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# FIXED POINTS OF HOLOMORPHIC MAPPINGS IN THE CARTESIAN PRODUCT OF $n$ UNIT HILBERT BALLS

BY

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ABSTRACT. Every continuous mapping $T = (T_1, \ldots, T_n): \bar{B}^n \to \bar{B}^n$ holomorphic in $B^n$ has a fixed point.

In the recent book "Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings" by K. Goebel and S. Reich [4] the authors study geometry of an open unit Hilbert ball $B$ with hyperbolic metric and apply obtained results to the fixed point theory of holomorphic selfmappings in $B$. In this paper we are concerned with the problem of fixed points of holomorphic mappings in the Cartesian product of $n$ unit Hilbert balls.

Let $B^n$ ($\bar{B}^n$) be the Cartesian product of an open (closed) unit ball $B$ ($\bar{B}$) in a complex Hilbert space $H$. It is well known that $B$ can be furnished with the invariant hyperbolic metric $\rho_1$ ([2], [4], [6]), which generates the Carathéodory metric in $B^n$ ([8], [9]):

$$\rho_n((x_1, y_1), (x_2, y_2)) = \max\{\rho_1(x_1, x_2), \rho_{n-1}(y_1, y_2)\} =$$

$$\max\left\{\tanh^{-1}\left(1 - \frac{(1 - \|x_1\|^2)(1 - \|x_2\|^2)}{|1 - \langle x_1, x_2\rangle|^2}\right)^{1/2}, \rho_{n-1}(y_1, y_2)\right\}$$

for $(x_1, y_1), (x_2, y_2) \in B \times B^{n-1} = B^n$.

Let us notice that in $(B, \rho_1)$ the Möbius transforms $M_a: B \to B$ ($a \in B$) given by

$$M_a(x) = ((1 - \|a\|^2)^{1/2} P_a^{\perp} + P_a)\left(\frac{x + a}{1 + \langle x, a\rangle}\right),$$

where $P_a$ is the orthogonal projection in the direction $a$ and $P_a^{\perp} = Id - P_a$, are $\rho_1$-isometries ([2, Chapter VI], [4]). Generally in $(B^n, \rho_n)$ every holomorphic mapping $T: B^n \to B^n$ is nonexpansive ([2], [6]) and if $T(B^n)$ lies "strictly inside" $B^n$ then $T$ is an $\rho_n$-contraction and has a unique fixed point ([1]).

In this paper for $x \in B \setminus \{0\}$ $\mathrm{Proj}_x$ denotes a metrical projection onto a geodesic line $\{\mu x: \mu \in (-1/\|x\|, 1/\|x\|)\}$ ([5]).

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First we give a few simple lemmas.

LEMMA 1. *If* $x \in B \setminus \{0\}$, *then the set*

$$A = \{y \in B: \langle \mathrm{Proj}_x(y), x\rangle \geqslant \|x\|^2\}$$

*is the image under* $M_x$ *of the set*

$$C = \{z \in B: \mathrm{re}\,\langle z, x\rangle \geqslant 0\}.$$

PROOF. See either [10] or the proof of Lemma 1 in [5].

LEMMA 2. *If* $-1 < \epsilon \leqslant 0$, $x \in B \setminus \{0\}$, $\|x\| \geqslant |\epsilon|$ *and*

$$D = \{y \in B: \mathrm{re}\,\langle y, x\rangle \geqslant \epsilon\|x\|\}$$

*then for* $z \in M_x(D)$ *we have*

$$\text{(i) } \mathrm{re}\,\langle z, x\rangle \geqslant \left(\frac{\|x\| + \epsilon}{1 + \epsilon\|x\|}\right)\|x\|,$$

$$\text{(ii) } \left\|z - M_x\left(\epsilon\frac{x}{\|x\|}\right)\right\| \leqslant \sqrt{1 - \left(\frac{\|x\| + \epsilon}{1 + \epsilon\|x\|}\right)^2},$$

$$\text{(iii) } \operatorname*{diam}_{\|\cdot\|} M_x(D) \leqslant 2\sqrt{1 - \left(\frac{\|x\| + \epsilon}{1 + \epsilon\|x\|}\right)^2}.$$

PROOF. Put $z = M_x(w) \in M_x(D)$. Then the inequality

$$\mathrm{re}\,\langle z, x\rangle \geqslant \left(\frac{\|x\| + \epsilon}{1 + \epsilon\|x\|}\right)\|x\|$$

is equivalent to the following one

$$|\langle w, x\rangle|^2 + (1 - \epsilon\|x\|)\,\mathrm{re}\,\langle w, x\rangle - \epsilon\|x\| \geqslant 0$$

which is easy to verify and then the other inequalities follow.

