and (2.26) of the following theorem stated in the introduction.

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Theorem 2.2. *Denoted by $[n]$, the set of elements $\{1,\ldots,n\}$, let $\emptyset \neq I \subsetneq [n]$ be a subset of $[n]$, $|I|$ be its cardinality, and denoted by $I^{\complement}$, the complementary set of I in $[n]$. Then for constant positive integers d, n and integer l with restriction $1-|I| \le l \le n-|I|-1$, let $A_d(\vec{x},I,l)$ and $B_d(\vec{x},I,l)$ be two rational functions in $\vec{x}$ and q with an extra data l:*

$$A_d(\vec{x},I,l) = \sum_{|d_I|=d} \frac{\left(\prod_{i\in I} x_i^{d_i} q^{\frac{d_i(d_i-1)}{2}}\right)^l}{\prod_{i,j\in I}(q^{d_{ij}+1}x_{ij};q)_{d_j}\prod_{i\in I}\prod_{j\in I^{\complement}}(qx_{ij};q)_{d_i}}, \tag{2.24}$$

$$B_d(\vec{x},I,l) = \sum_{|\vec{d}_I|=d} \frac{\left(\prod_{i\in I} x_i^{-d_i} q^{\frac{d_i(d_i+1)}{2}}\right)^l}{\prod_{i,j\in I}(q^{d_{ij}+1}x_{ji};q)_{d_j}\prod_{i\in I}\prod_{j\in I^{\complement}}(qx_{ji};q)_{d_i}}, \tag{2.25}$$

where $\vec{d}_I$ is an $|I|$-tuple of nonnegative integers and $|\vec{d}_I| := \sum_{i\in I} d_i$. $x_i, i=1,\ldots,n$ are parameters. For convenience, we use the notation $x_{ij} := x_i/x_j$ and $d_{ij} := d_i - d_j$. Then we have

$$A_d(\vec{x},I,l) = B_d\left(\vec{x},I^{\complement},-l\right). \tag{2.26}$$

2.2. *Examples*

In the following two examples, we show how the proof of Theorem 2.2 works.

Example 2.3 (d=1). For the case l=0, d=1, r=2, n=3, equation (2.3) becomes the following simple form:

$$f(w) = \frac{1}{(1-q)}\frac{w^{-1}}{(1-x_1/w)(1-x_2/w)(1-qw/x_3)}.$$

Consider the integration in equation (2.5). Then there are simple poles of type $(1,0)$ and $(0,1)$ in the counter C_ρ:

- type (1,0): $w = x_1$,
- type (0,1): $w = x_2$.

Then the residue for each type is as follows:

- type $(1,0)$:

$$E_{(1,0)} = \operatorname*{Res}_{\hat{w}=x_1} f = \frac{1}{(1-q)(1-x_{21})(1-qx_{13})},$$

- type $(0,1)$:

$$E_{(0,1)} = \operatorname*{Res}_{\hat{w}=x_2} f = \frac{1}{(1-q)(1-x_{12})(1-qx_{23})}.$$

And there is only one simple pole $w = q^{-1}x_3$ in the counter C'_ρ, so

- type 1:

$$E'_1 = \operatorname*{Res}_{\hat{w}=q^{-1}x_3} f = \frac{-1}{(1-q)(1-qx_{13})(1-qx_{23})}.$$

It is easy to obtain

$$\frac{1}{(1-q)(1-x_{21})(1-qx_{13})} + \frac{1}{(1-q)(1-x_{12})(1-qx_{23})} = \frac{1}{(1-q)(1-qx_{13})(1-qx_{23})},$$

which agrees with equation (2.26).

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Example 2.4 (d=2). For the case l=0, d=2, r=2, n=3, equation (2.3) becomes the following simple form:

$$f(\vec{w}) = \frac{1}{2(1-q)^2} \prod_{i\neq j}^{2} \frac{1-w_i/w_j}{1-qw_i/w_j} \prod_{i=1}^{2} \frac{w_i^{-1}}{\prod_{j=1}^{2}(1-x_j/w_i)\cdot(1-qw_i/x_3)}.$$

Consider the integration in equation (2.5). Then there are simple poles of type $(2,0)$, $(1,1)$ and $(0,2)$ in the counter C_{ρ_i}:

- type (2,0): $\{w_1,w_2\}=\{x_1,x_1q\}$,
- type (1,1): $\{w_1,w_2\}=\{x_1,x_2\}$,
- type (0,2): $\{w_1,w_2\}=\{x_2,x_2q\}$.

Then the residue for each type is as follows:

- type $(2,0)$:

$$\begin{aligned}2!E_{(2,0)} &= \operatorname*{Res}_{\hat{w}_2=qx_1}\operatorname*{Res}_{\hat{w}_1=x_1} f + \operatorname*{Res}_{\hat{w}_2=x_1}\operatorname*{Res}_{\hat{w}_1=qw_2} f\\ &= \frac{1}{2(1-q)^2}\frac{1}{(1+q)(1-qx_{13})(1-q^2x_{13})(1-x_{21})(1-q^{-1}x_{21})}\\ &\quad+\frac{1}{2(1-q)^2}\frac{1}{(1+q)(1-q^2x_{13})(1-qx_{13})(1-q^{-1}x_{21})(1-x_{21})}\\ &= \frac{1}{(1-q)^2}\frac{1}{(1+q)(1-q^2x_{13})(1-qx_{13})(1-q^{-1}x_{21})(1-x_{21})},\end{aligned}$$

- type $(1,1)$:

$$\begin{aligned}2!E_{(1,1)} &= \operatorname*{Res}_{\hat{w}_2=x_2}\operatorname*{Res}_{\hat{w}_1=x_1} f + \operatorname*{Res}_{\hat{w}_2=x_1}\operatorname*{Res}_{\hat{w}_1=x_2} f\\ &= \frac{1}{2(1-q)^2}\frac{1}{(1-qx_{12})(1-qx_{21})(1-qx_{13})(1-qx_{23})}\\ &\quad+\frac{1}{2(1-q)^2}\frac{1}{(1-qx_{21})(1-qx_{12})(1-qx_{23})(1-qx_{13})}\\ &= \frac{1}{(1-q)^2}\frac{1}{(1-qx_{21})(1-qx_{12})(1-qx_{23})(1-qx_{13})},\end{aligned}$$

- type $(0,2)$:

$$\begin{aligned}2!E_{(0,2)} &= \operatorname*{Res}_{\hat{w}_2=qx_2}\operatorname*{Res}_{\hat{w}_1=x_2} f + \operatorname*{Res}_{\hat{w}_2=x_2}\operatorname*{Res}_{\hat{w}_1=qw_2} f\\ &= \frac{1}{2(1-q)^2}\frac{1}{(1+q)(1-x_{12})(1-qx_{23})(1-q^{-1}x_{12})(1-q^2x_{23})}\\ &\quad+\frac{1}{2(1-q)^2}\frac{1}{(1+q)(1-q^{-1}x_{12})(1-q^2x_{23})(1-x_{12})(1-qx_{23})}\\ &= \frac{1}{(1-q)^2}\frac{1}{(1+q)(1-q^{-1}x_{12})(1-q^2x_{23})(1-x_{12})(1-qx_{23})}.\end{aligned}$$

Consider the integration in equation (2.23). Then there are simple poles of type 2 in the counter C'_{ρ_i}:

- type 2: $\{w_1,w_2\}=\{q^{-1}x_3,q^{-2}x_3\}$.

