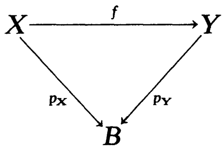
Canad. Math. Bull. Vol. **21** (3), 1978

NOTE ON ATTACHING DOLD FIBRATIONS

BY
PHILIP R. HEATH AND KLAUS HEINER KAMPS

In this note, we patch up the proof of a Theorem due to Handel on the characterization of homotopy epimorphisms ([6], 2.2) and generalize a Theorem due to Ibisch on attaching disk-bundles to Dold fibrations ([7], Satz 1).

1. Cofibrations over B. We work throughout in the category Top_B of spaces over B for some fixed topological space B. Thus, the objects are maps $p_X : X \to B$, $p_Y : Y \to B$ in Top and the arrows $f : p_X \to p_Y$ in Top_B are commutative diagrams



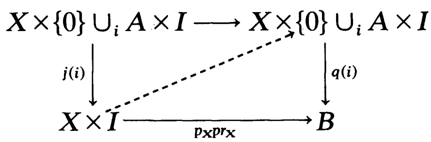
in Top. In Top_B, we have notions of fibre homotopy and fibre homotopy equivalence (see [3], (0.22)). The notion of cofibration in Top_B can be defined as follows.

DEFINITION 1.1. A map $i : p_A \to p_X$ in Top_B is said to be a *cofibration over B* if there exists a fibre retraction of the canonical inclusion $j(i)$ of the mapping cylinder

$$M_i = X \times \{0\} \cup_i A \times I \xrightarrow{q(i)} B$$

into $X \times I \xrightarrow{p_X pr} B$, i.e. the dotted arrow exists in the diagram

(1.2)



EXAMPLE 1.3. If $i : A \to X$ is a closed cofibration in Top and if further p_A and p_X are Hurewicz fibrations, then $i : p_A \to p_X$ is a cofibration over B.

EXAMPLE 1.4. If $p : E \to B$ is any map in Top and $k : \dot{I} = \{0, 1\} \to I = [0, 1]$ the inclusion, then $1_E \times k : p_{E \times \dot{I}} \to p_{E \times I}$ is a cofibration over B where $p_{E \times \dot{I}} : E \times \dot{I} \to B$, $p_{E \times I} : E \times I \to B$ are given by $(e, t) \to p(e)$.

Received by the editors April 4, 1977.

EXAMPLE 1.5. Any map $f:p_X \to p_Y$ in Top_B can be factored as a cofibration over B, $p_X \to q(f)$, followed by a fibre homotopy equivalence $q(f) \to p_Y$.

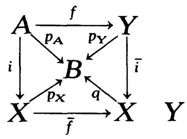
Under the conditions of 1.3, the map $q(i)$ in (1.2) is a Hurewicz fibration ([1], 3.8) and M_i is a subspace of $X \times I$, so 1.3 is a consequence of [11], Theorem 4.

Example 1.4 follows from the existence of a retraction $r:I\times I \to I \times \{0\} \cup \dot{I} \times I$, while 1.5 is the generalization of a standard topological result (cf. [3], (1.27)).

We remark that in 1.5, $q(f)$ is a Dold fibration, i.e. a map which has the WCHP ([4], 5), if and only if p_Y is.

In the following Lemma, we regard p_X as attached to p_Y via f.

LEMMA 1.6. *Let*



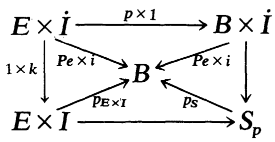
be a pushout diagram in Top_B, *in which* i *is a cofibration over* B. *If* p_A, p_X, p_Y *are Dold fibrations, then so also is* q.

We sketch the proof. Using the glueing Theorem for homotopy equivalences in Top_B ([9], (8.2)), we see that $q:X \sqcup Y \to B$ is fibre homotopy equivalent to the projection $q(i, f)$ from the double mapping cylinder of i and f. This double mapping cylinder can be thought of as union of a numerable cover of two subspaces homeomorphic to the mapping cylinders of i and f with intersection $A \times [\frac{1}{3}, \frac{2}{3}]$. By the remark preceding this lemma, the result follows from [2], Theorem 3.

