Canad. Math. Bull. Vol. 14 (2), 1971

ON INTEGRAL INEQUALITIES RELATED TO HARDY'S

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1. Introduction. The purpose of this note is to provide integral inequalities which are related to Hardy's ([2] and [3, Theorem 330]). This latter result we state as

THEOREM 1. *Let* $p>1$, $r\neq 1$, *and* $f(x)$ *be nonnegative and Lebesgue integrable on* $[0, a]$ *or* $[a, \infty]$ *for every* $a>0$, *according as* $r>1$ *or* $r<1$. *If* $F(x)$ *is defined by*

$$F(x) = \begin{cases} \int_0^x f(t)\,dt & (r > 1), \\ \int_x^\infty f(t)\,dt & (r < 1), \end{cases} \tag{1}$$

then

$$\int_0^\infty x^{-r}F^p\,dx < \left(\frac{p}{|r-1|}\right)^p \int_0^\infty x^{-r}(xf)^p\,dx \tag{2}$$

unless $f\equiv 0$. *The constant is the best possible.*

Our main result is stated in Theorem 2, the proof of which will be carried out by modifying a method given in a recent paper by Benson [1].

THEOREM 2. *Let* $p>1$, $r\neq 1$, *and* $f(x)$ *be nonnegative and Lebesgue integrable on* $[0, b]$ *or on* $[a, \infty]$ *according as* $r>1$ *or* $r<1$, *where* $a>0$, $b>0$. *If* $F(x)$ *is defined by* (1), *then if* $\int_0^b x^{-r}(xf)^p\,dx<\infty$ *in* (3a) *and* $\int_a^\infty x^{-r}(xf)^p\,dx<\infty$ *in* (3b),

$$\int_0^b x^{-r}F^p\,dx+\frac{p}{r-1}\,b^{1-r}F^p(b) \leq \left(\frac{p}{r-1}\right)^p \int_0^b x^{-r}(xf)^p\,dx \quad \text{for } r > 1, \tag{3a}$$

$$\int_a^\infty x^{-r}F^p\,dx+\frac{p}{1-r}\,a^{1-r}F^p(a) \leq \left(\frac{p}{1-r}\right)^p \int_a^\infty x^{-r}(xf)^p\,dx \quad \text{for } r < 1, \tag{3b}$$

with equality in (3a) *or* (3b) *only for* $f\equiv 0$, *where the constant* $[p/(r-1)]^p$ *or* $[p/(1-r)]^p$ *is the best possible when the left side of* (3a) *or* (3b) *is unchanged respectively.*

2. Preliminary lemmas. The elementary algebraic inequality from which we can obtain (3a) and (3b) is as follows:

$$\begin{aligned} g(t) &= t^p-ct+\left(\frac{c}{p}\right)^{p/(p-1)}(p-1) > 0 && \text{for all } t \geq 0 \text{ and } t \neq \left(\frac{c}{p}\right)^{1/(p-1)} \\ &= 0 && \text{for } t = \left(\frac{c}{p}\right)^{1/(p-1)}, \end{aligned} \tag{4}$$

Received by the editors December 23, 1969 and, in revised form, June 25, 1970.

where c is any positive number, and $p>1$. This can be verified by using elementary calculus.

LEMMA 1. *Let $u(x)$ be absolutely continuous on $[a, b]$ with $u'(x) \geq 0$ a.e. Also, suppose that $Q(x)$ is positive and continuous on (a, b), and $G(u, x)$ is continuously differentiable for x in $[a, b]$ and u in the range of the function $u(x)$, with $G_u(u, x)>0$. Then, if the integral exists,*

$$(5) \qquad \int_a^b \{Qu'^p + \left(\frac{c}{p}\right)^{p/(p-1)} (p-1) G_u^{p/(p-1)} Q^{-1/(p-1)} + cG_x\}\, dx \geq c\{G(u(b), b) - G(u(a), a\}$$

where c is any positive number, $p>1$ and $G_u = (\partial/\partial u)G(u, x)$, $G_x = (\partial/\partial x)G(u, x)$. Equality holds if and only if the differential equation

$$(6) \qquad u' = \left(\frac{c}{p}\right)^{1/(p-1)} \left(\frac{G_u}{Q}\right)^{1/(p-1)}$$

is satisfied almost everywhere.

Proof. By taking $t = u'(G_u/Q)^{-1/(p-1)}$ in (4), and then multiplying by $Q(G_u/Q)^{p/(p-1)}$ we have

$$Qu'^p + \left(\frac{c}{p}\right)^{p/(p-1)} (p-1) G_u^{p/(p-1)} Q^{-1/(p-1)} \geq cu'G_u,$$

that is,

$$Qu'^p + \left(\frac{c}{p}\right)^{p/(p-1)} (p-1) G_u^{p/(p-1)} Q^{-1/(p-1)} + cG_x \geq c \frac{d}{dx} G(u, x),$$

proving (5) by integrating both sides of the above inequality from a to b.