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Then the residue for each type 2 is as follows:

- type 2:

$$
\begin{aligned}
(-1)^2 2! E_2' &= \operatorname*{Res}_{\hat{w}_2=q^{-2}x_3}\operatorname*{Res}_{\hat{w}_1=q^{-1}x_3} f + \operatorname*{Res}_{\hat{w}_2=q^{-1}x_3}\operatorname*{Res}_{\hat{w}_1=q^{-1}w_2} f \\
&= \frac{1}{(1+q)(1-q)^2(1-q^2x_{13})(1-qx_{13})(1-q^2x_{23})(1-qx_{23})}.
\end{aligned}
$$

By a little computation, we have

$$E_2 = 2!E_{(2,0)} + 2!E_{(1,1)} + 2!E_{(0,2)} = E_2'.$$

Example 2.5. From Proposition 1.2, if we take $n = 3$, $l = 0$ and $I = [2]$, we know that $A_d(\vec{x}, [2], 0) = B_d(\vec{x}, [3]\backslash[2], 0)$. By the following computation, there is a phenomenon that we can extract from $A_d(\vec{x}, [2], 0)$ to get $B_d(\vec{x}, [3]\backslash[2], 0)$ times another factor when $d = 1, 2$: that is, $A_d(\vec{x}, [2], 0) = B_d(\vec{x}, [3]\backslash[2], 0) \times G(\vec{x})$, $d = 1, 2$. Thus we can conclude that $G(\vec{x}) = 1$. Furthermore, this is a general phenomenon for all d; see the following Corollary 2.1.

By definition, $\vec{x} = \{x_1, x_2, x_3\}$, so

$$A_d(\vec{x}, [2], 0) = \sum_{d_1+d_2=d} \frac{1}{(q;q)_{d_1}(q;q)_{d_2}(q^{d_{12}+1}x_{12};q)_{d_2}(q^{d_{21}+1}x_{12};q)_{d_1}(qx_{13};q)_{d_1}(qx_{23};q)_{d_2}}, \tag{2.27}$$

$$B_d(\vec{x}, [3]\backslash[2], 0) = \frac{1}{(q;q)_d(qx_{13};q)_d(qx_{23};q)_d}. \tag{2.28}$$

For $d = 1$ – that is, $(d_1, d_2) = (1, 0)$ or $(0, 1)$ – we have

$$
\begin{aligned}
A_1(\vec{x}, [2], 0) &= \sum_{d_1+d_2=1} \frac{1}{(q;q)_{d_1}(q;q)_{d_2}(q^{d_{12}+1}x_{12};q)_{d_2}(q^{d_{21}+1}x_{12};q)_{d_1}(qx_{13};q)_{d_1}(qx_{23};q)_{d_2}} \\
&= \sum_{d_1+d_2=1} \frac{1}{(q;q)_1(qx_{13};q)_1(qx_{23};q)_1} \\
&\quad \cdot \frac{(q;q)_1(qx_{13};q)_1(qx_{23};q)_1}{(q;q)_{d_1}(q;q)_{d_2}(q^{d_{12}+1}x_{12};q)_{d_2}(q^{d_{21}+1}x_{12};q)_{d_1}(qx_{13};q)_{d_1}(qx_{23};q)_{d_2}} \\
&= B_1(\vec{x}, [3]\backslash[2], 0) \times \sum_{(d_1,d_2)=(1,0),(0,1)} \frac{(q;q)_1}{(q;q)_{d_1}(q;q)_{d_2}} \prod_{i=1}^{2}\left(\prod_{j\neq i}^{2} \frac{(q^{d_i+1}x_{i3};q)_{1-d_i}}{(q^{d_{ij}+1}x_{ij};q)_{d_j}}\right) \\
&= B_1(\vec{x}, [3]\backslash[2], 0) \times \left(\frac{1-qx_{23}}{1-x_{21}} + \frac{1-qx_{13}}{1-x_{12}}\right) \\
&= B_1(\vec{x}, [3]\backslash[2], 0).
\end{aligned}
$$

For $d = 2$ – that is, $(d_1, d_2) = (2, 0), (1, 1)$ or $(0, 2)$ – similarly we have

$$
\begin{aligned}
A_2(\vec{x}, [2], 0) &= B_2(\vec{x}, [3]\backslash[2], 0) \times \sum_{(d_1,d_2)=(2,0),(1,1),(0,2)} \frac{(q;q)_2}{(q;q)_{d_1}(q;q)_{d_2}} \prod_{i=1}^{2}\left(\prod_{j\neq i}^{2} \frac{(q^{d_i+1}x_{i3};q)_{2-d_i}}{(q^{d_{ij}+1}x_{ij};q)_{d_j}}\right) \\
&= B_2(\vec{x}, [3]\backslash[2], 0) \\
&\quad \times \left(\frac{(1-qx_{13})(1-q^2x_{13})}{(1-q^{-1}x_{21})(1-x_{21})} + \frac{(1+q)(1-q^2x_{13})(1-q^2x_{23})}{(1-qx_{12})(1-qx_{21})} + \frac{(1-qx_{13})(1-q^2x_{13})}{(1-q^{-1}x_{12})(1-x_{12})}\right) \\
&= B_2(\vec{x}, [3]\backslash[2], 0).
\end{aligned}
$$

More generally, we have the following corollary.

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Corollary 2.1.

$$\sum_{d_1+d_2=d} \frac{(q;q)_d}{(q;q)_{d_1}(q;q)_{d_2}} \prod_{j\neq i}^{2} \frac{(q^{d_i+1}x_{i3};q)_{d-d_i}}{(q^{d_i-d_j+1}x_{ij};q)_{d_j}} = 1.$$

