RESTRICTED PARTITIONS OF FINITE SETS

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Introduction. In this paper we consider the following combinatorial problem. In how many ways can n distinguishable objects be placed into an unrestricted number of indistinguishable boxes, if each box can hold at most r objects? Let us denote this number by $G_{n,r}$.

Special cases of this problem have been the object of considerable study. In the case $r = 2$ we have the numbers $G_{n,2} = T_n$ which have been treated by Rothe [12] as early as 1800. T_n is also the number of solutions of $x^2 = 1$ in the symmetric group on n letters, and in this and related guises has been studied by Touchard [13], Chowla, Herstein and Moore [3] and two of the present authors [7].

The case $G_{n,n} = G_n$ has received even more attention. Indeed, these numbers have been the subject of a recent Master's thesis of Finlayson [4], who lists over fifty references dealing with these numbers. Two of the present authors have recently published a complete asymptotic expansion for these numbers [8]. We take this opportunity to acknowledge that the first term of this expansion was obtained earlier, though without a rigorous justification given, in a paper by Gernuschi and Castegnetto [2]. We have determined tables of G_n up to $n = 51$, but have not included them in this paper. Tables of G_n up to $n = 50$ have been published earlier by Gupta [5].

The case of $G_{n,r}$ for arbitrary r has been treated briefly by Hadwiger [6]. Becker has informed the authors that he has considered these numbers, together with many related sets of numbers, in connection with his general theory of rhyme [1]. We believe however that many of the results of the present paper are new.

In §1 we derive several recurrence formulae for $G_{n,r}$ and an exponential generating function. In §2 we obtain some congruences for $G_{n,r}$, useful in checking tables of these

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numbers. In §3 we derive an asymptotic formula for $G_{n,r}$, valid for r fixed and n tending to infinity. In §4 we deal with some related numbers and polynomials.

1. Recurrence Formulae. By the elementary theory of combinatory analysis it is easily seen that the number of ways of placing n distinguishable objects into indistinguishable boxes, with a_1 boxes containing one object each, a_2 boxes containing two objects each, ..., a_r boxes containing r objects each, is

$$n!/(1!^{a_1}2!^{a_2}\ldots r!^{a_r}\, a_1!a_2!\ldots a_r!) \quad . \tag{1.1}$$

Hence we have the following explicit formula for $G_{n,r}$.

THEOREM 1.

$$G_{n,r} = \sum n!/(1!^{a_1}2!^{a_2}\ldots r!^{a_r}a_1!a_2!\ldots a_r!)$$

where the summation is taken over all $a_1, a_2, \ldots a_r$ such that $1a_1 + 2a_2 + \ldots + ra_r = n$.

Although this expression can be used to calculate $G_{n,r}$ for r and n small, it is clearly not very useful for large r and n.

To obtain a recurrence formula for $G_{n,r}$ we observe that if we distribute n + 1 objects and specialize one of these, then this one can go together with s others $(0 \leq s \leq r - 1)$ in $\binom{n}{s}$ ways and the remaining n-s objects can be distributed in $G_{n-s,r}$ ways. Summing over s yields the recurrence given in

THEOREM 2. $G_{n,1} = 1$,

$$G_{n+1,r} = G_{n,r} + \binom{n}{1}G_{n-1,r} + \binom{n}{2}G_{n-2,r} + \ldots + \binom{n}{r-1}G_{n-r+1,r}.$$

From Theorem 2 an exponential generating function for $G_{n,r}$ is easily obtained. Indeed we have

THEOREM 3.

$$\sum_{n=0}^{\infty} G_{n,r}x^n/n! = \exp(x + x^2/2! + \ldots + x^r/r!) \quad .$$

Proof. Differentiating both sides of the above with respect to x and comparing coefficients of x^n yields the recurrence of Theorem 2. Since the coefficients of $x^n/n!$ in the expansion of

e^x are all 1, the identification of the $G_{n,r}$ in Theorems 2 and 3 is complete.