Equality holds in (5) if, and only if,

$$t = u'\left(\frac{G_u}{Q}\right)^{-1/(p-1)} = \left(\frac{c}{p}\right)^{1/(p-1)}$$

almost everywhere. This is equivalent to (6).

LEMMA 2. *Let $f(x)$ be nonnegative and Lebesgue integrable on $[0, b]$ or on $[a, \infty]$ according as $r>1$ or $r<1$, and let $F(x)$ be defined by (1). Then if $\int_0^b x^{p-r} f^p(x)\, dx < \infty$, where $p>1$ and $b>0$,*

$$\lim_{x \to 0^+} x^{1-r} F^p(x) = 0 \quad \textit{if } r > 1.$$

Similarly if $\int_a^\infty x^{p-r} f^p(x)\, dx < \infty$, where $p>1$ and $a>0$, then

$$\lim_{x \to \infty} x^{1-r} F^p(x) = 0 \quad \textit{if } r < 1.$$

Proof. By using Hölder's inequality we have, in case $r>1$,

$$F(x) = \int_0^x f(t)\,dt = \int_0^x (t^{(p-r)/p}f(t))t^{(r-p)/p}\,dt$$

$$\leq \left(\int_0^x t^{p-r}f^p(t)\,dt\right)^{1/p}\left(\int_0^x t^{(r-p)/(p-1)}\,dt\right)^{(p-1)/p}.$$

Hence, by using the fact that $(r-p)/(p-1)>-1$ we obtain as $x\to 0^+$

$$x^{1-r}F^p(x) \leq o(1)(x^{(1-r)/(p-1)}\int_0^x t^{(r-p)/(p-1)}\,dt)^{p-1}$$

$$= o(1)(x^{(1-r)/(p-1)}x^{(r-1)/(p-1)})^{p-1} = o(1),$$

proving the part of the lemma with $r>1$. The second part of the lemma can be proved by using precisely the same argument as above.

LEMMA 3. *Let f and F be as noted in Theorem* 1, *and suppose that $p>1$. If $r>1$ and $\int_1^\infty x^{-r}F^p\,dx<\infty$, then $\lim_{x\to\infty} x^{1-r}F^p(x)=0$, while if $r<1$ and $\int_0^b x^{-r}F^p\,dx<\infty$, then $\lim_{x\to 0^+} x^{1-r}F^p(x)=0$.*

Proof. Assume the first part of the lemma is false. Therefore there exists a sequence $\{b_n\}(b_n\to\infty)$ such that $b_n^{1-r}F^p(b_n)\geq\delta$ for some $\delta>0$, where we may assume $2\leq 2b_n\leq b_{n+1}$. Then,

$$\int_1^\infty x^{-r}F^p(x)\,dx \geq \sum_{i=1}^\infty \int_{b_i}^{2b_i} x^{-r}F^p(x)\,dx \geq \sum_{i=1}^\infty F^p(b_i)\int_{b_i}^{2b_i} x^{-r}\,dx$$

$$= \frac{2^{1-r}-1}{1-r}\sum_{i=1}^\infty b_i^{1-r}F^p(b_i) = \infty,$$

which is a contradiction.

The proof of the second part of the lemma is similar.

3. Proof of Theorem 2. We first prove (3a). Let

$$u(x) = F(x) = \int_0^x f(x)\,dt, \qquad Q(x) = x^{p-r}, \qquad G(u,x) = \frac{u^p}{x^{r-1}}.$$

Then, from (5) with $a>0$, we obtain

$$\int_a^b \{x^{p-r}f^p+\left(\frac{c}{p}\right)^{p/(p-1)}(p-1)p^{p/(p-1)}F^px^{-r}-c(r-1)F^px^{-r}\}\,dx$$

$$\geq c(F^p(b)b^{1-r}-F^p(a)a^{1-r}),$$

or

$$\int_a^b \{x^{-r}(xf)^p+[c^{p/(p-1)}(p-1)-c(r-1)]x^{-r}F^p\}\,dx \tag{7}$$

$$\geq c(F^p(b)b^{1-r}-F^p(a)a^{1-r})$$

https://doi.org/10.4153/CMB-1971-038-4 Published online by Cambridge University Press

for all $c>0$. Now on letting $a \to 0^+$, and using the first part of Lemma 2, we obtain

$$(8) \qquad \int_0^b \{x^{-r}(xf)^p+[c^{p/(p-1)}(p-1)-c(r-1)]x^{-r}F^p\}\,dx \ge cF^p(b)b^{1-r}$$