Proof. Set $l=0, r=2, n=3$ in equation (2.26). We have

$$\begin{aligned}
A_d(\vec{x},[2],0) &= \sum_{d_1+d_2=d} \prod_{i,j=1}^{2} \frac{1}{(q^{d_{ij}+1}x_{ij};q)_{d_j}} \prod_{i=1}^{2} \frac{1}{(qx_{i3};q)_{d_i}} \\
&= \sum_{d_1+d_2=d} \prod_{i=1}^{2} \left(\frac{1}{(q;q)_{d_i}} \prod_{j\neq i}^{2} \frac{1}{(q^{d_{ij}+1}x_{ij};q)_{d_j}} \cdot \frac{1}{(qx_{i3};q)_{d_i}} \right) \\
&= \sum_{d_1+d_2=d} \frac{(q^{d_1+1}x_{13};q)_{d-d_1}(q^{d_2+1}x_{23};q)_{d-d_2}}{(qx_{13};q)_d(qx_{23};q)_d} \prod_{j\neq i}^{2} \left(\frac{1}{(q;q)_{d_i}} \frac{1}{(q^{d_i-d_j+1}x_{ij};q)_{d_j}} \right) \\
&= \sum_{d_1+d_2=d} \frac{(q;q)_d}{(q;q)_d(qx_{13};q)_d(qx_{23};q)_d} \prod_{j\neq i}^{2} \left(\frac{(q^{d_i+1}x_{i3};q)_{d-d_i}}{(q;q)_{d_i}(q^{d_{ij}+1}x_{ij};q)_{d_j}} \right) \\
&= \sum_{d_1+d_2=d} B_d(\vec{x},[n]\backslash[2],0) \cdot \frac{(q;q)_d}{(q;q)_{d_1}(q;q)_{d_2}} \prod_{j\neq i}^{2} \left(\frac{(q^{d_i+1}x_{i3};q)_{d-d_i}}{(q^{d_{ij}+1}x_{ij};q)_{d_j}} \right).
\end{aligned}$$

Since we know $A_d(\vec{x},I,0)$ equals $B_d(\vec{x},I^{\complement},0)$, we get the conclusion. □

2.3. *Boundary cases*

For $l=-|I|$, $l=n-|I|$, equation (2.26) no longer holds, since the residue at infinity is nonzero, but we can compare the behaviour of some special limit in equation (2.26) to obtain following results.

Corollary 2.2. ○ *For $l=n-|I|$, we have*

$$A_d(\vec{x},I,l) = \sum_{s=0}^{d} C_s(\vec{x},I^{\complement},d) B_{d-s}\left(\vec{x},q,I^{\complement},-l\right), \tag{2.29}$$

where $C_s(\vec{x},I,d)$ is defined as

$$C_s(\vec{x},I,d) = \frac{(-1)^{|I|\cdot s} \prod_{i\in I^{\complement}} x_i^s}{(q;q)_s q^{s(d-s+|I|)}}.$$

○ *For $l=-|I|$, we have*

$$B_d\left(\vec{x},I^{\complement},-l\right) = \sum_{s=0}^{d} D_s(\vec{x},I,d) A_{d-s}(\vec{x},q,I,l), \tag{2.30}$$

where

$$D_s(\vec{x},I,d) = \frac{(-1)^{|I|\cdot s} \prod_{i\in I} x_i^{-s}}{(q;q)_s q^{s(d-s)}}.$$

Proof. Consider $[n+1]$, $I \subsetneq [n+1]$, $\{n+1\} \notin I$, $l = n - |I|$ in equation (2.26). Then we have

$$\sum_{|\vec{d}_I|=d} \frac{\left(\prod_{i\in I} x_i^{d_i} q^{\frac{d_i(d_i-1)}{2}}\right)^l}{\prod_{i,j\in I}(q^{d_{ij}+1}x_{ij};q)_{d_j}\prod_{i\in I}\prod_{j\in I^{\complement}}(qx_{ij};q)_{d_i}} \tag{2.31}$$

$$= \sum_{|\vec{d}_{I^{\complement}}|=d} \frac{\left(\prod_{i\in I^{\complement}} x_i^{-d_i} q^{\frac{d_i(d_i+1)}{2}}\right)^{-l}}{\prod_{i,j\in I^{\complement}}(q^{d_{ij}+1}x_{ji};q)_{d_j}\prod_{i\in I^{\complement}}\prod_{j\in I}(qx_{ji};q)_{d_i}}. \tag{2.32}$$

It is easy to see that taking $\lim_{x_{n+1}\to\infty}$ in equation (2.31), we obtain

$$\lim_{x_{n+1}\to\infty}(2.31) = A_d(\vec{x}, I, l), \quad \text{for } l = n - |I|.$$

Now let's take limit $\lim_{x_{n+1}\to\infty}$ in equation (2.32):

$$\sum_{|\vec{d}_{I^{\complement}}|=d} \frac{\left(\prod_{i\in I^{\complement}} x_i^{-d_i} q^{\frac{d_i(d_i+1)}{2}}\right)^{-l}}{\prod_{i,j\in I^{\complement}}(q^{d_{ij}+1}x_{ji};q)_{d_j}\prod_{i\in I^{\complement}}\prod_{j\in I}(qx_{ji};q)_{d_i}}$$

$$= \sum_{|\vec{d}_{I^{\complement}}|=d} \frac{1}{(q;q)_{d_{n+1}}}\cdot\frac{1}{\prod_{j\in I}(qx_{j,n+1};q)_{d_{n+1}}}\cdot\frac{1}{\prod_{j\in\{[n]\setminus I\}}(q^{d_{n+1}-d_j+1}x_{j,n+1};q)_{d_j}} \tag{2.33}$$

$$\times \frac{(x_{n+1}^{d_{n+1}} q^{-\frac{d_{n+1}(d_{n+1}+1)}{2}})^l}{\prod_{i\in\{[n]\setminus I\}}(q^{d_i-d_{n+1}+1}x_{n+1,i};q)_{d_{n+1}}} \tag{2.34}$$

$$\times \prod_{i\in\{[n]\setminus I\}}\left(\frac{1}{\prod_{j\in\{[n]\setminus I\}}(q^{d_i-d_j+1}x_{ji};q)_{d_j}}\cdot\frac{(x_i^{d_i}q^{\frac{-d_i(d_i+1)}{2}})^l}{\prod_{j\in I}(qx_{ji};q)_{d_i}}\right).$$

The limits of the last two terms in equation (2.33) equal 1, and by a little computation, we obtain that equation (2.34) equals

$$(-1)^{d_{n+1}(n-r)}\cdot q^{-(\sum_{i\in\{[n]\setminus I\}} d_i)d_{n+1}-(n-r)d_{n+1}}\prod_{i\in\{[n]\setminus I\}} x_i^{d_{n+1}}.$$

Then we obtain

$$\lim_{x_{n+1}\to\infty}\sum_{|\vec{d}_{I^{\complement}}|=d}\prod_{i\in I^{\complement}}\left(\frac{1}{\prod_{j\in I^{\complement}}(q^{d_i-d_j+1}x_{ji};q)_{d_j}}\cdot\frac{(x_i^{d_i}q^{\frac{-d_i(d_i+1)}{2}})^l}{\prod_{j\in I}(qx_{ji};q)_{d_i}}\right)$$

$$= \sum_{|\vec{d}_{I^{\complement}}|=d}\frac{1}{(q;q)_{d_{n+1}}}\cdot\frac{(-1)^{d_{n+1}(n-|I|)}}{q^{(\sum_{i\in\{[n]\setminus I\}} d_i)d_{n+1}+(n-|I|)d_{n+1}}}\cdot\frac{1}{\prod_{i\in\{[n]\setminus I\}} x_i^{-d_{n+1}}}$$

$$\times\prod_{i\in\{[n]\setminus I\}}\left(\frac{1}{\prod_{j\in\{[n]\setminus I\}}(q^{d_i-d_j+1}x_{ji};q)_{d_j}}\cdot\frac{(x_i^{d_i}q^{\frac{-d_i(d_i+1)}{2}})^l}{\prod_{j\in I}(qx_{ji};q)_{d_i}}\right)$$

$$= \sum_{\alpha=0}^{d}\frac{1}{(q;q)_{d-\alpha}}\cdot\frac{(-1)^{(d-\alpha)(n-|I|)}}{q^{(n-|I|+\alpha)(d-\alpha)}}\cdot\frac{1}{\prod_{i=r+1}^{n} x_i^{-(d-\alpha)}}\cdot B_\alpha\left(\vec{x}, I^{\complement}, -l\right).$$

We obtain the conclusion.

