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THE KÖNIGSBERG BRIDGE PROBLEM FOR PEANO CONTINUA

W. BULA, J. NIKIEL AND E. D. TYMCHATYN

ABSTRACT. Peano continua which are images of the unit interval [0, 1] or the circle S under a continuous and irreducible map are investigated. Necessary conditions for a space to be the irreducible image of [0, 1] are given, and it is conjectured that these conditions are sufficient as well. Also, various results on irreducible images of [0, 1] and S are given within some classes of regular curves. Some of them involve inverse limits of inverse sequences of Euler graphs with monotone bonding maps.

1. **Introduction.** All spaces considered in this paper are metrizable and all mappings are continuous. A *continuum* is a compact and connected space.

We shall say that a mapping $f: X \longrightarrow Y$ is *irreducible* if f is surjective (*i.e.*, $f(X) = Y$) and $f(F) \neq Y$ for each proper closed subset F of X (see *e.g.* [2]; in some papers the mappings with the above property were called *strongly irreducible*, while irreducibility meant that no proper subcontinuum of X is mapped onto Y by the given map).

We are going to investigate Hausdorff spaces which are irreducible images of the closed unit interval [0, 1] and the circle S. Of course, each such space is a metrizable locally connected continuum (that is an obvious consequence of the easier part of the well-known Hahn-Mazurkiewicz theorem) but the converse fails, as the example of the simple triod shows.

Let Irr denote the class of all (Hausdorff) spaces which are irreducible images of [0, 1], and let Irr_0 be the class of all spaces which are irreducible images of the circle S. Obviously, $\mathrm{Irr}_0 \subset \mathrm{Irr}$. The problem to characterize the class Irr was posed in [12]. In this section we shall recall three results closely related to that problem. Also, we put here most of the definitions and preliminary properties. In Section 2 we shall get some necessary conditions for a Peano continuum X to belong to Irr or Irr_0. They are expressed in terms of monotone decompositions of X onto cyclicly completely regular continua. And we conjecture that those conditions are also sufficient. Section 3 contains various results concerning the members of Irr among regular continua. Some of results proved there involve inverse sequences of Euler graphs with monotone bonding surjections.

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By a *graph* we shall mean a continuum which is a 1-dimensional polyhedron. Unless otherwise stated, a *vertex* of a graph is a point of order $\neq 2$ in that graph. The following result due to Leonard Euler can be found in each textbook on graph theory.

THEOREM A. *Let X be a graph. Then* $X \in \mathrm{Irr}$ *if and only if X has at most two vertices of odd order, and* $X \in \mathrm{Irr}_{\mathbf{0}}$ *if and only if all the vertices of X have even orders.*

We also recall here that graphs which belong to $\mathrm{Irr}_{\mathbf{0}}$ are called *Euler graphs*, and that each graph has an even number of vertices of odd order. By *irr-graphs* we shall mean graphs that belong to Irr.

Let X be a locally connected continuum and $Y \subset X$. Y is said to be a *cyclic element* of X if Y is connected and maximal with respect to the property that no point disconnects it, see *e.g.* [8] or [15]. It follows that each cyclic element of X is a locally connected continuum itself, and two different cyclic elements have at most one point in common. Also, a non-degenerate subset Z of X is a cyclic element of X if and only if it is maximal with respect to the property that every pair of points of Z is contained in a copy of S in Z.

Let X be a non-degenerate continuum. We shall say that X is

- *regular* if it has a basis $\mathcal{B}$ of open sets such that $\mathrm{bd}(U)$ is finite for each $U \in \mathcal{B}$;
- *totally regular* if, for each countable set $C \subset X$, there exists a basis $\mathcal{B}_C$ of open sets such that $C \cap \mathrm{bd}(U) = \emptyset$ and $\mathrm{bd}(U)$ is finite for all $U \in \mathcal{B}_C$;
- *completely regular* if each non-degenerate subcontinuum of X has a non-empty interior in X;
- *cyclicly completely regular* if X is locally connected and each non-degenerate cyclic element of X is a completely regular continuum;
- a *dendrite* if for every $x \neq y \in X$ there is $z \in X$ which separates x from y in X; equivalently, if X is locally connected and each cyclic element of X is degenerate.

All of these classes of continua were considered with respect to various properties in many old and newer papers (see *e.g.* [4], [15], [8] or [10]). They also admit nice generalizations in the class of compact connected Hausdorff spaces (see [11]). One easily gets the following inclusions between those classes (see *e.g.* [4]):

- each graph is a completely regular continuum;
- each completely regular continuum is cyclicly completely regular;
- each dendrite is a cyclicly completely regular continuum;
- each cyclicly completely regular continuum is totally regular;
- each totally regular continuum is regular.

None of the inclusions above can be reversed (for example [4, Figure 5 on p. 238] represents a totally regular continuum which is not cyclicly completely regular). It can be also shown that a locally connected continuum is totally regular (resp. regular) if and only if each of its cyclic elements is totally regular (resp. regular).

Now, let X be a locally connected continuum. A point x of X is said to be a *local separating point* of X if there is a connected open subset U of X such that $U - \{x\}$ is not connected. We shall denote by $N(X)$ the set of all $x \in X$ such that x is *not* a local separating point of X.

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A subset J of X is a *free arc* in X if J is homeomorphic to $]0,1[$ and J is an open set in X. By local connectedness of X, $|\operatorname{bd}(J)| \leq 2$ for each free arc J. Let $A \subset X$. We shall say that A is a *T-set* in X if A is closed and $|\operatorname{bd}(K)| = 2$ for each component K of $X - A$. And A is said to be a *strong T-set* in X if it is a T-set and each component of $X - A$ is a free arc in X.

