J. Austral. Math. Soc. 20 (Series B) (1978), 175–181

GREEN'S FUNCTION OF THE CLAMPED PUNCTURED DISK

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(Received 1 December 1976)

Abstract

If a thin elastic circular plate B: $|z|<1$ is clamped (simply supported, respectively) along its edge $|z| = 1$, its deflection at $z \in B$ under a point load at $\zeta \in B$, measured positively in the direction of the gravitational pull, is the biharmonic Green's function $\beta(z, \zeta)$ of the clamped plate ($\gamma(z, \zeta)$ of the simply supported plate, respectively). We ask: how do $\beta(z, \zeta)$ and $\gamma(z, \zeta)$ compare with the corresponding deflections $\beta_0(z, \zeta)$ and $\gamma_0(z, \zeta)$ of the punctured circular plate B_0: $0<|z|<1$ that is "clamped" or "simply supported", respectively, also at the origin? We shall show that $\gamma(z, \zeta)$ is not affected by the puncturing, that is, $\gamma(\cdot, \zeta) = \gamma_0(\cdot, \zeta)$, whereas $\beta(\cdot, \zeta)$ is:

$$\beta_0(z, \zeta) = \beta(z, \zeta) - 16\pi\beta(z, 0)\,\beta(\zeta, 0)$$

on $B_0 \times B_0$. Moreover, while $\beta(\cdot, \zeta)$ is of constant sign, $\beta_0(\cdot, \zeta)$ is not. This gives a simple counterexample to the conjecture of Hadamard [6] that the deflection of a clamped thin elastic plate be always of constant sign:

The biharmonic Green's function of a clamped concentric circular annulus is not of constant sign if the radius of the inner boundary circle is sufficiently small.

Earlier counterexamples to Hadamard's conjecture were given by Duffin [2], Garabedian [4], Loewner [7] and Szegö [9]. Interest in the problem was recently revived by the invited address of Duffin [3] before the Annual Meeting of the American Mathematical Society in 1974. The drawback of the counterexample we will present is that, whereas the classical examples are all simply connected, ours is not. In the simplicity of the proof, however, there is no comparison.

1. Clamping and simple supporting

First we make precise what we mean by clamping and simple supporting at the isolated point O. Denote by B_s the annulus $s<|z|<1$ for $s \in (0, 1)$. The corresponding biharmonic Green's function $\beta_s(z, \zeta)$ ($\gamma_s(z, \zeta)$, respectively) of the clamped (simply supported, respectively) annulus B_s is characterized by

$$\Delta^2 \beta_s(\cdot, \zeta) = \delta_\zeta \quad (\Delta^2 \gamma_s(\cdot, \zeta) = \delta_\zeta, \text{ respectively}) \tag{1}$$

https://doi.org/10.1017/S0334270000001557 Published online by Cambridge University Press

on B_s, and

$$\beta_s(\cdot, \zeta) = \frac{\partial}{\partial n}\beta_s(\cdot, \zeta) = 0 \quad (\gamma_s(\cdot, \zeta) = \Delta\gamma_s(\cdot, \zeta) = 0, \text{ respectively}) \tag{2}$$

on the boundary ∂B_s of B_s. Here Δ is the Laplace–Beltrami operator $-(\partial^2/\partial x^2 + \partial^2/\partial y^2)$, δ_ζ is the Dirac delta at $\zeta \in B_s$, and $\partial/\partial n$ denotes the inner normal derivative (for example, Bergman and Schiffer [1]). We will define $\beta_0(\cdot, \zeta)$ and $\gamma_0(\cdot, \zeta)$ as the limits of $\beta_s(\cdot, \zeta)$ and $\gamma_s(\cdot, \zeta)$, respectively, as $s \to 0$.

2. Simply supported punctured disk

Denote by $g_s(\cdot, \zeta)$ the harmonic Green's function of B_s with pole $\zeta \in B_s$, and by $g(\cdot, \zeta)$ that of B. By the maximum principle and the Riemann removability theorem, $\{g(\cdot, \zeta) - g_s(\cdot, \zeta)\}$ converges decreasingly and uniformly to zero on each compact subset of $\bar{B} - 0$ as $s \to 0$, and

$$g(z, \zeta) = \frac{1}{2\pi}\log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right|.$$