Similarly, consider $A_d(\vec{x} \cup x_{n+1}, \tilde{I}, l)$ and $B_d\left(\vec{x} \cup x_{n+1}, \tilde{I}^{\complement}, -l\right)$. For $\tilde{I} = I \cup \{n+1\}$ and $l = -|I|$, from equation (2.26), we have

$$\sum_{|\vec{d}_{\tilde{I}}|=d} \prod_{i \in \tilde{I}} \left(\frac{1}{\prod_{j \in \tilde{I}} (q^{d_i - d_j + 1} x_{ij}; q)_{d_j}} \cdot \frac{(x_i^{d_i} q^{\frac{d_i(d_i-1)}{2}})^l}{\prod_{j \in \tilde{I}^{\complement}} (q x_{ij}; q)_{d_i}} \right) \tag{2.35}$$

$$= \sum_{|\vec{d}_{\tilde{I}^{\complement}}|=d} \prod_{i \in \tilde{I}^{\complement}} \left(\frac{1}{\prod_{j \in \tilde{I}^{\complement}} (q^{d_i - d_j + 1} x_{ji}; q)_{d_j}} \cdot \frac{(x_i^{d_i} q^{\frac{-d_i(d_i+1)}{2}})^{-l}}{\prod_{j \in \tilde{I}} (q x_{ji}; q)_{d_i}} \right). \tag{2.36}$$

It is easy to see that after taking $\lim_{x_{n+1} \to 0}$ in equation (2.36), we obtain

$$B_d\left(\vec{x}, I^{\complement}, l\right), \quad \text{for } l = -|I|.$$

First, rewrite equation (2.35) as follows:

$$\begin{aligned}
&\sum_{|\vec{d}_{\tilde{I}}|=d} \prod_{i \in \tilde{I}} \left(\frac{1}{\prod_{j \in \tilde{I}} (q^{d_i - d_j + 1} x_{ij}; q)_{d_j}} \cdot \frac{(x_i^{d_i} q^{\frac{d_i(d_i-1)}{2}})^{-|I|}}{\prod_{j \in \tilde{I}^{\complement}} (q x_{ij}; q)_{d_i}} \right) \\
&= \sum_{|\vec{d}_I|=d} \frac{(\prod_{i \in I} x_i^{d_i} q^{\frac{d_i(d_i-1)}{2}})^{-|I|}}{\prod_{i,j \in I} (q^{d_{ij}+1} x_{ij}; q)_{d_j} \prod_{i \in I} \prod_{j \in \{[n] \setminus I\}} (q x_{ij}; q)_{d_i}} \\
&\times \frac{(x_{n+1}^{d_{n+1}} q^{\frac{d_{n+1}(d_{n+1}-1)}{2}})^{-|I|}}{(q;q)_{d_{n+1}} \prod_{i \in I} (q^{d_i - d_{n+1} + 1} x_{i,n+1}; q)_{d_{n+1}} \prod_{j \in I} (q^{d_{n+1} - d_j + 1} x_{n+1,j}; q)_{d_j} \prod_{j \in \{[n] \setminus I\}} (q x_{n+1,j}; q)_{d_{n+1}}}.
\end{aligned}$$

Now let's take limit $\lim_{x_{n+1} \to 0}$ in the above formula. We obtain

$$\begin{aligned}
&\lim_{x_{n+1} \to 0} \sum_{|\vec{d}_{\tilde{I}}|=d} \prod_{i \in \tilde{I}} \left(\frac{1}{\prod_{j \in \tilde{I}} (q^{d_i - d_j + 1} x_{ij}; q)_{d_j}} \cdot \frac{(x_i^{d_i} q^{\frac{d_i(d_i-1)}{2}})^{-|I|}}{\prod_{j \in \tilde{I}^{\complement}} (q x_{ij}; q)_{d_i}} \right) \\
&= \sum_{|\vec{d}_I|=d} \frac{(\prod_{i \in I} x_i^{d_i} q^{\frac{d_i(d_i-1)}{2}})^{-|I|}}{\prod_{i,j \in I} (q^{d_{ij}+1} x_{ij}; q)_{d_j} \prod_{i \in I} \prod_{j \in \{[n] \setminus I\}} (q x_{ij}; q)_{d_i}} \\
&\times \frac{(-1)^{|I| \cdot d_{n+1}}}{(q;q)_{d_{n+1}} q^{d_{n+1}(d-d_{n+1})} \prod_{i \in I} x_i^{d_{n+1}}} \\
&= \sum_{s=0}^{d} \frac{(-1)^{|I| \cdot s}}{(q;q)_s q^{s(d-s)} \prod_{i \in I} x_i^s} \times A_{d-s}(\vec{x}, I, -|I|).
\end{aligned}$$

□

3. *K*-theoretic *I*-function with level structure

3.1. *Definitions*

Let X be a GIT quotient $V//_\theta G$, where V is a vector space and G is a connected reductive complex Lie group. Let $\mathcal{Q}^{\epsilon}_{g,n}(X, \beta)$ be the moduli stack of ϵ-stable quasimaps [3] parametrising data

$(C, p_1,\dots, p_n, \mathcal{P}, s)$, where C is an n-pointed genus g Riemann surface, $\mathcal{P}$ is a principal G-bundle over C, s is a section and $\beta \in \mathrm{Hom}(\mathrm{Pic}^G(V))$. There are natural maps

$$ev_i : \mathcal{Q}^{\epsilon}_{g,n}(X,d) \to X, \quad i = 1,\dots,n$$

given by evaluation at the ith marked point: that is,

$$ev_i(C, p_1,\dots, p_n, \mathcal{P}, s) = s(p_i) \in X.$$

There are line bundles

$$L_i \longrightarrow \mathcal{Q}^{\epsilon}_{g,n}(X,d), \quad i = 1,\dots,n,$$

which are called universal cotangent line bundles. The fibre of L_i over the point $(C^{\epsilon}, p_1,\dots, p_n, \mathcal{P}, s)$ is the cotangent line to C at the point p_i.