We shall say that a collection $\mathcal{F}$ of subsets of X is a *null-family* if for each $\epsilon > 0$ the subcollection $\{F \in \mathcal{F} : \operatorname{diam} F \geq \epsilon\}$ is finite. By compactness of X, the latter notion does not depend on the choice of a particular metrization of X.

Recall that a (continuous) mapping $f: X \longrightarrow Y$ is *monotone* if $f^{-1}(y)$ is connected for each $y \in Y$.

Harrold proved the following general result in 1940 (see also [5] for somewhat related considerations).

THEOREM B [3]. *Let X be a locally connected continuum. If the set of all non-local separating points is dense in X, i.e., if $\operatorname{cl}(N(X)) = X$, then $X \in \mathrm{Irr}_0$.*

In [3], it was claimed that $X \in \mathrm{Irr}$ but the argument given there establishes the stronger inclusion $X \in \mathrm{Irr}_0$ as well. Harrold's proof of Theorem B is rather simple. He uses the facts that locally connected continua are locally arcwise connected and are continuous images of $[0,1]$ to prove that, for a given countable dense subset P of $N(X)$, the collection $\mathbf{J}$ of continuous surjections $f: [0,1] \longrightarrow X$ with the properties that $|f^{-1}(x)| = 1$ for each $x \in P$ and $f^{-1}(P)$ is dense in $[0,1]$ constitutes a dense G_δ-set in the completely metrizable space of all continuous surjections $[0,1] \longrightarrow X$. Clearly, each $f \in \mathbf{J}$ is irreducible and the argument remains valid when $[0,1]$ is replaced by S.

While there are many Peano continua which do not belong to Irr, it is interesting to see the following theorem proved by Ward in 1977.

THEOREM C [14]. *Each locally connected continuum is an irreducible image of some dendrite.*

2. Peano continua in general.

We start by proving the following basic result:

THEOREM 1. *If $X \in \mathrm{Irr}$ (resp. $X \in \mathrm{Irr}_0$), A is a proper closed subset of X and $\mathcal{G}$ is the decomposition of X into components of A and points, then the quotient space $X/\mathcal{G}$ belongs to* Irr *(resp. to* Irr_0*).*

More precisely, let T be one of $[0,1]$ and S and let $g: X \longrightarrow X/\mathcal{G}$ denote the quotient map. If $f: T \longrightarrow X$ is an irreducible map, then there exist a monotone surjection $m: T \longrightarrow T$ and an irreducible map $h: T \longrightarrow X/\mathcal{G}$ such that $g \circ f = h \circ m$.

PROOF. Since the decomposition of A into its components is upper semi-continuous, $\mathcal{G}$ is an upper semi-continuous decomposition of X. Therefore, $Y = X/\mathcal{G}$ is a Hausdorff space, whence it is a locally connected (metric) continuum. Let $g: X \longrightarrow Y$ denote the quotient map. Let f be an irreducible map of $T \in \{[0,1], S\}$ onto X.

Let $\mathcal{F}$ denote the decomposition of T into components of $(g \circ f)^{-1}(y)$, $y \in Y$. Since $A \neq X$, $\mathcal{F} \neq \{T\}$. Also, the members of $\mathcal{F}$ are closed intervals (most of them degenerate).

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Hence $\mathcal{F}$ is an upper semi-continuous decomposition of T and the quotient space $I = T/_{\mathcal{F}}$ is homeomorphic to T. Let $m: T \longrightarrow I$ denote the quotient map. Then there is an unique $h: I \longrightarrow Y$ such that $g \circ f = h \circ m$. Obviously, h is surjective and continuous (because if C is a closed subset of Y then $h^{-1}(C) = m\Big(f^{-1}\big(g^{-1}(C)\big)\Big)$ is closed in I; it is easy to see that $h \circ m$ constitutes the monotone-light factorization of $g \circ f: T \longrightarrow Y$, see *e.g.* [15]). We shall prove that h is irreducible.

Suppose that H is a proper closed subset of I such that $h(H) = Y$. Let U be a component of the open set $I - H$.

First, consider the case when $m^{-1}(U) \subset f^{-1}(A)$. Since $m^{-1}(U)$ is connected, $f\big(m^{-1}(U)\big)$ is contained in a component K of A. Therefore, U consists of a single point. This contradicts the fact that U is open in I.

Now, suppose that the set $V' = m^{-1}(U) - f^{-1}(A)$ is non-empty. Clearly, V' is open in T. Let V be a component of V'. Then $V = m^{-1}\big(m(V)\big)$ and $m|_V: V \longrightarrow m(V)$ is one-to-one. Also, $f^{-1}(A) \subset T - V$ and $m(V) \subset I - H$. Since $h(H) = Y$, it follows that $f(T - V) = X$. Thus f is not irreducible, a contradiction.

Now, let X be a locally connected continuum. Let $\mathcal{G}_X$ be the decomposition of X into components of the set $\mathrm{cl}\big(N(X)\big)$ and points. Also, let Y_X denote the quotient space $X/_{\mathcal{G}_X}$ and let $g_X: X \longrightarrow Y_X$ be the quotient map. Clearly, Y_X is a locally connected continuum and g_X is a monotone surjection.

The following Lemma 1 is a part of [4, (5.1)].

