THE ASYMPTOTIC BEHAVIOUR OF $\mu(z, \beta, \alpha)$

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1. Introduction. The function $\mu(z, \beta, \alpha)$ defined by

$$\mu(z, \beta, \alpha) = \int_0^\infty F(z, t)\, dt, \tag{1.1}$$

where

$$F(z, t) = z^{\alpha+t}t^\beta / \Gamma(\beta + 1)\Gamma(\alpha + t + 1) \tag{1.2}$$

plays an important role in Volterra's theory of convolution-logarithms, and also in the Paley-Wiener inversion formula for the Laplace transformation. These and other properties of μ are briefly described in (**3**).

Our aim in this paper is to find the asymptotic behaviour of μ as $|z| \to \infty$, a result which, as far as we are aware, has not been obtained. It is of interest to note that the procedure to be used also gives, with some minor modification, the asymptotic behaviour of μ as $z \to 0$, a result that is well known (**3**, p. 219). When this behaviour becomes known, it becomes possible to write a significant generalization of Watson's Lemma.

In 1906, Barnes (**1**) published a significant paper containing many asymptotic results of major importance. It is of passing interest to note that this paper contains a general theorem from which the result of Watson's Lemma, published in 1918, can easily be obtained. It would seem that this latter result is misnamed in mathematical literature, and might well be called Barnes' Lemma. From a very broad point of view, the pattern of the present paper will follow the pattern set by Barnes. However, the details of proof will differ so significantly from those used by Barnes that no detailed use will be made of the work of Barnes.

The paper mentioned above obtains the complete asymptotic behaviour of a function defined by

$$G_\beta(z, \alpha) = \sum_{n=0}^{\infty} z^n/n!\,(n + \alpha)^\beta, \tag{1.3}$$

a function that is not the same as $\mu(z, \beta, \alpha)$. However, their asymptotic behaviours are closely related.

2. General considerations. In (**3**, p. 222), the Laplace transformation of $\mu(z, \beta, \alpha)$ is given by

$$\int_0^\infty \mu(z, \beta, \alpha) \exp(-sz)\, dz = s^{-\alpha-1}(\log s)^{-\beta-1}, \tag{2.1}$$

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with $\operatorname{Re}\alpha > -1$ and $\operatorname{Re} s > 1$. The usual inversion formula yields:

$$\mu(z, \beta, \alpha) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} H \exp(sz)\, ds, \tag{2.2}$$

where $H = s^{-\alpha-1}(\log s)^{-\beta-1}$, and c is an arbitrary real number exceeding 1. In (2.2), z is real and positive.

It is possible to use Cauchy's Theorem to deform the path of integration in a variety of ways. The symbol L will be used as a generic symbol to denote paths of integration that begin and end at the point at infinity with the direction of approach to infinity being restricted to the third and fourth quadrants of the complex s-plane. The contour L must loop around the origin in a counter-clockwise direction so that both $s = 0$ and $s = 1$ are contained within the region bounded by L, and the contour must not cross any cuts placed in the s-plane in order to make $s^{-\alpha-1}(\log s)^{-\beta-1}$ single-valued functions of s. The main purpose of the deformation is to find an integral representation of $\mu(z, \beta, \alpha)$ in which the restriction $\operatorname{Re}\alpha > -1$ can be removed, and the requirement that z be real and positive may be relaxed. An appeal to the principle of analytic continuation allows one to identify the function of (2.2) with the function defined when the path of integration becomes L.

Suppose that sz is replaced by s, and μ is defined by a suitable path of integration L, then

$$\mu(z, \beta, \alpha) = (2\pi i)^{-1} z^{\alpha} \int_L s^{-\alpha-1}\left(\log \frac{s}{z}\right)^{-\beta-1} \exp(s)\, ds. \tag{2.3}$$

If z is now allowed to be complex, there are three possibilities that must be considered when choosing the contour L.

If Δ denotes an arbitrarily small fixed positive number, the paths of integration and cuts in the s-plane will be illustrated only for the case

$$0 \leqq \arg z \leqq \pi - 2\Delta, \tag{2.4}$$

$$-\tfrac{1}{2}\pi - \Delta \leqq \arg s \leqq \tfrac{1}{2}\pi + \Delta. \tag{2.5}$$

In this range, $\arg(s/z) = \arg s - \arg z$, and $\log(s/z) = \log s - \log z$. Since $\mu(\bar{z}, \bar{\beta}, \bar{\alpha}) = \mu(z, \beta, \alpha)$, no other range of $\arg z$ need be considered. The choice of L and the cuts in the s-plane are illustrated below (Figures 2.1–2.3).