The permutation-equivariant K-theoretic quasimap invariants with level structures [14] are holomorphic Euler characteristics over $\mathcal{Q}^{\epsilon}_{g,n}(X,d)$ of the sheaves

$$\langle \mathbf{t}(L),\dots,\mathbf{t}(L)\rangle^{R,l,S_n,\epsilon}_{g,n,d} := \chi\left(\mathcal{Q}^{\epsilon}_{g,n}(X,d); \mathcal{O}^{virt}_{g,n,d} \otimes \prod_{m,i} L_i^k t_{k,i} \mathrm{ev}_i^*(\phi_i) \otimes \mathcal{D}^{R,l}\right), \tag{3.1}$$

where $\mathcal{O}^{vir}_{g,n,d}$ is called the virtual structure sheaf [9]. $\mathbf{t}(q)$ is defined as follows:

$$\mathbf{t}(q) = \sum_{m\in\mathbb{Z}} t_m q^m, \quad t_m = \sum_{\alpha} t_{m,\alpha}\phi_{\alpha},$$

$\{\phi_{\alpha}\}$ is a basis in $K^0(X) \otimes Q$ and $t_{k,\alpha}$ are formal variables. The last term in equation (3.1) is the level l determinant line bundle over $\mathcal{Q}^{\epsilon}_{g,n}(X,d)$ defined as

$$\mathcal{D}^{R,l} := (\det R^{\bullet}\pi_*(\mathcal{P} \times_G R))^{-l},$$

where π is the forgetful map from the universal curve: that is,

$$\pi : \mathcal{C} \longrightarrow \mathcal{Q}^{\epsilon}_{g,n}(X,d).$$

The bundle $\mathcal{P}$ is the universal principal bundle over the universal curve, and R is a G-representation.

Similarly, we can define a quasimap graph space $\mathcal{QG}^{\epsilon}_{0,n}(X,\beta)$, which parametrises quasimaps with parametrised component $\mathbb{P}^1$, so there is a natural $\mathbb{C}^*$-action on the quasimap graph space. It is denoted by $\mathrm{F}_{0,\beta}$, the special fixed loci in $(\mathcal{QG}^{\epsilon}_{0,n}(X,\beta))^{\mathbb{C}^*}$, and denoted by q, the weight of the cotangent bundle at $0 := [1,0]$ of $\mathbb{P}^1$; for details, see [3].

Definition 3.1 ([14].). The permutation-equivariant K-theoretic $\mathcal{J}^{R,l,\epsilon}$-function of $V//G$ of level l is defined as

$$\begin{aligned}
\mathcal{J}^{R,l,\epsilon}_{S_\infty}(\mathbf{t}(q),Q) &:= \sum_{k\geq 0,\beta\in \mathrm{Eff}(V,\mathbf{G},\theta)} Q^{\beta}(ev_{\bullet})_*[\mathrm{Res}_{\mathrm{F}_{0,\beta}}(\mathcal{QG}^{\epsilon}_{0,n}(V//\mathbf{G},\beta)_0)^{\mathrm{vir}} \otimes \mathcal{D}^{R,l} \otimes_{i=1}^n \mathbf{t}(L_i)]^{S_n} \\
&:= 1 + \frac{\mathbf{t}(q)}{1-q} + \sum_a \sum_{\beta\neq 0} Q^{\beta}\chi\left(\mathrm{F}_{0,\beta}, \mathcal{O}^{\mathrm{vir}}_{\mathrm{F}_{0,\beta}} \otimes ev_{\bullet}^*(\phi_a) \otimes \left(\frac{\mathrm{tr}_{\mathbb{C}^*}\mathcal{D}^{R,l}}{\lambda^{\mathbb{C}^*}_{-1}N^{\vee}_{\mathrm{F}_{0,\beta}}}\right)\right)\phi^a \\
&\quad + \sum_a \sum_{\substack{n\geq 1 or \beta(L_\theta)\geq \frac{1}{\epsilon} \\ (n,\beta)\neq(1,0)}} Q^{\beta}\left\langle \frac{\phi_a}{(1-q)(1-qL)}, \mathbf{t}(L),\dots,\mathbf{t}(L)\right\rangle^{R,l,\epsilon,S_n}_{0,n+1,\beta}\phi^a,
\end{aligned}$$

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where $\{\phi_\alpha\}$ is a basis of $K^0(V//G)$ and $\{\phi^\alpha\}$ is the dual basis with respect to twisted pairing $(\ ,\ )^{R,l}$: that is,

$$(u,v)^{R,l} := \chi\Big(X, u\otimes v\otimes \det{}^{-l}(V^{ss}\times_G R)\Big).$$

Definition 3.2 ([14].). When taking ϵ small enough, denoted by $\epsilon = 0^+$, we call $\mathcal{J}^{R,l,0^+}(0)$ the small I-function of level l: that is,

$$I^{R,l}(q;Q) := \mathcal{J}^{R,l,0^+}_{S_\infty}(0,Q) = 1 + \sum_{\beta\geq 0} Q^\beta (ev_\bullet)_*\left(\mathcal{O}^{\mathrm{vir}}_{\mathrm{F}_{0,\beta}} \otimes \left(\frac{\mathrm{tr}_{\mathbb{C}^*}\,\mathcal{D}^{R,l}}{\lambda^{\mathbb{C}^*}_{-1} N^\vee_{\mathrm{F}_{0,\beta}}}\right)\right)\cdot \det{}^l(V^{ss}\times_G R).$$

3.2. *Level correspondence in Grassmann duality*

Let V be $r\times n$ matrixes $M_{r\times n}$, G be the general linear group GL_r and θ be the $\mathbf{det} : GL_r \to \mathbb{C}^*$. Then we have

$$V//_{\mathbf{det}}G = M_{r\times n}//_{\mathbf{det}}G = Gr(r,n).$$

There is a natural $T = (\mathbb{C}^*)^n$-action on $\mathbb{C}^n$ with weights $\mathbb{C}^n = \Lambda_1+\cdots+\Lambda_n$. Then deducing an action on $Gr(r,n)$ by $T\cdot A = AT$, $A\in M_{r\times n}$. Using general abelian/nonabelian correspondence in [20] for $Gr(r,n)$, we have

$$\begin{aligned} I_T^{Gr(r,n)} =& 1 + \sum_d \sum_{|\vec{d}|=d} \sum_{\omega\in S_r/S_{r_1}\times\cdots\times S_{r_{h+1}}} \\ & \omega\left[\frac{\prod_{1\leqslant j<i\leqslant r}\prod_{1\leqslant m\leqslant d_i-d_j}\left(1-L_iL_j^{-1}q^m\right)}{\prod_{\substack{1\leqslant i<j\leqslant r_j\\ 1\leqslant m\leqslant d_j-d_i-1}}\left(1-L_iL_j^{-1}q^{-m}\right)\prod_{1\leqslant i<j\leqslant r}\left(1-L_i^{-1}L_j\right)}\prod_{i=1}^r\prod_{k=1}^{d_i}\prod_{m=1}^n\frac{1}{(1-q^kL_i\Lambda_m^{-1})}\right]Q^d, \end{aligned}$$

