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ON THE THREE-SPACE PROBLEM FOR THE DUNFORD-PETTIS PROPERTY

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A Banach space X is called a twisted sum of the Banach spaces Y and Z if it has a subspace isomorphic to Y in such a way that the corresponding quotient is isomorphic to Z. In this paper we study twisted sums of Banach spaces with either have the Dunford-Pettis property, are c_0-saturated or l_1-saturated. Amongst other things, we show that every Banach space is a complemented subspace of a twisted sum of two Banach spaces with the Dunford-Pettis property.

1. INTRODUCTION

Let $\mathcal{P}$ and $\mathcal{Q}$ be two properties of Banach spaces stable by isomorphisms. Following [7], a Banach space X is said to have the $\mathcal{P}$-by-$\mathcal{Q}$ property if it admits a subspace Y with property $\mathcal{P}$ so that X/Y has property $\mathcal{Q}$. In such case, we shall also say that X is a twisted sum of Y and Z (in this order). If Y is complemented in X we say that it is a trivial twisted sum. A property $\mathcal{P}$ is said to be a three-space property (3-space property in short) if $\mathcal{P}$-by-$\mathcal{P}$ implies $\mathcal{P}$. The monograph [5] contains rather complete information about 3-space problems in Banach spaces.

In the present paper we are interested in the choices

$$\mathcal{P}, \mathcal{Q} \in \{\text{Dunford-Pettis, } c_0\text{-saturation, } l_1\text{-saturation}\}.$$

Previously known results are that the Dunford-Pettis property is not a 3-space property, while c_0-saturation, l_1-saturation and the Schur property are well-known 3-space properties (see [5]). Let us briefly recall the meaning of those properties. The Dunford-Pettis property (DPP for short) means that given two weakly null sequences (x_n) and (f_n) in X and X^*, respectively, then $\lim_n f_n(x_n) = 0$. It is an important property of Banach spaces, notwithstanding the fact that very few examples of spaces with this property are known. The main examples are the $C(K)$ and $L_1(\mu)$-spaces; or, more generally, the $\mathcal{L}_\infty$

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and $\mathcal{L}_1$ spaces. Even more scarce are those spaces such that every closed subspace has DPP. They are called spaces with the hereditary Dunford-Pettis property (DPPh); $c_0(\Gamma)$ and $l_1(\Gamma)$ are perhaps the simplest examples. A Banach space X is said to be c_0-saturated (respectively l_1-saturated) if every closed subspace contains c_0 (respectively l_1). We shall study the implications between these notions in section 4. Now we simply recall that particular cases of DPPh-spaces are the Schur spaces: those in which weakly convergent sequences are norm convergent; that if X^* has the DPP then X has the DPP and that if X has the DPP then X^* is Schur if and only if X does not contain l_1.

2. The twisted Dunford-Pettis property

In [6] it was proved that the Dunford-Pettis property is not a 3-space property, (this was an open problem for some time). The following result shows that the world has plenty of counterexamples.

Theorem 1. *Every Banach space is a complemented subspace of a twisted sum of two Banach spaces with the Dunford-Pettis property.*

Proof: Let X be a Banach space. It is possible to find a compact space K such that X is isomorphic to a subspace of $C(K)$ and then an index set Γ such that $C(K)$ is isomorphic to a quotient of $l_1(\Gamma)$. Let $j : X \to C(K)$ be the isomorphic embedding and $Q : l_1(\Gamma) \to C(K)$ the quotient map. Since $Q^{-1}\big(j(X)\big)$ is a subspace of $l_1(\Gamma)$ it has the Schur property (hence, the hereditary Dunford-Pettis property). If $p \in Q^{-1}\big(j(X)\big)$ there exists a unique $x \in X$ such that $Qp = jx$. Thus, the operator $T : Q^{-1}\big(j(X)\big) \to l_1(\Gamma) \oplus X$ given by $Tp = (p, x)$ is well-defined and an isomorphic embedding. Moreover, the operator $l_1(\Gamma) \oplus X \to C(K)$ given by $(l, x) \to Ql - jx$ is a quotient map whose kernel is precisely $TQ^{-1}\big(j(X)\big)$. ∎

Hence, it makes sense to define the *twisted* Dunford-Pettis property as the property *DPP-by-DPP*. Reflexive spaces do not possess the twisted DPP. There exist c_0-saturated spaces without the twisted Dunford-Pettis property:

Proposition 1. *The Schreier space does not have the twisted DPP.*

Proof: The Schreier space S (see [16]) is c_0-saturated, and every quotient of S is c_0-saturated (see [14]). Thus, if S is the twisted sum of two DPP spaces Y and Z then both Y^* and Z^* would be Schur spaces. This being a 3-space property, also S^* should be a Schur space, which it is not since it fails the DPP. ∎

It would be interesting to know if the twisted DPP and the DPP coincide for c_0-saturated spaces; we remark in passing that quotients of c_0-saturated spaces need not be c_0-saturated (see [12]). As a corollary to the proof of Theorem 1, choosing $X = S$ and $C(K) = C(\omega^\omega)$ we obtain:

Proposition 2. *Schur-by-(DPP and c_0-saturated) does not imply DPP.*

This result shall be completed with Propositions 6 and 7. Exchanging the two properties yields (see [**5**, 6.6.c]).