In view of (1) and (2), we have

$$\gamma_s(z, \zeta) = \int_{B_s} g_s(w, z)\, g_s(w, \zeta)\, du\, dv \quad (w = u + iv) \tag{3}$$

on $B_s \times B_s$. On letting $s \to 0$ we see that

$$\gamma_0(z, \zeta) = \lim_{s \to 0} \gamma_s(z, \zeta) \tag{4}$$

exists uniformly on each compact subset of $B_0 \times B_0$, and

$$\gamma_0(z, \zeta) = \int_{B_0} g(w, z)\, g(w, \zeta)\, du\, dv. \tag{5}$$

On the other hand, since $\Delta^2\gamma(\cdot, \zeta) = \delta_\zeta$ on B and $\gamma(\cdot, \zeta) = \Delta\gamma(\cdot, \zeta) = 0$ on ∂B,

$$\gamma(z, \zeta) = \int_B g(w, z)\, g(w, \zeta)\, du\, dv. \tag{6}$$

On comparing the right-hand sides of (5) and (6) we conclude that

$$\gamma_0(z, \zeta) = \gamma(z, \zeta) \tag{7}$$

on $B_0 \times B_0$, that is, simple supporting at a single point does *not* have any effect on the deflection of a simply supported disk. This result agrees with physical intuition: if we place the tip of a needle under a very thin plate that is simply supported along its periphery, and then put a point load on the plate, the plate will be pierced by the needle.

3. Clamped punctured disk

In contrast with the above, what happens to $\beta_0(\cdot, \zeta)$ is somewhat surprising. Denote by $H_s(\cdot, \zeta) = \Delta\beta_s(\cdot, \zeta)$ the *β-density* of $\beta_s(\cdot, \zeta)$. It is readily deduced from Stokes' formula that $H_s(\cdot, \zeta) \perp H_2(B_s)$, that is

$$\int h(w)\, H_s(w, \zeta)\, du\, dv = 0 \tag{8}$$

for any h in the class $H_2(B_s)$ of square integrable harmonic functions on B_s (cf. [8]). As consequences of (8) we easily obtain

$$\begin{aligned} &\beta_s(z, \zeta) = \int_{B_s} H_s(w, z)\, H_s(w, \zeta)\, du\, dv, \\ &\| H_t(\cdot, \zeta) - H_s(\cdot, \zeta)\|^2 = \| H_t(\cdot, \zeta)\|^2 - \| H_s(\cdot, \zeta)\|^2 \quad (0 < t < s < 1), \\ &|\beta_t(z, \zeta) - \beta_s(z, \zeta)| \leqslant \| H_t(\cdot, z) - H_s(\cdot, z)\| \cdot \| H_t(\cdot, \zeta) - H_s(\cdot, \zeta)\|, \\ &\| H_s(\cdot, \zeta)\| \leqslant \| g(\cdot, \zeta)\|, \end{aligned} \tag{9}$$

where $\|\cdot\|$ is the L_2-norm on B and functions here and hereafter are defined to be zero outside their genuine domains of definition. It follows that

$$\beta_0(z, \zeta) = \lim_{s \to 0} \beta_s(z, \zeta) \tag{10}$$

exists uniformly on each compact subset of $\bar{B} - O$. If we denote by

$$H_0(\cdot, \zeta) = \Delta\beta_0(\cdot, \zeta)$$

the *β-density* of $\beta_0(\cdot, \zeta)$, then by (8) and (9),

$$\begin{aligned} &H_0(\cdot, \zeta) \perp H_2(B_0), \\ &\lim_{s \to 0} \| H_0(\cdot, \zeta) - H_s(\cdot, \zeta)\| = 0, \\ &\beta_0(z, \zeta) = \int_{B_0} H_0(w, z)\, H_0(w, \zeta)\, du\, dv = \int_{B_0} H_0(w, z)\, K(w, \zeta)\, du\, dv, \end{aligned} \tag{11}$$

where $K(\cdot, \zeta)$ is any square integrable function on B_0 with $\Delta K(\cdot, \zeta) = \delta_\zeta$ on B.