LEMMA 1. *A continuum X is cyclicly completely regular if and only if $Y - N(X)$ contains a non-degenerate continuum, for each non-degenerate subcontinuum Y of X.*

THEOREM 2. *Let X be a locally connected continuum. If Y_X is non-degenerate then*

(a) Y_X is a cyclicly completely regular continuum; and

(b) if $X \in \mathrm{Irr}$ then $Y_X \in \mathrm{Irr}$, and if $X \in \mathrm{Irr}_\mathbf{o}$ then $Y_X \in \mathrm{Irr}_\mathbf{o}$.

PROOF. (b) follows from Theorem 1. To prove (a) observe that the set $N(Y_X)$ of all non-local separating points of Y_X is contained in the 0-dimensional set $g_X\big(Nt(X)\big)$. By Lemma 1, Y_X is cyclicly completely regular.

In the next section some results on cyclicly completely regular continua which belong to Irr will be given. We conjecture that the converse of Theorem 2(b) is true, *i.e.*, the following problem has a positive solution:

PROBLEM 1. Let X be a locally connected continuum such that $Y_X \in \mathrm{Irr}$ or $Y_X \in \mathrm{Irr}_\mathbf{o}$. Does it follow that $X \in \mathrm{Irr}$ or $X \in \mathrm{Irr}_\mathbf{o}$?

EXAMPLE 1. One can not claim that if $f: X \longrightarrow Y$ is a monotone surjection and $X \in \mathrm{Irr}$ then $Y \in \mathrm{Irr}$ again (compare with Theorem 1; see also Theorem 6, below). In fact, let X be any one of the following spaces: the square $[0,1]^2$, the Sierpiński universal plane curve (see *e.g.* [8]), or the Janiszewski universal dendrite (see *e.g.* [8]). Then $X = \mathrm{cl}\big(N(X)\big)$ and so $X \in \mathrm{Irr}_\mathbf{o}$. Observe that X admits a monotone retraction onto the triod ($\equiv$ the only acyclic graph with 4 vertices).

The following auxiliary fact was proved in a more general setting in [9, (3.2)].

LEMMA 2. *If A is a T-set in a locally connected continuum X, then $\{K : K$ is a component of $X - A\}$ is a null-family.*

THEOREM 3. *Suppose that X is a Peano continuum and Z is a strong T-set in X such that* $\mathrm{cl}(X - Z) = X$. *If Z is a Peano continuum too, then* $X \in \mathrm{Irr}_0$.

Observe that Y_X is the Hawaiian earring (*i.e.*, the wedge of a countable null-family of circles), and so $Y_X \in \mathrm{Irr}_0$.

PROOF. Let d be a metric on X such that all open balls of X are connected and all open balls in d restricted to Z are connected too (*e.g.* d can be a simple extension to X of the Mazurkiewicz metric on Z, see [15]). For each positive integer n, let A_n be a finite subset of Z which is 2^{-n}-dense in Z.

By Lemma 2, the collection $\mathcal{F}$ of all components of $X-Z$ is a null-family. In particular, $\mathcal{F}$ is at most countable. Since $\bigcup \mathcal{F} = X - Z$ is dense in X, $\mathcal{F}$ is infinite. Let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ be a representation of $\mathcal{F}$ as the union of finite pairwise disjoint subcollections $\mathcal{F}_n$ such that $\operatorname{diam} F < 2^{-n}$ for each $F \in \mathcal{F}_n$, $n = 2, 3, \ldots$. Also, let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$ be a representation of $\mathcal{F}$ as the union of pairwise disjoint collections $\mathcal{K}_n$ such that $Z \subset \mathrm{cl}(\bigcup \mathcal{K}_n)$ for each n.

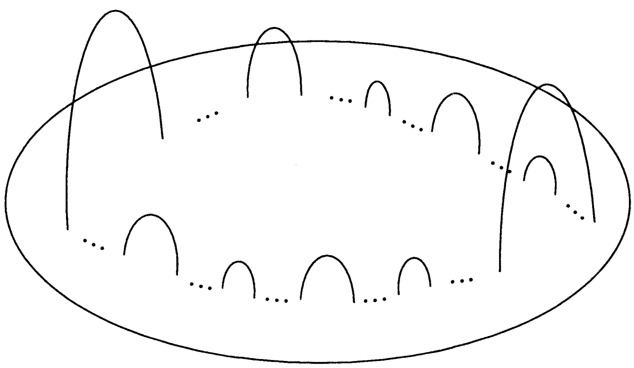

FIGURE 1

We are going to need the following fact (see Figure 1).

(*) If $F_1, \ldots, F_k \in \mathcal{F}$, $z_1, \ldots, z_l \in Z$ and $\mathcal{K} \subset \mathcal{F}$ is such that $Z \subset \mathrm{cl}(\bigcup \mathcal{K})$, then there exists $g: S \longrightarrow X$ such that

(1) $g: S \longrightarrow g(S)$ is irreducible,

(2) $g^{-1}(Z)$ is 0-dimensional,

(3) $F_1 \cup \cdots \cup F_k \cup \{z_1, \ldots, z_l\} \subset g(S) \subset F_1 \cup \cdots \cup F_k \cup Z \cup \bigcup \mathcal{K}$, and

(4) $\operatorname{diam} g(S) \leq 2 \operatorname{diam}(F_1 \cup \cdots \cup F_k \cup \{z_1, \ldots, z_l\})$.