2. **The Results.** In order to deduce his main Theorem [6], 1.1, Handel gives a fallacious proof that for any Hurewicz fibration $p:E \to B$, the projection $p_S:S_p \to B$ of the suspension overspace ([8], 4) is also a Jurewicz fibration. Handel exhibits a lifting function which fails in general to be continuous. In order to fix up Handel's work, it is necessary only to prove that p_S is a Dold fibration. We are able to prove a stronger result.

PROPOSITION 2.1. *If* $p:E \to B$ *is a Dold fibration, then so also is* $p_S:S_p \to B$.

Proof. The overspace S_p can be exhibited as the following pushout in Top_B.



The proposition follows from 1.4 and 1.6.

REMARK. Proposition 2.1 enables one to generalize [6], 2.2, converting the

https://doi.org/10.4153/CMB-1978-064-2 Published online by Cambridge University Press

question of whether or not a Hurewicz fibration is a homotopy epimorphism to a suitable cross-sectioning problem, from Hurewicz fibrations to Dold fibrations.

For the second result, let

$$\begin{array}{ccc}\dot{E} & \xrightarrow[\subset]{i} & E \\ & {\scriptstyle p_{\dot{E}}}\searrow \quad \swarrow{\scriptstyle p_E} & \\ & B & \end{array}$$

be a fibre-bundle pair with fibre pair $(F, \dot{F})$ ([10], p. 256) over a space B which is paracompact. Let $\dot{F} \subset F$ be a closed cofibration.

Proposition 2.2. *If $p_E : E \to B$ is attached to a Dold fibration $p_Y : Y \to B$ via $f : p_{\dot{E}} \to p_Y$. Then the projection $q : E \sqcup Y \to B$ is a Dold fibration.*

Proof. In view of [3], (9.4) and 1.3 and 1.6 of this note, it is sufficient to know that $i : \dot{E} \subset E$ is a closed cofibration. But this follows from [5], Satz 3.

Remark. In the case $(F, \dot{F}) = (E^n, S^{n-1}) =$ (disk, sphere), 2.2 generalizes [7], Satz 1.

References

1. J. E. Arnold, Jr., *Attaching Hurewicz fibrations with fiber preserving maps.* Pacific J. Math. **46,** 325–335 (1973).

2. T. tom Dieck, *Partitions of unity in homotopy theory.* Compositio Math. **23,** 159–167 (1971).

3. T. tom Dieck, K. H. Kamps, D. Puppe, *Homotopietheoie. Lecture Notes in Mathematics* **157,** Berlin-Heidelberg-New York: Springer, 1970.

4. A. Dold, *Partitions of unity in the theory of fibrations.* Ann. of Math, **78,** 223–255 (1963).

5. A. Dold, *Die Homotopieerweiterungseigenschaft (= HEP) ist eine lokale Eigenschaft.* Inventiones Math. **6,** 185–189 (1968).

6. D. Handel, *Epimorphism plus monomorphism implies equivalence in the homotopy category.* J. Pure Appl. Algebra **6,** 357–360 (1975).

7. H. D. Ibisch, *Anheften von Kugelbündeln an Faserräume.* Math. Z. **91,** 294–299 (1966).

8. I. M. James, *Overhomotopy theory.* Symposia Mathematica **4,** 219–229 (1970).

9. K. H. Kamps, *Kan-Bedingungen und abstrakte Homotopietheorie.* Math. Z. **124,** 215–236 (1972).

10. E. H. Spanier, *Algebraic Topology,* New York: McGraw-Hill, 1966.

11. A. Strøm, *Note on cofibrations.* Math. Scand. **19.** 11–14 (1966).

Department of Mathematics,
Memorial University of Newfoundland,
St. John's, Newfoundland,
Canada. A1B 3X7

Fachbereich Mathematik,
Fernuniversitat,
Postfach 940,
D-5800 Hagen.

https://doi.org/10.4153/CMB-1978-064-2 Published online by Cambridge University Press