LEMMA 3. *For* $x \in B \setminus \{0\}$ *let us define the following sets*

$$A_x = \{y \in B: \langle \mathrm{Proj}_x(y), x\rangle \geqslant \|x\|^2\},$$

$$C_x = \bigcup_{y \in A_x} \bar{K}(y, \rho_1(x, y)),$$

*where* $\bar{K}(y, \rho_1(x, y))$ *is the closed ball in* $(B, \rho_1)$. *Then there exists* $-1 < \delta < 0$ *such that we have*

$$\operatorname*{diam}_{\|\cdot\|} C_x \leq 2\sqrt{1 - \left(\frac{\|x\| + \delta}{1 + \delta\|x\|}\right)^2}$$

*for every* $x \in B$ *with* $\|x\| > |\delta|$.

PROOF. The image of the set $A_x$ under the transform $M_{-x}$ is

$$\tilde{A} = \{z \in B: \langle \mathrm{Proj}_x(z), x\rangle \geqslant 0\} = \{z \in B: \mathrm{re}\,\langle z, x\rangle \geqslant 0\}$$

and the image of the set $C_x$ is

$$\tilde{C} = \bigcup_{z \in \tilde{A}} \bar{K}(z, \rho_1(0, z)).$$

Now there exists $-1 < \delta < 0$ such that $\mathrm{re}\,\langle v, x\rangle > \delta\|x\|$ for all $v \in \tilde{C}$. The negation of this statement leads quickly to a contradiction. Notice that this $\delta$ is independent of a choice of $x \in B \setminus \{0\}$. Now we return to $C_x$ using the transform $M_x$.

LEMMA 4. *The Möbius transform $M_a$ is lipschitzian in norm sense, i.e.*

$$\|M_a(x) - M_a(y)\| \leqslant \left(\frac{1 + \|a\|}{1 - \|a\|}\right)^2 \|x - y\|.$$

LEMMA 5. *Suppose that $x, y \in B \setminus \{0\}$, $\|x\| > |\delta|$ ($\delta$ is taken from Lemma 3) and $\varphi$: $B \to B$ is a holomorphic mapping such that*

(i) $\varphi(y) \neq \varphi(0)$,

(ii) $\langle \mathrm{Proj}_x(\varphi(y)), x\rangle \geqslant \|x\|^2$.

*Then*

$$\operatorname*{diam}_{\|\cdot\|} \bar{K}(y, \rho_1(x, \varphi(y))) \leqslant \frac{2\|y\|(1 + \|\varphi(0)\|^2}{\|M_{-\varphi(0)}(\varphi(y))\|(1 - \|\varphi(0)\|)^2} \sqrt{1 - \left(\frac{\|x\| + \delta}{1 + \delta\|x\|}\right)^2}.$$

PROOF. Putting $\psi = M_{-\varphi(0)} \circ \varphi$ we have $\|M_{-\varphi(0)}(\varphi(y))\| = \|\psi(y)\| \leqslant \|y\|$ (Th. III.2.3 in [2]) and

$$M_{-\varphi(0)}(C_x) = M_{-\varphi(0)}\left(\bigcup_{w \in A_x} \bar{K}(w, \rho_1(x, w))\right)$$

$$= \bigcup_{w \in A_x} \bar{K}(M_{-\varphi(0)}(w), \rho_1(x, w)) \supset \bar{K}(M_{-\varphi(0)}(\varphi(y)), \rho_1(x, \varphi(y))).$$

Choosing a unitary transform $U$ such that

$$Uy = \frac{\|y\|}{\|M_{-\varphi(0)}(\varphi(y))\|} M_{-\varphi(0)}(\varphi(y))$$

we get

$$U[\bar{K}(y, \rho_1(x, \varphi(y)))] = \bar{K}(Uy, \rho_1(x, \varphi(y)))$$

and

$$\frac{\|M_{-\varphi(0)}(\varphi(y))\|}{\|y\|} \bar{K}(Uy, \rho_1(x, \varphi(y))) \subset \bar{K}(M_{-\varphi(0)}(\varphi(y)), \rho_1(x, \varphi(y))) \subset M_{-\varphi(0)}(C_x)$$

which implies the desired result.