We now obtain a second recurrence formula for $G_{n,r}$, namely

THEOREM 4.

$$G_{n,r} = \sum_{s=0}^{[n/r]} \frac{n!}{s!(r!)^s(n-rs)!} G_{n-rs,r-1}$$

Proof. In the distributions of n objects into boxes we isolate those distributions in which s boxes are full, i.e. contain r elements each. These s boxes, $(0 \leq s \leq n/r)$, can be filled in

(1.2) $$\frac{1}{s!} \binom{n}{r} \binom{n-r}{r} \cdots \binom{n-rs}{r} = \frac{n!}{s!(r!)^s(n-rs)!}$$

ways, and the remaining n-rs objects disposed of in $G_{n-rs,r-1}$ ways. Summing over s yields the required result.

Theorem 4 is particularly useful in case r is large in comparison with n. Since $G_{n,r} = G_{n,n} = G_n$ for $r > n$ we have in particular,

THEOREM 5.

$$G_{n,r} = G_{n,r-1} + \binom{n}{r} G_{n-r} \qquad (r > [\tfrac{1}{2}n]).$$

This recurrence can be iterated to obtain a managable explicit formula for the case $r > [\frac{1}{2}n]$, namely,

THEOREM 6.

$$G_{n,r} = G_n - \binom{n}{r+1}G_{n-r-1} - \binom{n}{r+2}G_{n-r-2} - \cdots - \binom{n}{n}G_0, \quad r > [\tfrac{1}{2}n].$$

This is essentially an explicit formula since it is well known [8] that

(1.3) $$G_n = \sum_{s=1}^{n} \frac{(-1)^r}{s!} \sum_{i=1}^{s} (-1)^i \binom{s}{i} i^n .$$

2. Congruences. In order to check tables of $G_{n,r}$ it is useful to have congruence relations for these numbers. We first prove

THEOREM 7. For p a prime and $r < p$,

$$G_{p,r} \equiv 1 \pmod{p} \quad .$$

Proof. This follows from Theorem 1 with $n = p$, since in this case all the terms of the sum with the exception of the one corresponding to $p = 1 + 1 + \ldots + 1$ will be divisible by p.

We can obtain another proof of this result using the following representation of a distribution. Represent the n objects by n points at the vertices of a regular convex n-gon. If a set of k objects go into one box, then join the corresponding k points by line segments to form a convex k-gon. Now, if $n = p$, all the distributions except the one where every object is in a box by itself, will come in sets of p, since the corresponding geometric configurations come in sets of p by rotation through multiples of $2\pi/p$.

We next use induction over n to prove

THEOREM 8. For p a prime and $r < p$,

$$G_{n+p,r} \equiv G_{n,r} \pmod{p} \quad .$$

Proof. Since $G_{0,r} = G_0 = 1$, the case $n=0$ is simply Theorem 7. Suppose now the theorem is true for $k \leq n$. By Theorem 2,

$$(2.1) \quad G_{n+1+p,r} = G_{n+p,r} + \binom{n+p}{1} G_{n+p-1,r} + \ldots + \binom{n+p}{r-1} G_{n+p-r+1,r}.$$

Using

$$(2.2) \quad \binom{n+p}{k} \equiv \binom{n}{k} \pmod{p} \qquad (k < p)$$

and the induction hypothesis we obtain

$$(2.3) \quad G_{n+1+p,r} \equiv G_{n,r} + \binom{n}{1} G_{n-1,r} + \ldots + \binom{n}{r-1} G_{n-r+1,r}$$

$$\equiv G_{n+1,r} \pmod{p} \quad ,$$

and the induction is complete.

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3. Asymptotic Expansions. Let $a_1, a_2, \ldots, a_m$ be a set of real non-negative numbers and let

$$P(x) = a_1 x + a_2 x^2 + \ldots + a_m x^m \qquad , \; a_m \neq 0 .$$

Let R be the positive number determined by

$$(3.1) \quad \sum_{k=1}^{m} k a_k R^k \quad ,$$

and let Θ be the operator defined by

$$(3.2) \quad \Theta = R \frac{d}{dR} \quad .$$

In a previous paper [10] two of the present authors proved that if

$$(3.3) \quad \sum_{n=0}^{\infty} B_n x^n / n! = e^{P(x)}$$

then

$$(3.4) \quad B_n \sim \frac{n! \, e^{P(R)}}{R^n} \left[\frac{1}{2\pi \Theta^2 P(R)} \right]^{\frac{1}{2}}$$

This result is immediately applicable to the problem of finding an asymptotic expansion for $G_{n,r}$. Indeed, using this result and Theorem 3, we obtain