Although three cases are listed, permitting the possibility that $\arg a = 0$ allows the analysis to be accomplished in two steps. In every case it is necessary to discuss the asymptotic behaviour of

$$F(z, \beta, \alpha) = (2\pi i)^{-1} \int_{L_1} s^{-\alpha-1}\left[\log \frac{s}{z}\right]^{-\beta-1} \exp(s)\, ds. \tag{2.6}$$

The path of integration, L_1, is divided into two parts $L_1 = A + B$, where A is that portion of L_1 contained within $|s| \leqq |z|^{\delta}$ for a fixed δ in $0 < \delta < 1$. Clearly, as $|z| \to \infty$, A will include all of the circular portion of L_1 and part of the straight line parts of L_1. Under these circumstances, B will always consist of two disjointed straight line portions of L_1.

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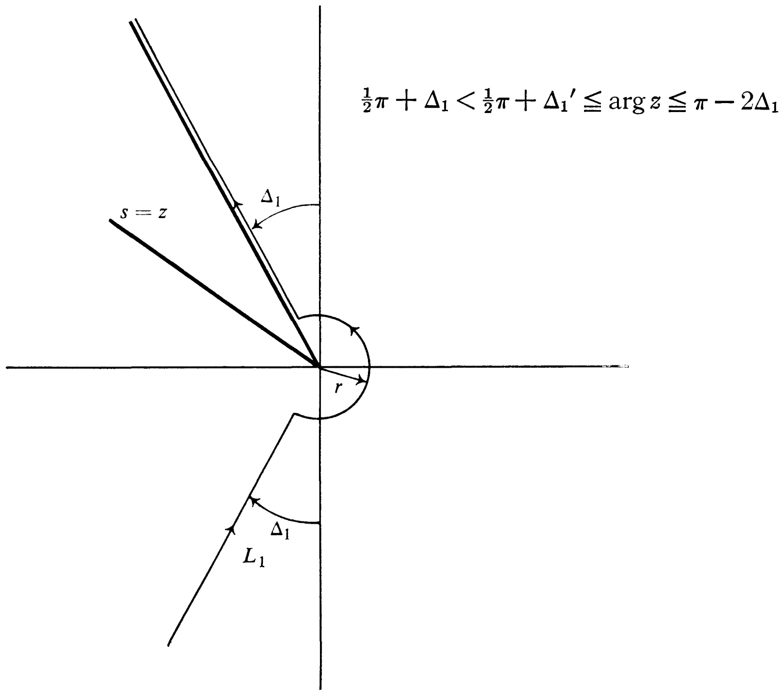

s-plane

FIGURE 2.1

On B, $\log(s/z)$ satisfies the inequalities

$$\left|\arg\left(\frac{s}{z}\right)\right| \leqq \left|\log\left(\frac{s}{z}\right)\right| \leqq \left|\log\left|\frac{s}{z}\right|\right| + \left|\arg\left(\frac{s}{z}\right)\right|. \tag{2.7}$$

Since $|\arg(s/z)|$ is uniformly bounded away from zero, $|\log(s/z)|$ is similarly uniformly bounded away from zero. Although $|\log(s/z)|$ becomes unbounded on B, it is either bounded by $\log|z|$ or by $\log|s|$, depending on which is the larger. An easy estimate yields, for $|s| \leqq |z|^{\delta}$, that a fixed $\epsilon > 0$ must exist for which

$$\int_B s^{-\alpha-1}\left[\log\left(\frac{s}{z}\right)\right]^{-\beta-1} \exp(s)\, ds = O(\exp(-\epsilon|z|^{\delta})), \tag{2.8}$$

as $z \to \infty$, with α, β unrestricted. The order relation holds uniformly.

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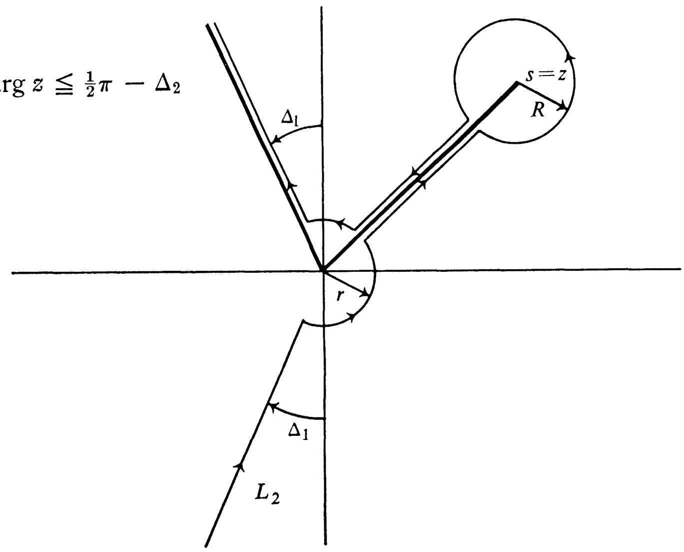
$0 \leqq \arg z \leqq \frac{1}{2}\pi - \Delta_2$