The desired map g can be obtained as the limit of a uniformly convergent sequence $(g_n)_{n=1}^{\infty}$ of mappings $S \longrightarrow X$ such that

(i) $g_n: S \longrightarrow g(S)$ is irreducible,

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(ii) $\operatorname{diam} K \leq \frac{1}{n}$ for each component K of $g_n^{-1}(Z)$,
(iii) $F_1 \cup \cdots \cup F_k \cup \{z_1, \ldots, z_l\} \subset g_n(S) \subset F_1 \cup \cdots \cup F_k \cup Z \cup \bigcup \mathcal{K}$, and
(iv) $\operatorname{diam} g_n(S) \leq 2 \operatorname{diam}(F_1 \cup \cdots \cup F_k \cup \{z_1, \ldots, z_l\})$.

Observe that $Z \cup \bigcup \mathcal{K}$ is a locally connected continuum and intersections of $Z \cup \bigcup \mathcal{K}$ with d-balls of X are connected again. These observations make it possible to get a simple construction of $(g_n)_{n=1}^{\infty}$ as required. The details are left to the reader.

Now, let $S_1, S_2, \ldots$ be copies of S. We are going to find mappings $f_n: S_n \longrightarrow X$ and $m_n: S_{n+1} \longrightarrow S_n$ such that

(a) $\bigcup(\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n) \cup A_1 \cup \cdots \cup A_n \subset f_n(S_n)$;
(b) $f_n: S_n \longrightarrow f_n(S_n)$ is irreducible;
(c) $f_n^{-1}(Z)$ is 0-dimensional;
(d) $X - f_n(S_n)$ contains Z in its closure;
(e) m_n is a monotone surjection;
(f) if $s \in S_n$ and $m_n^{-1}(s) = \{t\}$ then $f_n(s) = f_{n+1}(t)$;
(g) if $s \in S_n$ and $m_n^{-1}(s)$ is a non-degenerate arc with end-points t and t', then $f_{n+1}(t) = f_{n+1}(t') = f_n(s)$; and
(h) if $n > 1$ and $s \in S_n$ then $\operatorname{diam} f_{n+1}\big(m_n^{-1}(s)\big) < 2^{-n+2}$.

We get f_1 as g of (*), where $\{F_1, \ldots, F_k\} = \mathcal{F}_1$, $\{z_1, \ldots, z_l\} = A_1$ and $\mathcal{K} = \mathcal{K}_1$.

Suppose that f_n is already given for some n. Let G be either a member of $\mathcal{F}_{n+1}$ or a point of A_{n+1}. Let $x_G \in Z \cap f_n(S_n)$ be such that $d(x_G, G) < 2^{-n}$ and let $s_G \in f_n^{-1}(x_G)$.

Define a mapping $m_n: S_{n+1} \longrightarrow S_n$ to be such that the only possibly non-degenerate point preimages are arcs $m_n^{-1}(s_G)$, where G is as above. For $t \in S_{n+1} - \bigcup m_n^{-1}(s_G)$, let $f_{n+1}(t) = f_n\big(m_n(t)\big)$. Also, by (*), it is easy to find mappings $f_{n+1}|_{m_n^{-1}(s_G)}$ from the arcs $m_n^{-1}(s_G)$ into X such that all the conditions (a)–(h) are satisfied.

Since all the maps m_n are monotone, $T = \operatorname{liminv}(S_n, m_n)$ is a copy of S. Let $\pi_n: T \longrightarrow S_n$ denote the natural projections. Observe that if $\pi_n^{-1}(s)$ is non-degenerate for some n and $s \in S_n$, then $\pi_n^{-1}(s)$ is an arc with end-points t and t' such that $f_k\big(\pi_k(t)\big) = f_k\big(\pi_k(t')\big)$ for each $k \geq n$.

Define $h_n: T \longrightarrow X$ by $h_n(t) = f_n\big(\pi_n(t)\big)$. By (h), the sequence $(h_n)_{n=1}^{\infty}$ uniformly converges to some mapping $h: T \longrightarrow X$. By (a), h is surjective. It is not difficult to prove that, by (c), $h^{-1}(Z)$ is 0-dimensional. By (b), it follows that h is irreducible. Thus, $X \in \mathrm{Irr}_{\mathbf{o}}$.

In contrast with Theorem 3, the following particular versions of Problem 1 show how much topology interferes with combinatorics in investigations of the class Irr.

PROBLEM 2. Does Theorem 3 remain true if the hypothesis that Z is a Peano continuum is replaced by the hypothesis that Z is merely a continuum?

PROBLEM 3. Let X be a Peano continuum and Z a strong T-set in X such that $\operatorname{cl}(X - Z) = X$. Suppose that Z is homeomorphic to the product of a continuum Z' and the Cantor set. Does then $Y_X \in \mathrm{Irr}$ or $Y_X \in \mathrm{Irr}_{\mathbf{o}}$ imply that $X \in \mathrm{Irr}$ or $X \in \mathrm{Irr}_{\mathbf{o}}$?

EXAMPLE 2. Let X be a Peano continuum and Z a nowhere dense subcontinuum of X. Suppose that $\operatorname{bd}(K)$ consists of a single point a_K and $\operatorname{cl}(K) \in \mathrm{Irr}_{\mathbf{o}}$ for each component K of $X - Z$. Then $X \in \mathrm{Irr}_{\mathbf{o}}$.