4. Clamped disk

The function $\beta(\cdot, \zeta)$ is defined by $\Delta^2\beta(\cdot, \zeta) = \delta_\zeta$ on B and $\beta(\cdot, \zeta) = \partial\beta(\cdot, \zeta)/\partial n = 0$ on ∂B. An explicit expression for $\beta(\cdot, \zeta)$ is known (for example, Garabedian [5]):

$$\beta(z, \zeta) = \frac{1}{8\pi}\left[|z - \zeta|^2 \log\left|\frac{z - \zeta}{1 - \bar{\zeta}z}\right| + \tfrac{1}{2}(|z|^2 - 1)(|\zeta|^2 - 1)\right] \tag{12}$$

on $B \times B$. Our immediate aim is to express $\beta_0(z, \zeta)$ in terms of $\beta(z, \zeta)$. The basis of our computation is the first relation (11) and its counterpart $H(\cdot, \zeta) \perp H_0(B)$, where $H(\cdot, \zeta) = \Delta\beta(\cdot, \zeta)$ is the β-density of $\beta(\cdot, \zeta)$. The latter orthogonality relation implies that

$$\beta(z, \zeta) = \int_B H(w, z)\, H(w, \zeta)\, du\, dv = \int_B H(w, z)\, g(w, \zeta)\, du\, dv \tag{13}$$

on $B \times B$. Since $H(\cdot, \zeta) - H_0(\cdot, \zeta)$ is harmonic on B_0 and square integrable over B_0, we have

$$H(re^{i\theta}, \zeta) - H_0(re^{i\theta}, \zeta) = ag(r) + b + \sum_{n=1}^{\infty} \left(\sum_{m=1}^{2} c_{nm} S_{nm}(\theta) \right) r^n, \tag{14}$$

with uniform convergence on each compact subset of $\bar{B} - 0$. Here a, b and c_{nm} are constants, $g(r) = g(r, 0) = -(1/2\pi)\log r$, $S_{n1}(\theta) = \cos n\theta$, and $S_{n2}(\theta) = \sin n\theta$ for $n = 1, 2, \ldots$. We denote by $(\cdot, \cdot)$ the inner product on $L_2(B)$ and by $\|\cdot\|_1$ the norm on $L_1(B)$. Since $h_{nm}(re^{i\theta}) = S_{nm}(\theta) r^n$ is in the class $H_2(B) \subset H_2(B_0)$, and $\|h_{nm}\| \neq 0$,

$$c_{nm} \|h_{nm}\|^2 = (H(\cdot, \zeta) - H_0(\cdot, \zeta), h_{nm}) = 0$$

and $c_{nm} = 0$ for every n and m. Observe that

$$(H(\cdot, \zeta) - H_0(\cdot, \zeta), 1) = a\|g\|_1 + b\pi,$$

$$(H(\cdot, \zeta) - H_0(\cdot, \zeta), g) = a\|g\|^2 + b\|g\|_1.$$

By virtue of $1 \in H_2(B) \subset H_2(B_0)$ and $g \in H_2(B_0)$ (but $g \notin H_2(B)$), these equations take the form

$$\|g\|_1 a + \pi b = 0,$$

$$\|g\|^2 a + \|g\|_1 b = \beta(\zeta, 0).$$

In view of $\|g\|_1 = 1/4$ and $\|g\|^2 = 1/8\pi$, we obtain $a = 16\pi\beta(\zeta, 0)$ and $b = -4\beta(\zeta, 0)$. By (14), we conclude that

$$H_0(\cdot, \zeta) = H(\cdot, \zeta) - 16\pi\beta(\zeta, 0)\, g(\cdot, 0) + 4\beta(\zeta, 0) \tag{15}$$

on B_0. We take the inner product of each side of (15) with $H(\cdot, z)$ and obtain by (13) the following *main identity* of the present study:

$$\beta_0(z, \zeta) = \beta(z, \zeta) - 16\pi\beta(z, 0)\, \beta(\zeta, 0) \tag{16}$$

on $B_0 \times B_0$. This is the required representation of β_0 in terms of β.

By (12), $\beta(z, \zeta) > 0$ on $B \times B$, and *a fortiori*,

$$\beta_0(z, \zeta) < \beta(z, \zeta)$$

on $B_0 \times B_0$. Thus adding to the clamping at the periphery, the clamping at a single point O *does* have a noticeable effect on the resulting deflection. Compared with the case of γ_0, this result is quite intriguing.

We now analyse the boundary behavior of β_0 in some more detail, with a view on our main identity (16).

5. Boundary behaviour

By (12) and (16), we have $\Delta^2 \beta_0(\cdot, \zeta) = \delta_\zeta$ on B_0 and $\beta_0(\cdot, \zeta) = \partial\beta_0(\cdot, \zeta)/\partial n = 0$ on ∂B: $|z| = 1$. Thus both clamping conditions are satisfied at the outer boundary ∂B. By (12), $\beta(0,0) = 1/16\pi$, and by (16) and the symmetry of β,

$$\begin{aligned}\beta_0(0, \zeta) &= \lim_{z\to 0} \beta_0(z, \zeta)\\ &= \beta(0, \zeta) - 16\pi\beta(0,0)\beta(\zeta, 0) = 0,\end{aligned} \tag{17}$$

that is, the first clamping condition is satisfied at the inner boundary $z = 0$.