THEOREM 9. For r fixed and $n \longrightarrow \infty$

$$G_{n,r} \sim \frac{n! \exp(R/1! + R^2/2! + \ldots + R^r/r!)}{R^n \left[(2\pi)(1^2 R/1! + 2^2 R^2/2! + \ldots + r^2 R^r/r!)\right]^{\frac{1}{2}}}$$

where R is the positive number determined by

$$(3.5) \qquad R + R^2/1! + R^3/2! + \ldots + R^r/(r-1)! = n.$$

This asymptotic formula can be simplified in various ways. In particular using the definition of R we have

$$(3.6) \qquad R/1! + R^2/2! + \ldots + R^r/r! = n/R + R^r/r! - 1$$

$$(3.7) \qquad 1^2 R/1! + 2^2 R^2/2! + \ldots + r^2 R^r/r! \sim nr$$

and by Stirling's formula

(3.8) $n! \sim n^{n+\frac{1}{2}} e^{-n} (2\pi)^{\frac{1}{2}}$.

Combining these estimates with Theorem 9 gives

THEOREM 10. For r fixed and $n \to \infty$

$$G_{n,r} \sim (n/R)^n r^{-\frac{1}{2}} \exp(n/R + R^r/r! - n - 1)$$

where R is determined by (3.5) .

If r is quite small the equation for R can be conveniently solved by iteration starting with $R = (n(r-1)!)^{1/r}$ while if r is fairly large it is convenient to begin the iteration with an approximate solution to $Re^R = n$.

4. The polynomials $G_{n,r}(t)$. Following Hadwiger [6] , let us define a set of polynomials $G_{n,r}(t)$ by

(4.1) $$\sum_{n=0}^{\infty} G_{n,r}(t)x^n/n! = \exp(t(x + x^2/2! + \dots + x^r/r!)) ,$$

$$G_{n,n}(t) = G_n(t),$$

and let $\sigma^s_{n,r}$ be defined by

(4.2) $$G_{n,r}(t) = \sum_{s=1}^{n} \sigma^s_{n,r} t^s .$$

Clearly

(4.3) $$G_{n,r}(1) = \sum_{s=1}^{n} \sigma^s_{n,r} = G_{n,r}$$

Also, comparing coefficients of t in (4.1) and using (4.2), we obtain

(4.4) $$\sum_{n=0}^{\infty} \sigma^s_{n,r} x^n/n! = (x + x^2/2! + \dots + x^r/r!)^s .$$

From (4.4) it follows that $\sigma^s_{n,r}$ is the number of ways of placing n distinguishable objects into s indistinguishable boxes, where no box is to contain more than r objects. The $\sigma^s_{n,r}$ are thus generalizations of the well known Stirling numbers of the second kind σ^s_n with $\sigma^s_{n,n} = \sigma^s_n$.

We now obtain some recurrence formulae for the $G_{n,r}(t)$ and the $\sigma^s_{n,r}$. Differentiating (4.1) with respect to x and equating coefficients of x yields

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THEOREM 11.

$$G_{n+1,r}(t) = t(G_{n,r}(t) + \binom{n}{1} G_{n-1,r}(t) + \ldots + \binom{n}{r-1} G_{n-r+1,r}(t)) .$$

On the other hand, differentiation of (4.1) with respect to t yields

$$(4.5)\quad \frac{d}{dt} G_{n,r}(t) = \binom{n}{1} G_{n-1,r}(t) + \binom{n}{2} G_{n-2,r}(t) + \ldots + \binom{n}{r} G_{n-r,r}(t).$$

Combining (4.5) with Theorem 11, we obtain a three term recurrence formula for $G_{n,r}(t)$, namely,

THEOREM 12.

$$G_{n+1,r}(t) = tG_{n,r}(t) + \frac{d}{dt} G_{n,r}(t) - t\binom{n}{r} G_{n-r,r}(t).$$

A generalization of Theorem 4 can be obtained as follows.

$$(4.6)\quad \sum_{n=0}^{\infty} G_{n,r}(t)x^n/n! = \exp(t(x + x^2/2! + \ldots + x^{r-1}/(r-1)!))\exp(tx^r/r!)$$
$$= \sum G_{n,r-1}(t)\, x^n/n!\ (1 + tx^r/r! + t^2x^{2r}/2!\,r!^2 + \ldots).$$