Indeed, let $A = \{a_K : K \text{ is a component of } X - Z\}$. Then A is dense in X. Let $r: X \longrightarrow Z$ be defined by $r(z) = z$ for $z \in Z$ and $r(x) = a_K$ if $x \in K$ for some component of $X - Z$. Then r is a monotone retraction. In particular, Z is a locally connected continuum. By the well-known Hahn-Mazurkiewicz theorem, there exists a continuous surjection $f: S \longrightarrow Z$. Since A is dense in Z, it is possible to modify f and get a continuous surjection $g: S \longrightarrow Z$ such that g is not constant on an open subset of S and $g^{-1}(A)$ is dense in S. There exists a dense subset $B = \{b_K : K \text{ is a component of } X - Z\}$ of S such that $g(b_K) = a_K$ for each K. Let T be another copy of S and $m: T \longrightarrow S$ be a continuous monotone map such that $m^{-1}(s)$ is non-degenerate if and only if $s = b_K$ for some K. Since $\mathrm{cl}(K) \in \mathrm{Irr}_{\mathbf{o}}$ for each component K of $X - Z$, there exists an irreducible mapping $h: T \longrightarrow X$ such that $g \circ m = r \circ h$. Therefore, $X \in \mathrm{Irr}_{\mathbf{o}}$.

In particular, one can add a null-family of simple closed curves (or copies of the universal dendrite, etc.) to a given Peano continuum Z and get a continuum $X \in \mathrm{Irr}_{\mathbf{o}}$, see Figure 2.

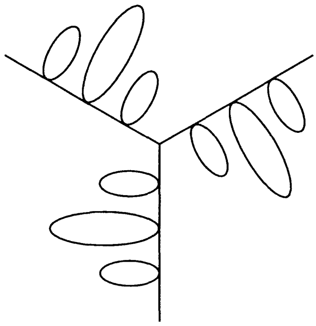

FIGURE 2

Thus addition of cyclic elements can produce a continuum which belongs to Irr or $\mathrm{Irr}_{\mathbf{o}}$.

PROBLEM 4. Relate the properties of cyclic elements and A-sets (see [15]) of a Peano (or regular, or cyclicly regular) continuum X to the property $X \in \mathrm{Irr}$ or $X \in \mathrm{Irr}_{\mathbf{o}}$.

3. **Various classes of regular continua.**

LEMMA 3 (SEE [6] OR [7]). *A continuum X is completely regular if and only if there exists a 0-dimensional subset of X which is a strong T-set in X.*

THEOREM 4. *If X is a completely regular continuum and $X \in \mathrm{Irr}$, then there exists an inverse sequence (X_n, h_n) of irr-graphs X_n with monotone bonding surjections $h_n: X_{n+1} \longrightarrow X_n$ such that $X = \mathrm{liminv}(X_n, h_n)$. If, moreover, $X \in \mathrm{Irr}_{\mathbf{o}}$ then one may assume that each X_n is an Euler graph.*

PROOF. The sequence (X_n, h_n) is exactly the one which was constructed in the proof of [10, Theorem 3.8] (see also [10, Remark 3.10 (i)] for some additional properties). Namely, by Lemma 3, there exists a strong T-set A in X such that A is 0-dimensional. Let

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$\{K_1, K_2, \ldots\}$ be an enumeration of all components of $X - A$ (by Lemma 2 and metrizability of X, that collection is countable).

For each positive integer n let $\mathcal{F}_n$ denote the family of all components of the set $Y_n = X - (K_1 \cup \cdots \cup K_n)$. Since Y_n is compact and locally connected, $\mathcal{F}_n$ is finite. Let $\mathcal{G}_n$ be the decomposition of X into points and members of $\mathcal{F}_n$. Let X_n denote the quotient space, $X_n = X/_{\mathcal{G}_n}$, and let $f_n: X \longrightarrow X_n$ be the quotient map. Note that each X_n is a graph, each f_n is a monotone surjection, and $\mathcal{F}_{n+1}$ refines $\mathcal{F}_n$ for each n. Then, for each n, we have a (unique and monotone) mapping $h_n: X_{n+1} \longrightarrow X_n$ such that $f_n = h_n \circ f_{n+1}$. It follows that X is homeomorphic to $\operatorname{liminv}(X_n, h_n)$ (see [10] for more details). By Theorem 1, $X_n \in \mathrm{Irr}$ for each n, and if $X \in \mathrm{Irr}_{\mathbf{o}}$ then $X_n \in \mathrm{Irr}_{\mathbf{o}}$ for each n.

EXAMPLE 3. Theorem 4 does not generalize to wider classes of continua.

(a) In Theorem 4 it is not enough to assume that X is cyclicly completely regular. Indeed, let X denote the Janiszewski universal dendrite. By Theorem B, $X \in \mathrm{Irr}_{\mathbf{o}}$. However, if (X_n, h_n) is an inverse sequence of graphs X_n with monotone bonding surjections h_n such that X is homeomorphic to $\operatorname{liminv}(X_n, h_n)$, then each X_n is an acyclic graph and there is n_0 such that X_{n_0} is not homeomorphic to $[0, 1]$. It follows that $X_{n_0+i} \notin \mathrm{Irr}$ for $i = 0, 1, \ldots$.

(b) Let G be a graph with $G \notin \mathrm{Irr}$ (*e.g.* a triod, see Figure 2). Let X be a Peano continuum such that $G \subset X$, $X = \mathrm{cl}(X - G)$ and each component K of $X - G$ is a copy of $]0, 1[$ with $|\mathrm{bd}(K)| = 1$. By Example 2, $X \in \mathrm{Irr}_{\mathbf{o}}$. Note that X is cyclicly completely regular and $N(X)$ is finite (whence $Y_X = X$). Observe that there is no inverse sequence (X_n, h_n) of Euler graphs X_n with monotone bonding surjections h_n such that X is homeomorphic to $\operatorname{liminv}(X_n, h_n)$.

(c) A continuum given in [4, Figure 5, p. 238] is totally regular, cyclic and not completely regular. It can be easily factorized as the inverse limit of an inverse sequence of Euler graphs with monotone bonding surjections. By Theorem 5, below, that continuum belongs to the class $\mathrm{Irr}_{\mathbf{o}}$.