We proceed to examine the second condition. Denote by $\partial/\partial n_\theta$ the directional derivative in the direction $e^{i\theta}$, that is,

$$\frac{\partial}{\partial n_\theta}\beta_0(0, \zeta) = \lim_{t\to+0} \frac{\beta_0(te^{i\theta}, \zeta) - \beta_0(0, \zeta)}{t}. \tag{18}$$

Again by (16),

$$\frac{\partial}{\partial n_\theta}\beta_0(0, \zeta) = \frac{\partial}{\partial n_\theta}\beta(0, \zeta) - 16\pi\beta(\zeta, 0)\frac{\partial}{\partial n_\theta}\beta(0,0).$$

Since $\beta(te^{i\theta}, 0) = (8\pi)^{-1}[t^2 \log t + \frac{1}{2}(t^2 - 1)]$ for $t > 0$, we have

$$\frac{\partial}{\partial n_\theta}\beta(0,0) = \lim_{t\to+0} \frac{\beta(te^{i\theta}, 0) - \beta(0,0)}{t} = 0$$

and, therefore,

$$\frac{\partial}{\partial n_\theta}\beta_0(0, \zeta) = \frac{\partial}{\partial n_\theta}\beta(0, \zeta). \tag{19}$$

By (12) we see that $\beta(z, \zeta)$ is real analytic in the neighbourhood $|z| < |\zeta|$ of $z = 0$ and *a fortiori*,

$$\frac{\partial}{\partial n_\theta}\beta(0, \zeta) = \left[\frac{\partial}{\partial t}\beta(te^{i\theta}, \zeta)\right]_{t=0}.$$

Using the explicit representation (12) for $\beta(z, \zeta)$, we obtain by direct calculation

$$\frac{\partial}{\partial n_\theta}\beta_0(0, \zeta) = \frac{1}{8\pi}|\zeta|(|\zeta|^2 - 2\log|\zeta| - 1)\cos(\theta - \arg\zeta). \tag{20}$$

Thus the "normal derivative" of $\beta_0(z, \zeta)$ at $z = 0$ does *not* vanish identically, and the second clamping condition is not satisfied. However, this failed "clamping" will conveniently serve to disprove Hadamard's conjecture, as we shall now see.

6. Hadamard's conjecture

Hadamard [6] conjectured that the Green's function of a clamped thin elastic plate cannot take on negative values. We give here a simple counterexample based on $\beta_0(z, \zeta)$. Observe that for any $\zeta \in B_0$,

$$|\zeta|^2 - 2\log|\zeta| - 1 > 0.$$

From this and (20) we see that

$$\operatorname{sign} \frac{\partial \beta_0(0, \zeta)}{\partial n_\theta}\bigg|_{\theta=\pi+\arg\zeta} \neq \operatorname{sign} \frac{\partial \beta_0(0, \zeta)}{\partial n_\theta}\bigg|_{\theta=\arg\zeta} = +1. \tag{21}$$

This means that $\beta_0(z, \zeta)$ takes *values of opposite sign* on line segments

$$\{z;\ 0<|z|<\sigma, \arg z = \pi + \arg\zeta\} \quad \text{and} \quad \{z;\ 0<|z|<\sigma, \arg z = \arg\zeta\}$$

for a sufficiently small $\sigma \in (0, 1)$. This rather agrees with our intuition provided $\beta_0(0, \zeta) = 0$, a fact which, however, is not clear *a priori.*

By (10) we see that $\beta_s(\cdot, \zeta)$ converges to $\beta_0(\cdot, \zeta)$ uniformly on each compact subset of B_0 and therefore, $\beta_s(\cdot, \zeta)$ takes on values of nonconstant sign along with $\beta_0(\cdot, \zeta)$ if $s > 0$ is sufficiently small. Thus we have the following

COUNTEREXAMPLE TO HADAMARD'S CONJECTURE. *The Green's function of a clamped concentric circular annulus is of nonconstant sign if the radius of the inner boundary circle is sufficiently small.*

Acknowledgement

The work was sponsored by the U.S. Army Research Office, Grant DA-ARO-31-124-73-G39, University of California, Los Angeles. MOS Classification 31 B 30.

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https://doi.org/10.1017/S0334270000001557 Published online by Cambridge University Press