Comparing coefficients of x^n yields

THEOREM 13.

$$G_{n,r}(t) = \sum_{s=0}^{[n/r]} \frac{n!}{s!\,(r!)^s(n-rs)!} G_{n-rs,r-1}(t)\ t^s .$$

In case $r > \left[\frac{1}{2}n\right]$ we can easily derive from this generalizations of Theorems 5 and 6, namely

THEOREM 14.

$$G_{n,r}(t) = G_{n,r-1}(t) + \binom{n}{r} G_{n-r}(t), \qquad r > \left[\tfrac{1}{2}n\right] ,$$

THEOREM 15.

$$G_{n,r}(t) = G_n(t) - t\sum_{s=r+1}^{n} \binom{n}{s} G_{n-s}(t), \quad r > \left[\tfrac{1}{2}n\right] .$$

Equating coefficients of t^s in Theorem 12 yields a three term recurrence for the $\sigma_{n,r}^{s}$, namely

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THEOREM 16.

$$\sigma^{s}_{n+1,r} = s\sigma^{s}_{n,r} + \sigma^{s-1}_{n,r} - \binom{n}{r}\sigma^{s-1}_{n-r,r} \qquad (1 \leq s \leq n).$$

This result may also be obtained by a direct combinatorial argument.

From (4.4) we have

(4.7) $$\sum_{n=0}^{\infty} \sigma^{s}_{n,r} x^n/n! = (x + x^2/2! + \ldots + x^r/r!)^{s-1}(x + x^2/2! + \ldots x^r/r!) = \sum_{n=0}^{\infty} \sigma^{s-1}_{n,r} x^n/n!\,(x + x^2/2! + \ldots + x^r/r!) \ .$$

Equating coefficients in (4.5) yields

THEOREM 17.

$$\sigma^{s}_{n,r} = \frac{1}{s} \sum_{i=\max(n-r,s)}^{\min(n-1,rs)} \binom{n}{i} \sigma^{s-1}_{i,r}$$

This will be valid for all $n, r, s \geq 1$ if we adopt the convention that the summation is zero whenever $\max(n-r,s) > \min(n-1, rs)$.

Using 4.4 and Theorem 17, a few cases of $\sigma_{n,r}$ can easily be computed. For example, we have obtained

(4.8) $$\sigma^{1}_{n,r} = 1 \quad \text{if } 1 \leq n \leq r,$$

$$\sigma^{1}_{n,r} = 0 \quad \text{if } n > r,$$

(4.9) $$\sigma^{2}_{n,r} = \frac{1}{2} \sum_{i=\max(n-r,1)}^{\min(n-1,r)} \binom{n}{i},$$

(4.10) $$\sigma^{n}_{n,r} = 1,$$

(4.11) $$\sigma^{n-1}_{n,r} = \frac{n(n-1)}{2} \qquad (r > 1),$$

and

(4.12) $$\sigma^{n-2}_{n,r} = \frac{n(n-1)(n-2)(3n-5)}{24} \qquad (r > 2).$$

Finally, Theorem 17 can be iterated to give an explicit expression.

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THEOREM 18.

$$\sigma^{s+1}_{n,r} = \frac{1}{(s+1)!} \sum_{q_1} \sum_{q_2} \sum_{q_3} \cdots \sum_{q_s} \binom{n}{q_1}\binom{q_1}{q_2}\binom{q_2}{q_3} \cdots \binom{q_{s-1}}{q_s}$$

where the summations are taken over the ranges

$$\max(n-r, s) \leq q_1 \leq \min(n-1, rs)$$

$$\max(q_1-r, s-1) \leq q_2 \leq \min(q_1-1, r(s-1))$$

$$\max(q_2-r, s-2) \leq q_3 \leq \min(q_2-1, r(s-2))$$

$$\vdots$$

$$\max(q_{s-1}-r, 1) \leq q_s \leq \min(q_{s-1}-1, r).$$

For example:

$$\sigma^{3}_{6,4} = \frac{1}{3!} \sum_{q_1=2}^{5} \sum_{q_2=2}^{q_1-1} \binom{6}{q_1}\binom{q_1}{q_2} = 90 .$$

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