(d) Figure 3, below, illustrates a continuum X such that

(1) X is totally regular and cyclic,

(2) $X \in \mathrm{Irr}$, and

(3) X does not factorize as the inverse limit of an inverse sequence of irr-graphs with monotone bonding surjections.

Concerning the considerations above, recall that a continuum which is the inverse limit of an inverse sequence of graphs with monotone bonding surjections must be totally regular, [10, (3.7)]. Conversely, each totally regular continuum is homeomorphic to the inverse limit of an inverse sequence of graphs with monotone bonding surjections, [1].

We shall need the following lemma concerning graphs.

LEMMA 4. *Let X and Y be graphs and $f: X \longrightarrow Y$ be a monotone surjection. If $y \in Y$ is a vertex of odd order, then there is a vertex x of odd order in X such that $f(x) = y$.*

PROOF. Since f is monotone, the set $f^{-1}(y)$ has a closed connected neighbourhood U in X such that

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(a) each component of $U - f^{-1}(y)$ is homeomorphic to $[0, 1[$, and
(b) $f|_U: U \longrightarrow f(U)$ is monotone.

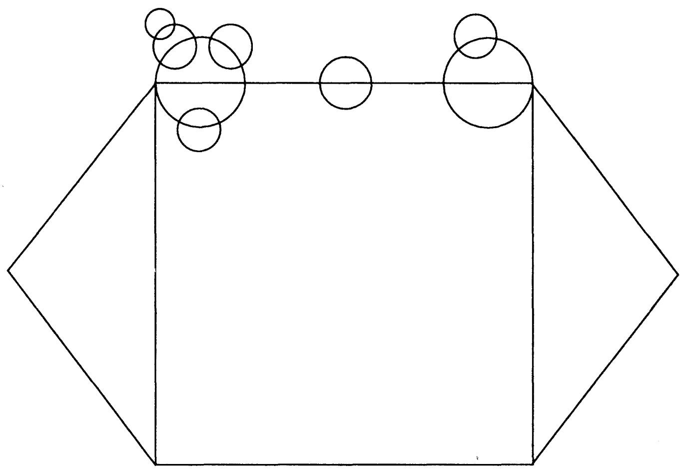


FIGURE 3

Since U is a subcontinuum of a graph X, it is a graph itself. Let $\mathcal{A} = \{K : K$ is a component of $U - f^{-1}(y)\}$. Then $\mathrm{cl}(K)$ is an arc from a_K to b_K, where $a_K \in K$ and $\{b_K\} = \mathrm{bd}(K) \subset f^{-1}(y)$. By (b), $f(U)$ is an $|\mathcal{A}|$-odd with the central point y and the end-points $f(a_K)$, $K \in \mathcal{A}$. Also, there is a one-to-one and onto correspondence between $\mathcal{A}$ and the family of all components of $f(U) - \{y\}$, which is given by $K \longmapsto f(K)$, $K \in \mathcal{A}$.

Note that $f(U)$ is a closed neighbourhood of y in Y. Hence, $|\mathcal{A}|$ coincides with the order of y in Y. Therefore, $|\mathcal{A}|$ is an odd number.

Let S and T denote the sets of all vertices of the graphs X and U, respectively. Observe that $T = \{a_K : K \in \mathcal{A}\} \cup \big(f^{-1}(y) \cap S\big)$, and the orders of z in X and in U coincide for each $z \in f^{-1}(y) \cap S$. Since $\{a_K : K \in \mathcal{A}\}$ are vertices of order 1 in U and $|\{a_K : K \in \mathcal{A}\}| = |\mathcal{A}|$ is an odd number, there is at least one more vertex x of U such that the order of x in U is odd. Then $x \in f^{-1}(y) \cap S$, and so the order of x in X is odd and $f(x) = y$.

The following fact follows immediately from Lemma 4 (another proof can be obtained from Theorem 1).

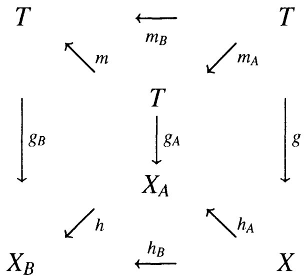
COROLLARY 1. *A monotone image of an Euler graph (resp. an irr-graph) is again an Euler graph (resp. an irr-graph).*

LEMMA 5. *Let (X_n, f_n) and (Y_n, g_n) be inverse sequences of compact spaces with surjective bonding maps. Also, let $h_n: X_n \longrightarrow Y_n$ be a sequence of mappings such that $h_n \circ f_n = g_n \circ h_{n+1}$ for each n. If all the maps h_n are irreducible, then the induced mapping h: $\mathrm{liminv}(X_n, f_n) \longrightarrow \mathrm{liminv}(Y_n, g_n)$ is irreducible.*

PROOF. Let $X = \mathrm{liminv}(X_n, f_n)$ and $Y = \mathrm{liminv}(Y_n, g_n)$. Also, let $\phi_n: X \longrightarrow X_n$ and $\psi_n: Y \longrightarrow Y_n$ denote the natural projections. Since f_n, g_n and h_n are surjective for all n, it follows that h_n, ϕ_n and ψ_n are also surjections. Since h is induced by h_n, $n = 1, 2, \ldots$, the equality $\psi_n \circ h = h_n \circ \phi_n$ holds for each n.