for all $c>0$. After setting $h_1(c)=c^{p/(p-1)}(p-1)-c(r-1)$, by elementary calculus we find that $h_1(c)$ attains its minimum at $c_0=[(r-1)/p]^{p-1}$ and that $h_1(c_0)=-[(r-1)/p]^p$. Putting $c=c_0$ in (7) and (8) respectively we obtain

$$(9) \qquad \int_a^b \{x^{-r}(xf)^p-[(r-1)/p]^p x^{-r}F^p\}\,dx \ge [(r-1)/p]^{p-1}[F^p(b)b^{1-r}-F^p(a)a^{1-r}],$$

$$(10) \qquad \int_0^b \{x^{-r}(xf)^p-[(r-1)/p]^p x^{-r}F^p\}\,dx \ge [(r-1)/p]^{p-1}F^p(b)b^{1-r}.$$

Equality holds in (9) if and only if $f(x)$ satisfies the differential equation determined by (6) with $c=c_0$, that is,

$$\begin{aligned} u'(x) = F'(x) &= \left(\frac{c_0}{p}\right)^{1/(p-1)}\left(\frac{G_u}{Q}\right)^{1/(p-1)} \\ &= \left(\frac{c_0}{p}\right)^{1/(p-1)}(pF(x)^{p-1}x^{1-r}x^{r-p})^{1/(p-1)} \\ &= \left(\frac{r-1}{p}\right)F(x)x^{-1}, \end{aligned}$$

and hence

$$xF'(x) = \left(\frac{r-1}{p}\right)F(x).$$

That is,

$$\frac{d}{dx}\{x^{(1-r)/p}F(x)\} = 0 \quad \text{for } x>0,$$

from which we have

$$f(x) = Kx^{[(r-1)/p]-1}$$

for some constant K, which is impossible unless $K=0$, since it would make $\int_0^b x^{-r}(xf)^p\,dx$ divergent. Hence $f\equiv 0$ if equality holds in (9). Now for $0\le\alpha<\beta\le b$ set

$$I(\alpha,\beta) = \int_\alpha^\beta \{x^{-r}(xf)^p-\left(\frac{r-1}{p}\right)^p x^{-r}F^p\}\,dx-\left(\frac{r-1}{p}\right)^{p-1}(F^p(\beta)\beta^{1-r}-F^p(\alpha)\alpha^{1-r}).$$

Then from (9) we see that $I(\alpha,\beta)\ge 0$ for $0<\alpha<\beta\le b$; but then this implies that $I(\alpha,\beta)$ does not decrease as the interval (α,β) expands, since

$$I(\alpha,\beta) = \int_\alpha^\beta \{x^{-r}(xf)^p-\left(\frac{r-1}{p}\right)^p x^{-r}F^p-\left(\frac{r-1}{p}\right)^{p-1}\frac{d}{dx}F^p(x)x^{1-r}\}\,dx$$

is a nonnegative, additive interval function. Now for any $f(x)\not\equiv 0$, $I(\alpha,\beta)>0$ for

$0<\alpha<\beta\leq b$ by what we have just proved, and hence $I(0, b)\geq I(\alpha, \beta)>0$ for any $f(x)\not\equiv 0$. This implies that equality holds in (10) if, and only if, $f(x)\equiv 0$. By multiplying both sides of (10) by $[p/(r-1)]^p>0$ we get (3a).

To complete the proof we only need to show that the constant is the best possible as stated in Theorem 2. Suppose now that $K(p, r, b)$ is the best possible constant before the integral of $x^{-r}(xf)^p$ for fixed p, r, and b when $p(r-1)^{-1}F^p(b)b^{1-r}$ is unchanged on the left side of (3a). Now let $f(x)=x^{-1+[(r-1)/p]+\varepsilon}$ $(\varepsilon>0)$; then

$$F(x) = x^{[(r-1)/p]+\varepsilon}\Big/\left(\frac{r-1}{p}+\varepsilon\right),$$

and hence from (3a) we have

$$K(p, r, b)\int_0^b x^{-1+\varepsilon p}\,dx-\int_0^b x^{-1+\varepsilon p}\left(\frac{r-1}{p}+\varepsilon\right)^{-p}dx > \left(\frac{p}{r-1}\right)b^{\varepsilon p}\left(\frac{r-1}{p}+\varepsilon\right)^{-p}$$

or

$$K(p, r, b)b^{\varepsilon p}(\varepsilon p)^{-1}-b^{\varepsilon p}(\varepsilon p)^{-1}\left(\frac{r-1}{p}+\varepsilon\right)^{-p} > \left(\frac{p}{r-1}\right)b^{\varepsilon p}\left(\frac{r-1}{p}+\varepsilon\right)^{-p}.$$

That is,

$$K(p, r, b)-\left(\frac{r-1}{p}+\varepsilon\right)^{-p} > \left(\frac{p}{r-1}\right)\varepsilon p\left(\frac{r-1}{p}+\varepsilon\right)^{-p}.$$

By letting $\varepsilon\to 0^+$ we have

$$K(p, r, b) \geq \left(\frac{r-1}{p}\right)^{-p} = \left(\frac{p}{r-1}\right)^{p}$$

for any fixed p, r, and b, thus completing the proof.