where $\vec{d} = \{d_1\leq d_2\leq\cdots\leq d_r\}$ such that $d_1 = d_2 = \cdots = d_{r_1} < d_{r_1+1} = \cdots = d_{r_1+r_2} < d_{r_1+\cdots+r_h}\cdots = d_{r_1+\cdots+r_h+r_{h+1}}$: that is, $r_1+\cdots+r_{h+1} = r$. ω is the Weyl group acting on L_i to change the index, $\{L_i\}_{i=1}^r$ come from the filtration of tautological bundle $\mathcal{S}_r$ of $Gr(r,n)$. We could rewrite the equivariant I-function in the following way

$$I_T^{Gr(r,n)} = 1 + \sum_d\sum_{|\vec{d}|=d}\sum_{\omega\in S_r/S_{r_1}\times\cdots\times S_{r_{h+1}}}\omega\left[\prod_{i,j=1}^r\frac{\prod_{k=-\infty}^{d_i-d_j}(1-q^kL_iL_j^{-1})}{\prod_{k=-\infty}^{0}(1-q^kL_iL_j^{-1})}\prod_{i=1}^r\prod_{k=1}^{d_i}\prod_{m=1}^n\frac{1}{(1-q^kL_i\Lambda_m^{-1})}\right]Q^d. \tag{3.2}$$

Suppose ω changes i_1 to i_2 and j_1 to j_2. Then one of the factors changes from

$$\frac{\prod_{k=-\infty}^{d_{i_1}-d_{j_1}}(1-q^kL_{i_1}L_{j_1}^{-1})}{\prod_{k=-\infty}^{0}(1-q^kL_{i_1}L_{j_1}^{-1})}\cdot\frac{\prod_{k=-\infty}^{d_{i_2}-d_{j_2}}(1-q^kL_{i_2}L_{j_2}^{-1})}{\prod_{k=-\infty}^{0}(1-q^kL_{i_2}L_{j_2}^{-1})}, \tag{3.3}$$

to

$$\frac{\prod_{k=-\infty}^{d_{i_1}-d_{j_1}}(1-q^kL_{i_2}L_{j_2}^{-1})}{\prod_{k=-\infty}^{0}(1-q^kL_{i_2}L_{j_2}^{-1})}\cdot\frac{\prod_{k=-\infty}^{d_{i_2}-d_{j_2}}(1-q^kL_{i_1}L_{j_1}^{-1})}{\prod_{k=-\infty}^{0}(1-q^kL_{i_1}L_{j_1}^{-1})}. \tag{3.4}$$

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Since $\omega \in S_r/S_{r_1} \times \cdots \times S_{r_{h+1}}$, we have $d_{i_1} \neq d_{i_2}, d_{j_1} \neq d_{j_2}$. In equation (3.2), we have an order of partition $\vec{d}$; one can see from equation (3.3) to equation (3.4) that ω-action is just $\{d_i\}$ rearranged without changing the form. There is a unique $\omega \in S_r/(S_{r_1} \times \ldots \times S_{r_{h+1}})$ whose inverse ω^{-1} arranges $(d_1, \ldots, d_r)$ in nondecreasing order $d_1 \leq d_2 \leq \ldots \leq d_r$. Then we have

$$I_T^{Gr(r,n)} = \sum_d \sum_{d_1+d_2+\cdots+d_r=d} Q^d \prod_{i,j=1}^r \frac{\prod_{k=-\infty}^{d_i-d_j}(1-q^k L_i L_j^{-1})}{\prod_{k=-\infty}^{0}(1-q^k L_i L_j^{-1})} \prod_{i=1}^r \prod_{k=1}^{d_i} \prod_{m=1}^n \frac{1}{(1-q^k L_i \Lambda_m^{-1})}.$$

Note that in [15], the author claimed a version of the mirror theorem with a different I-function.

If we consider the standard representation of GL_r, denoted by E_r, then the associated bundle $\mathcal{P} \times_G R|_{F_{0,\beta}}$ can be identified with $\oplus_{i=1}^r L_i \otimes \mathcal{O}_{\mathbb{P}^1}(-d_i)$

$$\begin{aligned}
\mathcal{D}^{E_r,l}|_{F_{0,\beta}} &= \det{}^{-l} R^\bullet \pi_*(\oplus_{i=1}^r L_i \otimes \mathcal{O}_{\mathbb{P}^1}(-d_i)) \\
&= \det{}^{-l}(\oplus_{i=1}^r [L_i \otimes R^1\pi_*(\mathcal{O}_{\mathbb{P}^1}(-d_i))]^{-1}) \\
&= \otimes_{i=1}^r \left(L_i^{d_i-1} \cdot q^{\frac{d_i(d_i-1)}{2}}\right)^l.
\end{aligned}$$

Similarly, if we take a dual standard representation, denoted by $E_r^\vee$, then

$$\begin{aligned}
\mathcal{D}^{E_r^\vee,l}|_{F_{0,\beta}} &= \det{}^{-l}(\oplus_{i=1}^r L_i^{-1} \otimes R^0\pi_*(\mathcal{O}_{\mathbb{P}^1}(d_i))) \\
&= \otimes_{i=1}^r \left(L_i^{d_i+1} \cdot q^{\frac{d_i(d_i+1)}{2}}\right)^l.
\end{aligned}$$

So the equivariant I-function of $Gr(r,n)$ with a level structure is as follows:

$$I_{T,d}^{Gr(r,n),E_r,l} = \sum_{d_1+d_2+\cdots+d_r=d} Q^d \prod_{i,j=1}^r \frac{\prod_{k=-\infty}^{d_i-d_j}(1-q^k L_i L_j^{-1})}{\prod_{k=-\infty}^{0}(1-q^k L_i L_j^{-1})} \prod_{i=1}^r \frac{(L_i^{d_i} q^{\frac{d_i(d_i-1)}{2}})^l}{\prod_{k=1}^{d_i}\prod_{m=1}^n (1-q^k L_i \Lambda_m^{-1})}, \tag{3.5}$$

and

$$I_{T,d}^{Gr(r,n),E_r^\vee,l} = \sum_{d_1+d_2+\cdots+d_r=d} Q^d \prod_{i,j=1}^r \frac{\prod_{k=-\infty}^{d_i-d_j}(1-q^k L_i L_j^{-1})}{\prod_{k=-\infty}^{0}(1-q^k L_i L_j^{-1})} \prod_{i=1}^r \frac{(L_i^{d_i} q^{\frac{d_i(d_i+1)}{2}})^l}{\prod_{k=1}^{d_i}\prod_{m=1}^n (1-q^k L_i \Lambda_m^{-1})}. \tag{3.6}$$