Suppose that there is a proper closed subset F of X such that $h(F) = Y$. Then $X - F$ is a non-empty open set, whence there exist an index m such that $\phi_n(F) \neq X_n$. Since h_n is irreducible, $h_n \circ \phi_n(F)$ is a proper subset of Y_n. On the other hand, $h_n \circ \phi_n(F) = \psi_n \circ h(F) = \psi_n(Y) = Y_n$, a contradiction.

LEMMA 6. *Let $T \in \{[0, 1], S\}$. Suppose that $g: T \longrightarrow X$ is an irreducible mapping and that $A \subset B$ are proper closed subsets of X. Let $\mathcal{G}_A$ and $\mathcal{G}_B$ denote the decompositions of X into points and components of A and B, respectively. Let $h_A: X \longrightarrow X_A = X/_{\mathcal{G}_A}$ and $h_B: X \longrightarrow X_B = X/_{\mathcal{G}_B}$ denote the corresponding quotient spaces and maps. Also, let $h: X_A \longrightarrow X_B$ denote the unique map such that $h \circ h_A = h_B$. Then there exist irreducible mappings $g_A: T \longrightarrow X_A$ and $g_B: T \longrightarrow X_B$, and monotone mappings m_A, m_B and m of T onto itself such that the following diagram commutes*



PROOF. The second part of Theorem 1 provides the following:

(a) an irreducible map $g_A: T \longrightarrow X_A$ and a monotone surjection $m_A: T \longrightarrow T$ such that $h_A \circ g = g_A \circ m_A$, and

(b) an irreducible map $g_B: T \longrightarrow X_B$ and a monotone surjection $m_B: T \longrightarrow T$ such that $h_B \circ g = g_B \circ m_B$.

Then there is the unique $m: T \longrightarrow T$ such that $m \circ m_A = m_B$. Since m_B is monotone, m is monotone too.

THEOREM 5. *If (X_n, h_n) is an inverse sequence of irr-graphs X_n with monotone bonding surjections h_n and $X = \text{liminv}(X_n, h_n)$, then $X \in \text{Irr}$. If all X_n's are Euler graphs then $X \in \text{Irr}_0$.*

PROOF. For all positive integers n and i let $h_n^{n+i} = h_n \circ \cdots \circ h_{n+i-1}: X_{n+i} \longrightarrow X_n$. Then each h_n^{n+i} is a monotone surjection. We may assume that for all n and i only finitely many point preimages $(h_n^{n+i})^{-1}(x)$, $x \in X_n$, are non-degenerate. Hence, each h_n^{n+i} may be considered as the quotient map $X_{n+i} \longrightarrow X_n = X_{n+i}/_{\mathcal{G}}$, where $\mathcal{G}$ is a decomposition of X_{n+i} into points and components of some proper closed subset. This allows applications of Theorem 1.

If there is some X_n which contains points a_n and b_n of odd order then, by Lemma 4, X_{n+i} contains points a_{n+i} and b_{n+i} of odd order such that $h_n^{n+i}(a_{n+i}) = a_n$ and $h_n^{n+i}(b_{n+i}) = b_n$, for $i = 1, 2, \ldots$. Hence, it suffices to consider the following two cases:

CASE 1. All the vertices of X_n are of even order for $n = 1, 2, \ldots$; and

CASE 2. Each X_n contains exactly two vertices of odd order.

We are going to consider Case 1 only. In Case 2 considerations are quite analogous provided the attention is restricted to irreducible maps $f: [0, 1] \longrightarrow X_n$ such that $f(0) = a_n$ and $f(1) = b_n$ for all n.

Suppose that all X_n's are Euler graphs. We may assume that all considered irreducible mappings $S \longrightarrow X_n$, $n = 1, 2, \ldots$, are piecewise linear, *i.e.*, for each k, the short subarc from $e^{\frac{2k\pi}{\alpha_n} i}$ to $e^{\frac{2(k+1)\pi}{\alpha_n} i}$ is mapped linearly onto an edge of X_n, where α_n denotes the number of edges of X_n and S is the unit circle of complex numbers (of course, we may assume that the set of vertices of each X_n is non-void, *i.e.*, X_n it is not a copy of S itself). Clearly, for each n, there are only finitely many piecewise linear irreducible maps $S \longrightarrow X_n$.

Let $f_n: S \longrightarrow X_n$ be an irreducible map for each n.

Since $h_1^n: X_n \longrightarrow X_1$ can be treated as g of Theorem 1, we get a monotone surjection $m_1^n: S \longrightarrow S$ and an irreducible map $f_{n1}: S \longrightarrow X_1$ such that $h_1^n \circ f_n = f_{n1} \circ m_1^n$. Clearly, f_{n1} may be assumed to be piecewise linear. Thus, a collection $\{f_{n1} : n = 1, 2, \ldots\}$ of piecewise linear irreducible maps $S \longrightarrow X_1$ is obtained. Therefore, there exist an irreducible and piecewise linear map $g_1: S \longrightarrow X_1$ and an infinite set N_1 of integers ≥ 2 such that $g_1 = f_{n1}$ for all $n \in N_1$.

Let i_1 denote the first element of N_1. By Theorem 1, for each $n \in N_1 - \{i_1\}$, we get a monotone surjection $m_{i_1}^n: S \longrightarrow S$ and an irreducible map $f_{ni_1}: S \longrightarrow X_{i_1}$ such that $h_{i_1}^n \circ f_n = f_{ni_1} \circ m_{i_1}^n$. Thus, a collection $\{f_{ni_1} : n \in N_1 - \{i_1\}\}$ of piecewise linear irreducible maps $S \longrightarrow X_{i_1}$ is obtained. Again, at least one member of that collection appears infinitely many often, *i.e.*, there are an irreducible and piecewise linear map $g_{i_1}: S \longrightarrow X_{i_1}$ and an infinite set $N_2 \subset N_1 - \{i_1\}$ such that $f_{ni_1} = g_{i_1}$ for all $n \in N_2$.