In order to prove (3b), let $u(x)=-F(x)=-\int_x^\infty f(t)\,dt$, $Q(x)=x^{p-r}$, and $G(u, x)=-(-u)^px^{1-r}$. Then from (5) with $a>0$, we have

$$\int_a^b \{x^{p-r}f^p+(p-1)c^{p/(p-1)}F^px^{-r}-c(1-r)F^px^{-r}\}\,dx \geq cF^p(a)a^{1-r}-cF^p(b)b^{1-r}$$

or

$$\text{(7')}\quad \int_a^b \{x^{-r}(xf)^p+[(p-1)c^{p/(p-1)}-c(1-r)]F^px^{-r}\}\,dx \geq cF^p(a)a^{1-r}-cF^p(b)b^{1-r}$$

for all $c>0$. After setting $h_2(c)=c^{p/(p-1)}(p-1)-c(1-r)$, one finds that $h_2(c)$ attains its minimum at $c'=[(1-r)/p]^{p-1}$ and $h_2(c')=-[(1-r)/p]^p$. Putting $c=c'$ in (7') we have

$$\int_a^b \{x^{-r}(xf)^p\left(\frac{1-r}{p}\right)^pF^px^{-r}\}\,dx \geq \left(\frac{1-r}{p}\right)^{p-1}\{F^p(a)a^{1-r}-F^p(b)b^{1-r}\}$$

or, multiplying by $[p/(1-r)]^p>0$,

$$\text{(11)}\quad \left(\frac{p}{1-r}\right)^p\int_a^b x^{-r}(xf)^p\,dx \geq \int_a^b x^{-r}F^p\,dx+p(1-r)^{-1}\{F^p(a)a^{1-r}-F^p(b)b^{1-r}\}$$

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for all $0<a<b<\infty$. Equality holds in (11) if and only if $f(x)$ satisfies the differential equation determined by (6) with $c=c'$, that is, $u'=-F'=(c'/p)^{1/(p-1)}(G_u/Q)^{1/(p-1)}$. After precisely the same argument as in the proof of (3a), one finds that for a given $a>0$, equality holds in (11) if and only if $f(x)\equiv 0$. Now taking the limit of both sides of (11) as $b\to\infty$, on using the second part of Lemma 2 we obtain (3b) without the equality clause. However the condition for equality follows by the same argument used before, so that equality holds in (3b) if and only if $f(x)\equiv 0$.

The proof of the best possible constant in (3b) is precisely the same as in the corresponding part of the proof of (3a) except that we now take $f(x)=x^{-1+[(r-1)/p]-\varepsilon}$ for all $x\geq a$ in the course of proof.

REMARK 1. It is easily seen that if in (2), with $r>1$, one sets $f(x)=0$ for $x>b$, then Hardy's inequality gives a slightly weaker result than (3a). For the right side of (2) reduces to the right side of (3a) while the left side of (2) becomes

$$\int_0^b x^{-r}F^p\,dx+\int_b^\infty x^{-r}F^p\,dx = \int_0^b x^{-r}F^p\,dx+\frac{1}{r-1}F^p(b)b^{1-r}.$$

Similarly, if one sets $f(x)=0$ for $x<a$ in (2), with $r<1$, then Hardy's inequality reduces to

$$\int_a^\infty x^{-r}F^p\,dx+\frac{1}{1-r}a^{1-r}F^p(a) \leq \left(\frac{p}{1-r}\right)^p\int_a^x x^{-r}(xf)^p\,dx,$$

and this is weaker than (3b).

REMARK 2. On the other hand, if we strengthen the hypothesis in Theorem 2 by assuming $\int_0^\infty x^{p-r}f^p\,dx<\infty$ in both cases $r>1$ and $r<1$, which is the hidden hypothesis in Theorem 1, then Hardy's inequality (2) follows as an immediate Corollary of (3a) and (3b) without the condition $f\equiv 0$ for equality, by using the appropriate part of Lemma 3 for $r>1$ and $r<1$ respectively. The strict inequality in (2) follows by the same argument involving $I(\alpha,\beta)$ used before. The proof that the constant is best possible is similar to that given in Theorem 2.

ACKNOWLEDGEMENT. The author wishes to express his sincere thanks to Professor Paul R. Beesack, Department of Mathematics, Carleton University, for his invaluable guidance and encouragement throughout this research. Thanks are also due to the referee for Lemma 3.

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