PROPOSITION 3. *DPP-by-Schur and DPP coincide.*

3. TWISTED SUMS OF c_0-SATURATED AND l_1-SATURATED SPACES

In [**5**, 6.6.d.], it was proved that the DPPh is equivalent to the property that every weakly null sequence admits a weakly 1-summable subsequence, that is, a subsequence satisfying an estimate

$$\sup_{F\in FIN(\mathbb{N})}\left\|\sum_{n\in F}\pm x_n\right\| < +\infty$$

where $FIN(\mathbb{N})$ denotes the set of finite subsets of $\mathbb{N}$. We say that a Banach space is $\{c_0, l_1\}$-saturated if every closed infinite dimensional subspace contains either c_o or l_1. It is clear that Banach spaces with the DPPh are $\{c_0, l_1\}$-saturated. Our first result clarifies the structure of twisted sums of c_0-saturated and l_1-saturated spaces.

PROPOSITION 4. *(c_0-saturated)-by-(l_1-saturated) implies $\{c_0, l_1\}$-saturated; also (l_1-saturated)-by-(c_0-saturated) implies $\{c_0, l_1\}$ saturated.*

PROOF: Let X be a twisted sum of a c_0-saturated space Y_0 and a l_1-saturated space Z_1. Let W be a subspace of X that does not contain c_0. Since Y_0 and W are totally incomparable, $Y_0 + W$ is closed (see [**15**]) in X; therefore W, which is isomorphic to $W/(W \cap Y_0)$, is also isomorphic to $(Y_0 + W)/Y_0$; this is a subspace of Z_1, and hence l_1-saturated. The proof of the other case is entirely analogous and thus we omit it. ∎

We show that the converse does not hold. Our counterexample is modelled upon a transfinite version of Schreier space.

PROPOSITION 5. *DPPh does not imply (c_0-saturated)-by-(l_1-saturated).*

PROOF: Let ω_1 be the first uncountable ordinal. We say that an element $g \in \omega_1^{\omega_1}$, that is, a function $g : \omega_1 \to \omega_1$, is *increasing* if for every countable ordinal $\alpha \in \omega_1$, $\alpha < g(\alpha)$ and whenever $\alpha < \beta$ then $g(\alpha) \leqslant g(\beta)$. Given an increasing element g, a countable subset A of the ordinal interval $[0, \omega_1)$ is said to be g-admissible if $\sup A \leqslant g(\min A)$.

We define the Schreier space S_g as the closure of the elements of $\mathbb{R}^{\omega_1}$ having countable support with respect to the norm

$$\|x\|_g = \sup\left\{\sum_{\alpha\in A}|x_\alpha| : \ A \text{ is } g\text{-admissible}\right\}.$$

The space S_g has the DPPh. Let (x_n) be a weakly null sequence in S_g. If $\alpha < \omega_1$, we denote by $S_g(\alpha)$ the subspace formed by those elements having support contained in $[0, \alpha)$. Since elements in S_g have countable support and ω_1 does not admit countable cofinal subsets, there is some $\Omega < \omega_1$, such that $(x_n) \subset S(\Omega)$. The vectors $(e_\gamma)_{\gamma<\Omega}$ form

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a basis for $S(\Omega)$ and thus by standard perturbation arguments the sequence (x_n) can be considered as formed by blocks (u_n) of $(e_\gamma)_{\gamma<\Omega}$. Now, either there is a sequence (n_j) of integers such that if $B_n =$ support ofu_n then $g\left(\max B_{n_j}\right) < \min B_{n_{j+1}}$ for all j, or there is no such sequence. If such sequence exists then for every $A \subset \mathbb{N}$ one has $\left\|\sum_{j\in A} u_{n_j}\right\|_g = 1$. If not, there is some index n_0 and some subsequence $B_{n_j} \subset \left(\max B_{n_0}, g\left(\max B_{n_0}\right)\right]$, in which case $\left\|\sum \lambda_j u_{n_j}\right\|_g = \sum |\lambda_j|$, and one gets a contradiction.