Let i_2 denote the first element of N_2 and proceed by induction.

The inductive construction above gives an increasing sequence $N = \{1, i_1, i_2, \ldots\}$ of integers, and irreducible maps $g_j: S \longrightarrow X_j$ and monotone surjections $m_j^n: S \longrightarrow S$ such that, by Lemma 6, $h_j^n \circ g_n = g_j \circ m_j^n$ for all $n, j \in N$ with $n > j$.

Note that $m_i^n = m_i^j \circ m_j^n$ for all $n, j, i \in N$ with $n > j > i$. Thus, (S, m_i^n, N) is an inverse sequence of copies of S with monotone bonding maps. Therefore, its inverse limit $Z = \operatorname{liminv}(S, m_i^n, N)$ is homeomorphic to S.

Of course, (X_n, h_i^n, N) is an inverse sequence the inverse limit of which is homeomorphic to X. We may assume that $\operatorname{liminv}(X_n, h_i^n, N) = X$. By Lemma 5, since $h_j^n \circ g_n = g_j \circ m_j^n$ and all the maps g_j are irreducible, the induced map $g: Z \longrightarrow X$ is irreducible. Since Z is a copy of S, $X \in \mathrm{Irr}_{\mathbf{o}}$.

COROLLARY 2. *A completely regular continuum X belongs to* Irr *(resp. to* $\mathrm{Irr}_{\mathbf{o}}$*) if and only if X is homeomorphic to* $\operatorname{liminv}(X_n, h_n)$*, where all* X_n*'s are irr-graphs (resp. Euler graphs) and all* f_n*'s are monotone surjections.*

EXAMPLE 4. Let C denote the Cantor ternary set constructed in the usual manner in $[0, 1]$ and let A be the two-point discrete space. Let $\mathcal{G}$ be the decomposition of $[0, 1] \times A$

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into the sets $\{c\} \times A$, $c \in C$, and points. Let $K = [0, 1] \times A/_{\mathcal{G}}$ denote the quotient space, see Figure 4.

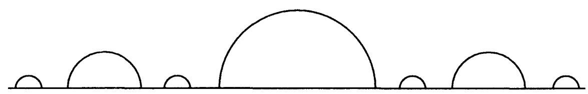

FIGURE 4

Clearly, K is a completely regular continuum which is a cyclic chain. Let a denote an end-point of K. Note that each non-degenerate cyclic element of K is a simple closed curve and non-degenerate cyclic elements of K are pairwise disjoint. Let $\mathcal{F}$ denote the decomposition of K into its non-degenerate cyclic elements and points. Then $\mathcal{F}$ is upper semi-continuous and the quotient space $K/_{\mathcal{F}}$ is an arc. Let k denote the quotient map $k: K \longrightarrow K/_{\mathcal{F}}$.

Let B be the three-point discrete space and let Z denote the quotient space $K \times B/_{\{a\} \times B}$, see Figure 5. Then Z is a completely regular continuum and $Z \in \mathrm{Irr}_0$. The mapping k of K can be used to construct a monotone surjection of Z onto a triod.

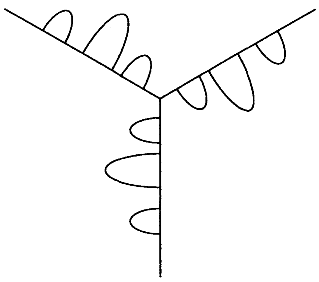

FIGURE 5

Exploring somewhat more, one can easily represent Z as the inverse limit of an inverse sequence (Z_n, h_n) of graphs Z_n with monotone bonding surjections h_n such that $Z_n \notin \mathrm{Irr}$ for $n = 1, 2, \ldots$. On the other hand recall that, by Theorem 4, there is another inverse limit representation of $Z = \mathrm{liminv}(Z'_n, h'_n)$, where all the graphs Z'_n are Euler.

Recall that if X is a regular continuum, then there exists a monotone mapping $m: Z \longrightarrow X$ of some completely regular continuum Z onto X (see [13, Remark on p. 232]). Furthermore, trivially, a monotone image of a regular continuum must be a regular continuum again.

THEOREM 6. *Suppose that X is a regular continuum. Then there exists a monotone mapping $m: Y \longrightarrow X$ of some completely regular continuum Y onto X such that $Y \in \mathrm{Irr}_0$.*

PROOF. We sketch the basic ideas of the construction of Y as required. The details are left to the reader.

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As remarked above, by a result of [13], there is a monotone map $m: Z \longrightarrow X$ of a completely regular continuum Z onto X. By Lemma 3, there is a 0-dimensional set A in Z which is a strong T-set.

Let K denote again the continuum constructed in Example 4. Let Y be the (unique) space formed from Z by replacing $\mathrm{cl}(J)$ by a copy of K, for all components J of $Z - A$. Then Y is a completely regular continuum which admits a monotone map onto X. Observe that $Y \in \mathrm{Irr}_{\mathbf{o}}$.

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Department of Mathematics
University of Saskatchewan
Saskatoon, Saskatchewan
S7N 0W0
e-mail: bula@snoopy.usask.ca

Current address:
American University of Beirut
Beirut, Lebanon
nikiel@layla.aub.ac.lb

Department of Mathematics
University of Saskatchewan
Saskatoon, Saskatchewan
S7N 0W0
tymchatyn@snoopy.usask.ca