Remark. For the dual Grassmannian $Gr(n-r,n)$, the $(\mathbb{C}^*)^n$-action on $\mathbb{C}^n$ is the dual action, so the weights are $\mathbb{C}^n = \Lambda_1^{-1} + \cdots + \Lambda_n^{-1}$. The deduced action on $Gr(n-r,n)$ is as follows: $T \cdot B = BT$, $B \in M_{n-r\times n}$. So the corresponding equivariant I-function is as follows

$$I_{T,d}^{Gr(n-r,n),E_{n-r},l} = \sum_{d_1+d_2+\cdots+d_{n-r}=d} Q^d \prod_{i,j=1}^{n-r} \frac{\prod_{k=-\infty}^{d_i-d_j}(1-q^k \tilde{L}_i \tilde{L}_j^{-1})}{\prod_{k=-\infty}^{0}(1-q^k \tilde{L}_i \tilde{L}_j^{-1})} \prod_{i=1}^{n-r} \frac{(\tilde{L}_i^{d_i} q^{\frac{d_i(d_i-1)}{2}})^l}{\prod_{k=1}^{d_i}\prod_{m=1}^n (1-q^k \tilde{L}_i \Lambda_m)}$$

and

$$I_{T,d}^{Gr(n-r,n),E_{n-r}^\vee,l} = \sum_{d_1+d_2+\cdots+d_{n-r}=d} Q^d \prod_{i,j=1}^{n-r} \frac{\prod_{k=-\infty}^{d_i-d_j}(1-q^k \tilde{L}_i \tilde{L}_j^{-1})}{\prod_{k=-\infty}^{0}(1-q^k \tilde{L}_i \tilde{L}_j^{-1})} \prod_{i=1}^{n-r} \frac{(\tilde{L}_i^{d_i} q^{\frac{d_i(d_i+1)}{2}})^l}{\prod_{k=1}^{d_i}\prod_{m=1}^n (1-q^k \tilde{L}_i \Lambda_m)},$$

where $\tilde{L}_i$ for $i = 1, \ldots, n-r$ come from the filtration of tautological bundle $\mathcal{S}_{n-r}$ over $Gr(n-r,n)$.

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Let T act on the Grassmannian $Gr(r,n)$ as before. Then there are $\binom{n}{r}$ fixed points: that is, denoted by $\{e_1,\ldots,e_n\}$, the basis of $\mathbb{C}^n$. Then the subspace V spanned by $\{e_{i_1},\ldots,e_{i_r}\}$ is a T-fixed point for $\{i_1,\cdots,i_r\}\subset[n]$. Let

$$\imath_*: K_T\left(Gr(r,n)^T\right)\to K_T(Gr(r,n))$$

be the map induced from the close embedding $\imath: Gr(r,n)^T\hookrightarrow Gr(r,n)$. The kernel and cokernel are $K_T(pt)$-modules and have some support in the torus T. From a very general localisation theorem of Thomason [16], we know

$$\text{supp Coker }\imath_*\subset\bigcup_{\mu}\{\mathrm{t}^{\mu}=1\},$$

where the union is over finitely many nontrivial characters μ. The same is true of $\ker\imath_*$, but since

$$K_T\left(Gr(r,n)^T\right)=K(Gr(r,n))\otimes_{\mathbb{Z}}K_T(pt)$$

has no such torsion, this forces $\ker\imath_*=0$, so after inverting finitely many coefficients of the form $t^{\mu}-1$, we obtain an isomorphism: that is,

$$K_T^{loc}(Gr(r,n)^T)\cong K_T^{loc}(Gr(r,n)).$$

We denote $K_T^{loc}(-)$ by

$$K_T^{loc}(-)=K_T(-)\otimes_{R(T)}\mathcal{R},$$

where $\mathcal{R}\cong\mathbb{Q}(t_1,\ldots,t_n)$ and $\{t_i\}$ are the charaters of torus T.

Similarly, $T=(\mathbb{C}^*)^n$-action on $Gr(n-r,n)$ also has $\binom{n}{n-r}=\binom{n}{r}$ isolated fixed points, which are indexed by $(n-r)$-element subsets of $[n]$, so identification of $Gr(r,n)^T$ with $Gr(n-r,n)^T$ gives an $\mathcal{R}$-module isomorphism of $K_T^{loc}(Gr(r,n))$ with $K_T^{loc}(Gr(n-r,n))$. Indeed, suppose W is a subspace of dimension r in a vector space V of dimension n. Then we have a natural short exact sequence

$$0\to W\to V\to V/W\to 0.$$

Taking the dual of this short exact sequence yields an inclusion of $(V/W)^*$ in V^* with quotient W^*

$$0\to(V/W)^*\to V^*\to W^*\to 0,$$

so $\psi: W\mapsto(V/W)^*$ gives a cannocial equivariant isomorphism $Gr(r,V)\cong Gr(n-r,V^*)$, where the action of $T=(\mathbb{C}^*)^n$ on V^* is induced from the action of T on V. Thus, ψ gives the canonical identification of fixed points

$$\psi: Gr(r,n)^T\longrightarrow Gr(n-r,n)^T,\qquad <e_j>_{j\in I}\longmapsto<e^j>_{j\in I^{\complement}},\tag{3.7}$$

where I is a set of $[n]$ with $|I|=r$ and $\{e^i\}_{i=1}^n$ is the dual basis of $\{e_i\}_{i=1}^n$. Now we can state the following Level correspondence in Grassmann duality.

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Theorem 3.1 (Level correspondence). *For the Grassmannian $Gr(r,n)$ and its dual Grassmannian $Gr(n-r,n)$ with standard $T=(\mathbb{C}^*)^n$ torus action, let E_r, E_{n-r} be the standard representation of* $\mathrm{GL}(r,\mathbb{C})$ *and* $\mathrm{GL}(n-r,\mathbb{C})$*, respectively. Consider the following equivariant I-function:*

$$I_T^{Gr(r,n),E_r,l} = 1+\sum_{d=1}^{\infty} I_{T,d}^{Gr(r,n),E_r,l} Q^d,$$
$$I_T^{Gr(n-r,n),E_{n-r}^\vee,-l} = 1+\sum_{d=1}^{\infty} I_{T,d}^{Gr(n-r,n),E_{n-r}^\vee,-l} Q^d.$$

Then we have the following relations between $I_{T,d}^{Gr(r,n),E_r,l}$ and $I_{T,d}^{Gr(n-r,n),E_{n-r}^\vee,-l}$ in $K_T^{loc}(Gr(r,n))\otimes \mathbb{C}(q)$ (which equals $K_T^{loc}(Gr(n-r,n))\otimes\mathbb{C}(q)$):

- *For $1-r\le l\le n-r-1$, we have*

$$I_{T,d}^{Gr(r,n),E_r,l} = I_{T,d}^{Gr(n-r,n),E_{n-r}^\vee,-l}.$$

- *For $l=n-r$, we have*

$$I_{T,d}^{Gr(r,n),E_r,l} = \sum_{s=0}^{d} C_s(n-r,d) I_{T,d-s}^{Gr(n-r,n),E_{n-r}^\vee,-l},$$

  where $C_s(k,d)$ is defined as

$$C_s(k,d)=\frac{(-1)^{ks}}{(q;q)_s q^{s(d-s+k)}\left(\bigwedge^{top}\mathcal{S}_{n-r}\right)^s},$$

  and $\mathcal{S}_{n-r}$ is the tautological bundle of $Gr(n-r,n)$.
- *For $l=-r$, we have*

$$I_{T,d}^{Gr(n-r,n),E_{n-r}^\vee,-l} = \sum_{s=0}^{d} D_s(r,d) I_{T,d-s}^{Gr(r,n),E_r,l},$$

  where

$$D_s(r,d)=\frac{(-1)^{rs}}{(q;q)_s q^{s(d-s)}\left(\bigwedge^{top}\mathcal{S}_r\right)^s},$$

  and $\mathcal{S}_r$ is the tautological bundle of $Gr(r,n)$.