We recall that for a holomorphic function $T: B^n \to B^n$ and $t \in [0, 1)$ the mapping $tT$ is a $\rho_n$-contraction and has exactly one fixed point $Z(t)$.

LEMMA 6 ([9]). *If $T: B^n \to B^n$ is holomorphic and has exactly one fixed point $Z$ in $B^n$, then* $\lim_{t \to 1} Z(t) = Z$.

The proof is based on the idea given in Theorem 13 in [5].

Now we can prove the following

THEOREM. *Every continuous mapping $T = (T_1, \ldots, T_n): \bar{B}^n \to \bar{B}^n$ holomorphic in $B^n$ has a fixed point.*

PROOF. For $n = 1$ the theorem is true ([5]). Thus let us consider the mapping $T: \bar{B}^n \to \bar{B}^n$ for $n \geqslant 2$. Then we may have the following three cases.

CASE 1. There exists a fixed point in $B^n$.

CASE 2. There exists a point $x_1 \in B$ such that the mapping $F_{1x_1}: \bar{B}^{n-1} \to \bar{B}^{n-1}$ given by

$$F_{1x_1}(x_2, \ldots, x_n) = (T_2(x_1, x_2, \ldots, x_n), \ldots, T_n(x_1, x_2, \ldots, x_n))$$

has a fixed point $y = (y_2, \ldots, y_n)$ which lies in $\partial B^{n-1}$. Without loss of generality we may assume that $\|y_2\| < 1, \ldots, \|y_k\| < 1$ and $\|y_{k+1}\| = \ldots = \|y_n\| = 1$. By The Maximum Principle (Th. II.3.4 in [2]) the mappings $T_{k+1}(\cdot, y_{k+1}, \ldots, y_n), \ldots, T_n(\cdot, y_{k+1}, \ldots, y_n)$ are constant and therefore we may apply induction.

Now notice that if the mapping $F_{1x_1}$ has two distinct fixed points in $B^{n-1}$ then either case 1 occurs or the situation is as in case 2 after eventual permutation of indices. Indeed, denote them by $a = (a_2, \ldots, a_n)$ and $b = (b_2, \ldots, b_n)$. After an application of the Identity Theorem ([6]) for the set

$$\bar{K}\left(x_1, \frac{1}{2}\rho_{n-1}(a, b)\right) \times \prod_{j=2} \left[\bar{K}\left(a_j, \frac{1}{2}\rho_{n-1}(a, b)\right) \cap \bar{K}\left(b_j, \frac{1}{2}\rho_{n-1}(a, b)\right)\right]$$

and the mapping $T$ we may continue the induction argument.

CASE 3. The above cases are not satisfied. Then every mapping $F_{jx}: \bar{B}^{n-1} \to \bar{B}^{n-1}$ $(1 \leqslant j \leqslant n)$ defined by

$$F_{jx} = (T_1, \ldots, T_{j-1}, T_{j+1}, \ldots, T_n) \circ I_{jx}$$

where

$$I_{jx}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) = (x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_n)$$

has exactly one fixed point

$$\Phi_j(x) = (\varphi_{1j}(x), \ldots, \varphi_{j-1,j}(x), \varphi_{j+1,j}(x), \ldots, \varphi_{nj}(x))$$

which moreover lies in $B^{n-1}$. Every $\Phi_j$ is holomorphic as a limit of approximating

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functions $Z_j(x, t)$ defined for $x$ in the same way as $Z(t)$ of Lemma 6. Those are also holomorphic ($Z_j(x, t) = \lim_{k\to\infty} (tF_{jx})^k (0)$, Th.3.18.1 in [7]).