The space S_g depends on the choice of the initial function g. If $g(\alpha) = \alpha + 1$ then g-admissible sets are just singletons and $S_g = c_0(\omega_1)$, while if one could (it is not an increasing function) choose the constant function $g = \omega_1$ then $S_g = l_1(\omega_1)$. In this way, a proper choice of the function g, for instance $\omega^\alpha < g(\alpha)$, makes some copies of l_1 appear. In what follows we assume that, for all countable ordinals α, $\omega^\alpha < g(\alpha)$. In such case one has:

The space S_g is not a twisted sum of a c_0-saturated space and a l_1-saturated space. Assume that S_g is a twisted sum of a c_0-saturated space Y_0 and a l_1-saturated space Z_1. Let us first remark that since S_g has the DPPh and Y_0 is c_0-saturated then Z_1 has the DPPh (see [**5**]); being l_1-saturated, it also has the Schur property.

Now, observe that if (γ_n) is a sequence of ordinals such that $g(\gamma_n) < \gamma_{n+1}$ then $\left\|\sum_n e_{\gamma_n}\right\|_g = 1$. thus, the sequence (e_{γ_n}) is weakly null in S_g. Therefore $\lim \|e_{\gamma_n} + Y_0\| = 0$. But this yields that for some $\gamma_0 < \omega_1$ and all $\gamma \geqslant \gamma_0$ one has $\|e_\gamma + Y_0\| = 0$. In this way, $e_\gamma \in Y_0$ for all $\gamma \geqslant \gamma_0$. This is absurd since then Y_0 would simultaneously be c_0-saturated and contain l_1. ∎

REMARK 1. In [**8**], it is proved that, given a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ that is compact in the topology of $\{0,1\}^{\mathbb{N}}$, it is possible to construct a (so-called) Schreier-like space $S_{\mathcal{F}}$ that is c_0-saturated. Some properties of those spaces appear related to properties of a hierarchy of functions g_n defined on the finite subsets of $\mathbb{N}$. Precisely, when $S_{\mathcal{F}}$ is an M-ideal in its bidual then the Dunford-Pettis and the hereditary Dunford-Pettis properties are equivalent, and equivalent to the finiteness of the function $g_0 : \mathbb{N} \to \mathbb{N}$ defined by $g_0(n) = \max\{\max A : n \in A \in \mathcal{F}\}$. It is clear that the family $\mathcal{F}$ is then defined as those sets such that $\max A < g_0(\min A)$; that is, the Schreier-like space induced by g_0. The spaces S_g can be considered as transfinite constructions of this type where the function g_0 takes values in $[0, \omega_1)$ instead of in $[0, \omega)$.

REMARK 2. Professor Kutzarova has kindly pointed out to us that a type of function like g_0 was considered in [**9**] and is at the basis of the construction shown in [**10**].

REMARK 3. The dependence of S_g on the initial choice of g means that there is an uncountable quantity of non-isomorphic S_g spaces: let S_g and S_h be the spaces constructed starting with the functions g and h, respectively. Since an isomorphism must transform sequences equivalent to the canonical basis of c_0 (respectively l_1) into the same class of sequences, a necessary condition to be isomorphic is that, for some $M \in \mathbb{N}$, all $n \in \mathbb{N}$

and all countable ordinals α

$$h\big(g^n(\alpha)\big) \leqslant g^{n+M}(\alpha).$$

The rest is easy since there are uncountably many ways to produce functions not satisfying that requirement; say, for ever ordinal β define by transfinite induction functions such that

$$h_\beta\big(h_\alpha^n(\omega)\big) \leqslant h_\alpha^{n+\beta}(\omega).$$

4. Further examples

A c_0-saturated space need not have the DPP: the Schreier space (see [**16**]) is c_0-saturated since it is a subspace of a $C(\omega^\omega)$ and it was shown in [**6**] that it fails the DPP. A l_1-saturated space does not necessarily have the *DPP* either: Lorentz sequence spaces $d(w,1)$ are l_1-saturated (see [**13**, Proposition 4.e.3]) and fail the DPP since they are separable duals without the Schur property.