Proof. From the discussion above, we prove the above identity by comparing $i_I^* I_T^{E_r,l}$ and $i_{I^\complement}^* I_T^{E_{n-r}^\vee,-l}$; here i_I and $i_{I^\complement}$ are inclusion maps from the corresponding fixed points: that is, we compare two I-functions by restricting them to corresponding fixed points. Let $I=(j_1,\cdots,j_r)$ be the subset of $[n]=\{1,\ldots,n\}$, with $|I|=r$. Denote $v_1,v_2,\cdots,v_r$, the fibre coordinates in the fibre of $\mathcal{S}$ at fixed point $<e_j>_{j\in I}$, $\forall(t_1,\cdots,t_n)\in(\mathbb{C}^*)^n$, with weights $\mathbb{C}^n=\Lambda_1+\cdots+\Lambda_n$ and

$$\begin{aligned}&(t_1,\cdots,t_n)\cdot(e_{j_1},\cdots,e_{j_r};v_1,v_2,\cdots,v_r)=(t_{j_1}e_{j_1},\cdots,t_{j_r}e_{j_r};v_1,v_2,\cdots,v_r)\\ &\sim \mathrm{diag}(t_{j_1},\cdots,t_{j_r})\cdot(t_{j_1}e_{j_1},\cdots,t_{j_r}e_{j_r};v_1,v_2,\cdots,v_r)=(e_{j_1},\cdots,e_{j_r};t_{j_1}v_1,t_{j_2}v_2,\cdots,t_{j_r}v_r).\end{aligned}$$

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So the weights of $i_I^*\mathcal{S}_r$ are $\{\Lambda_i\}_{i\in I}$ and the weights of $i_{I^\complement}^*\mathcal{S}_{n-r}$ are $\{\Lambda_i^{-1}\}_{i\in I^\complement}$. Since the I-function is symmetric with respect to $\{L_i\}$, we can take any choice of weights

$$i_I^* I_{T,d}^{Gr(r,n),E_r,l} = \sum_{|\vec{d}_I|=d} \prod_{i,j\in I} \frac{\prod_{k=-\infty}^{d_i-d_j}(1-q^k\Lambda_i\Lambda_j^{-1})}{\prod_{k=-\infty}^{0}(1-q^k\Lambda_i\Lambda_j^{-1})} \prod_{i\in I} \frac{(\Lambda_i^{d_i} q^{\frac{d_i(d_i-1)}{2}})^l}{\prod_{k=1}^{d_i}\prod_{m\in[n]}(1-q^k\Lambda_i\Lambda_m^{-1})},$$

and

$$i_{I^\complement}^* I_{T,d}^{Gr(n-r,n),E_{n-r}^\vee,-l} = \sum_{|\vec{d}_{I^\complement}|=d} \prod_{i,j\in I^\complement} \frac{\prod_{k=-\infty}^{d_i-d_j}(1-q^k\Lambda_i^{-1}\Lambda_j)}{\prod_{k=-\infty}^{0}(1-q^k\Lambda_i^{-1}\Lambda_j)} \prod_{i\in I^\complement} \frac{(\Lambda_i^{-d_i} q^{\frac{d_i(d_i+1)}{2}})^{-l}}{\prod_{k=1}^{d_i}\prod_{m\in[n]}(1-q^k\Lambda_i^{-1}\Lambda_m)}.$$

Using notation $\Lambda_{ij}=\Lambda_i\Lambda_j^{-1}$ and the following Lemma 3.2, we obtain

$$i_I^* I_{T,d}^{Gr(r,n),E_r,l} = \sum_{|\vec{d}_I|=d} \prod_{i\in I}\left(\frac{1}{\prod_{j\in I}(q^{d_i-d_j+1}\Lambda_{ij};q)_{d_j}} \cdot \frac{(\Lambda_i^{d_i} q^{\frac{d_i(d_i-1)}{2}})^l}{\prod_{j\in I^\complement}(q\Lambda_{ij};q)_{d_i}}\right), \tag{3.8}$$

and

$$i_{I^\complement}^* I_{T,d}^{Gr(n-r,n),E_{n-r}^\vee,-l} = \sum_{|\vec{d}_{I^\complement}|=d} \prod_{i\in I^\complement}\left(\frac{1}{\prod_{j\in I^\complement}(q^{d_i-d_j+1}\Lambda_{ji};q)_{d_j}} \cdot \frac{(\Lambda_i^{d_i} q^{\frac{-d_i(d_i+1)}{2}})^l}{\prod_{j\in I}(q\Lambda_{ji};q)_{d_i}}\right). \tag{3.9}$$

Comparing equations (3.8) and (3.9) with equations (2.24) and (2.25), we obtain the conclusion. □

Lemma 3.2. *Let I be the subset of $[n]=\{1,\ldots,n\}$. We have*

$$\prod_{i,j\in I}\left(\frac{\prod_{k=-\infty}^{d_{ij}}(1-q^k x_{ij})}{\prod_{k=-\infty}^{0}(1-q^k x_{ij})}\frac{1}{\prod_{k=1}^{d_i}(1-q^k x_{ij})}\right) = \prod_{i,j\in I}\frac{1}{(q^{d_{ij}+1}x_{ij};q)_{d_j}}.$$

Proof. It is sufficient to consider one term. If $d_i \geq d_j$, then

$$LHS = \frac{\prod_{k=1}^{d_{ij}}(1-q^k x_{ij})}{\prod_{k=1}^{d_i}(1-q^k x_{ij})} = \frac{1}{\prod_{k=d_{ij}+1}^{d_i}(1-q^k x_{ij})} = RHS.$$

If $d_i \leq d_j$, then

$$LHS = \frac{1}{\prod_{k=d_{ij}+1}^{0}(1-q^k x_{ij})\prod_{k=1}^{d_i}(1-q^k x_{ij})} = \frac{1}{\prod_{k=d_{ij}+1}^{d_i}(1-q^k x_{ij})} = RHS.$$

□

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