Now we introduce the following holomorphic functions ($j = 1, 2, \ldots, n$)

$$G_j(x) = (T_j \circ I_{jx} \circ \Phi_j)(x)$$

for $x \in B$. None of these functions has fixed points and therefore there exist points $e_1, \ldots, e_n \in \partial B$ for which

$$z_j(t) = tG_j(z_j(t)) \xrightarrow[0<t\to 1]{} e_j$$

([3]). Let us notice that

$$\rho_n[T((I_{1z_1(t)} \circ \Phi_1)(z_1(t))), T((I_{kz_k(t)} \circ \Phi_k)(z_k(t)))]$$
$$\leq \rho_n[(I_{1z_1(t)} \circ \Phi_1)(z_1(t)), (I_{kz_k(t)} \circ \Phi_k)(z_k(t))]$$

for $k \geq 2$, $t \in [0, 1)$. Considerations similar to the ones given above while discussing cases 1 and 2 show that

$$\max\{\rho_1(\varphi_{j1}(z_1(t)), \varphi_{jk}(z_k(t))): j \in \{2, \ldots, k-1, k+1, \ldots, n\}\}$$
$$\leq \max\{\rho_1(z_1(t), \varphi_{1k}(z_k(t))), \rho_1(\varphi_{k1}(z_1(t)), z_k(t))\}.$$

Taking a sequence $t_m \to 1$ we may assume (choosing a subsequence if necessary) that for every $m$, $\|z_1(t_m)\| > |\delta|$ ($\delta$ is taken from Lemma 3) and

$$\max\{\rho_1(z_1(t_m), \varphi_{1k}(z_k(t_m))), \rho_1(\varphi_{k1}(z_1(t_m)), z_k(t_m))\} = \rho_1(z_1(t_m), \varphi_{1k}(z_k(t_m))).$$

Now we have

$$\rho_1\left(\frac{1}{t_m} z_1(t_m), \varphi_{1k}(z_k(t_m))\right) = \rho_1[T_1((I_{1z_1(t_m)} \circ \Phi_1)(z_1(t_m))),$$
$$T_1((I_{kz_k(t_m)} \circ \Phi_k)(z_k(t_m)))] \leq \rho_1(z_1(t_m), \varphi_{1k}(z_k(t_m)))$$

and therefore (see the proof of Lemma 1)

$$\langle \mathrm{Proj}_{z_1(t_m)}(\varphi_{1k}(z_k(t_m))), z_1(t_m)\rangle \geq \|z_1(t_m)\|^2$$

and by Lemma 2 we get

$$\|\varphi_{1k}(z_k(t_m)) - z_1(t_m)\| \leq \sqrt{1 - \|z_1(t_m)\|^2},$$

so that $\varphi_{1k}(z_k(t_m)) \to e_1$ as $m \to \infty$. Hence we may assume additionally that $\varphi_{1k}(z_k(t_m)) \neq \varphi_{1k}(0)$ for each $m = 1, 2, \ldots$. From Lemma 5 we deduce that

$$\|\varphi_{k1}(z_1(t_m)) - z_k(t_m)\| \leq \frac{2\|z_k(t_m)\|(1 + \|\varphi_{1k}(0)\|)^2}{\|M_{-\varphi_{1k}(0)}\varphi_{1k}(z_k(t_m))\|(1 - \|\varphi_{1k}(0)\|)^2}$$
$$\times \sqrt{1 - \left(\frac{\|z_1(t_m)\| + \delta}{1 + \delta\|z_1(t_m)\|}\right)^2}.$$

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These inequalities yield

$$\varphi_{1k}(z_k(t_m)) \underset{m}{\longrightarrow} e_1$$

and

$$\varphi_{k1}(z_1(t_m)) \underset{m}{\longrightarrow} e_k.$$

Changing $k$ we obtain

$$\Phi_1(z_1(t)) \underset{t\to 1}{\longrightarrow} (e_2, \ldots, e_n),$$

$$F_{1z_1(t)}(\Phi_1(z_1(t))) \underset{t\to 1}{\longrightarrow} (e_2, \ldots, e_n)$$

and finally

$$T(e_1, e_2, \ldots, e_n) = (e_1, e_2, \ldots, e_n).$$

REMARK. It is easy to obtain the proof of the above theorem for $n = 2$ without the use of Lemmas 1–5. See [9].

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