One might guess that the DPP plus either c_0- or l_1-saturation implies the DPPh. There are several examples showing that c_0-saturated spaces with DPP do not need to have the DPPh; perhaps the simplest example is a c_0-sum of renormings of c_0. If $\|\cdot\|_N$ denotes the norm

$$\|x\|_N = \sup_{i_1\neq i_2\neq\cdots\neq i_N}\left\{\frac{1}{N}\sum_{j=1}^{N}|x_{i_j}|\right\}$$

on c_0 then $X = c_0\big(c_0, c_0(\|\cdot\|_2), , \cdots , , c_0(\|\cdot\|_N), , \cdots\big)$ works. By applying [**4**], this space has the DPP. It is c_0-saturated since it is a Schreier-like space [**8**]; or else by [**11**]. It is not difficult, however, to construct weakly null sequences (r_n) without weakly 1-summable subsequences in X as follows: If $\{A_i\}_{i\in\mathbb{N}}$ is a partition of $\mathbb{N}$ with every A_i infinite then X is isomorphic with the completion of the space of finite sequences with respect to the norm

$$\|x\| = \sup\left\{\sum_{j\in A}|x_j| : \ A \text{ is “admissible”}\right\};$$

where a finite set A is now "admissible" if $A \subset A_i$ for some i and $\operatorname{card} A \leqslant i$. Let (B_n) be a sequence of finite subsets of $\mathbb{N}$ such that $\max B_n < \min B_{n+1}$. Moreover, the set B_n is chosen so that $\operatorname{card} B_n = n$, and $\operatorname{card}(B_n \cap A_i) = 1$, for $1 \leqslant i \leqslant n$. If r_n is the characteristic function of B_n then $\left\|\sum_{j=1}^{j=N} r_{n_j}\right\| \geqslant N/2$.

Less obvious is the fact that there exist l_1-saturated spaces with the DPP without the hereditary DPP, as we show now. Bourgain and Pisier show in [**1**] that given a separable space E it is possible to embed it as a subspace of a certain $\mathcal{L}_\infty$-space $\mathcal{L}_\infty(E)$ in such a way that $\mathcal{L}_\infty(E)/E$ has the Schur property. If we take $E = d(w,1)$, a suitable Lorentz sequence space (which is l_1-saturated) with normalised weakly null canonical basis, one obtains a $\mathcal{L}_\infty$-space $\mathcal{L}_\infty\big(d(w,1)\big)$ that is l_1 -saturated, has the DPP but not the *DPPh*:

it is l_1-saturated since this is a 3-space property [**5**]; and since in a l_1-saturated spaces the DPPh and Schur properties are equivalent, it does not have the DPPh because it contains $d(w,1)$ which is not Schur.

A questions implicit in [**5**] can be answered now.

PROPOSITION 6. *(l_1-saturated and DPP) is not a 3-space property.*

PROOF: The method of Theorem 1, replacing the $C(K)$ space by the l_1-saturated DPP space $\mathcal{L}_\infty(d(w,1))$ we have just obtained, and the starting space X by $d(w,1)$, yields that $d(w,1)$, is a complemented subspace of two (l_1-saturated and DPP)-spaces. ∎

Nevertheless, since a Banach space X is (c_0-saturated and DPP) if and only if X^* is Schur, and the Schur property is a 3-space property, one has.

PROPOSITION 7. *(c_0-saturated and DPP) is a 3-space property.*

Proposition 4 is not complete unless one shows it applies to some space. Nontrivial twisted sums of l_1-saturated and c_0-saturated spaces are provided by Theorem 1 (let us remark that it is considerably harder, although possible, to show that there exist nontrivial twisted sums of l_1 and c_0; see [**3**] or [**2**]). Nontrivial twisted sums of c_0-saturated and l_1-saturated spaces are, however, harder to find. This is so since the universal property of l_1 (every surjective map $X \to l_1$ admits a linear continuous section) prevents the existence of nontrivial twisted sums of a Banach space and l_1; while Sobczyk's Theorem prevents the existence of nontrivial twisted sums of c_0 and a separable Banach space.

PROPOSITION 8. *There exists a separable nontrivial twisted sum of a c_0-saturated and a l_1-saturated space.*

PROOF: Let E be a c_0-saturated space without the Dunford-Pettis property. There is a separable $\mathcal{L}_\infty$-space, in which E cannot be complemented since it lacks the DPP, such that $\mathcal{L}_\infty(E)/E$ has the Schur property. ∎

The following question has remained elusive.

PROBLEM 1. Does a $\mathcal{L}_1$-by-$\mathcal{L}_\infty$-space have the Dunford-Pettis property?

Again, things are simpler when we exchange the properties.

PROPOSITION 9. *$\mathcal{L}_\infty$-by-$\mathcal{L}_1$ implies DPP.*

PROOF: If X is a twisted sum of a $\mathcal{L}_\infty$ space Y and a $\mathcal{L}_1$ space Z then $X^{**} = Y^{**} \oplus Z^{**}$ since Y^{**} is injective. Hence X^{**} has the DPP and so does X. ∎

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