s-plane

FIGURE 2.2

On the part A of the path of integration,

$$(2.9)\quad s^{-\alpha-1}\left[\log\left(\frac{s}{z}\right)\right]^{-\beta-1} = s^{-\alpha-1}[\log s - \log z]^{-\beta-1}$$

$$= (-\log z)^{-\beta-1} s^{-\alpha-1}\left[1 - \frac{\log s}{\log z}\right]^{-\beta-1}$$

$$= (-\log z)^{-\beta-1} s^{-\alpha-1} \sum_{n=0}^{\infty} \frac{(\beta+1)_n}{n!} \frac{(\log s)^n}{(\log z)^n}$$

$$= (-\log z)^{-\beta-1} s^{-\alpha-1}$$

$$\times \left[\sum_{n=0}^{N} \frac{(\beta+1)_n}{n!} \frac{(\log s)^n}{(\log z)^n} + O\left(\frac{(\log s)^{N+1}}{(\log z)^{N+1}}\right)\right],$$

as $z \to \infty$, for every fixed integer $N \geqq 0$. Since $\int_{L_1} s^{-\alpha-1}(\log s)^n \exp(s)\, ds$ exists as an absolutely convergent integral for each fixed integer $n \geqq 0$, one must have

$$(2.10)\quad (2\pi i)^{-1} \int_A s^{-\alpha-1}\left[\log \frac{s}{z}\right]^{-\beta-1} \exp(s)\, ds$$

$$= (-\log z)^{-\beta-1}\left[\sum_{n=0}^{N} \frac{(\beta+1)_n}{n!} \frac{(2\pi i)^{-1}}{(\log z)^n} \int_A s^{-\alpha-1}(\log s)^n \exp(s)\, ds\right.$$

$$\left. + O\left(\frac{1}{(\log z)^{N+1}}\right)\right], \quad \text{as } z \to \infty.$$

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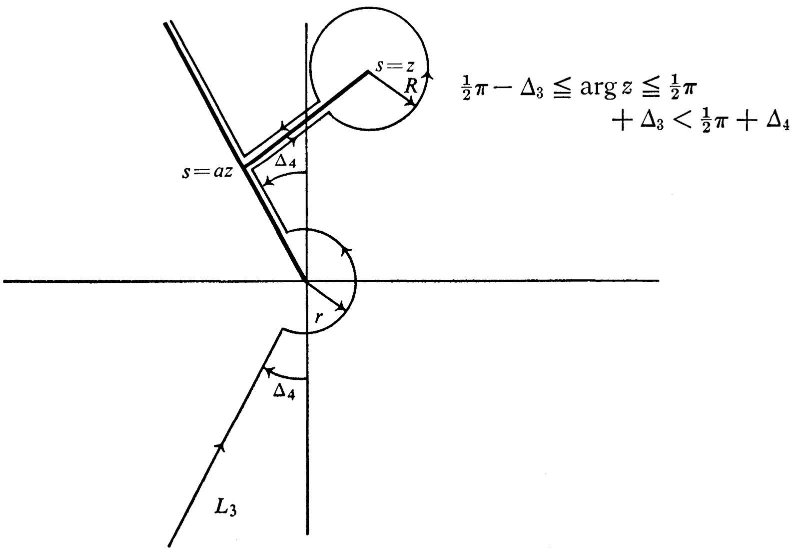

s-plane

FIGURE 2.3

However, by the same argument used to obtain (2.8),

$$(2.11)\qquad (2\pi i)^{-1}\int_A s^{-\alpha-1}(\log s)^n \exp(s)\,ds = (2\pi i)^{-1}\int_L s^{-\alpha-1}(\log s)^n \exp(s)\,ds + O(\exp(-\epsilon|z|^\delta)),\quad \text{as } z\to\infty,$$

and therefore

$$(2.12)\qquad (2\pi i)^{-1}\int_A s^{-\alpha-1}(\log s)^n \exp(s)\,ds = (-1)^n D^n[\Gamma^{-1}(\alpha+1)] + O(\exp(-\epsilon|z|^\delta)),$$

as $z\to\infty$, where $D = d/d\alpha$. These results coupled together yield:

$$(2.13)\qquad F(z,\beta,\alpha) \sim (-\log z)^{-\beta-1}\sum_{n=0}^{\infty}\frac{(\beta+1)_n D^n[\Gamma^{-1}(\alpha+1)]}{n!(-\log z)^n},$$

as $z\to\infty$ in $0 \leqq \arg z \leqq \pi - 2\Delta_1$. Further, this implies that

$$(2.14)\qquad \mu(x,\beta,\alpha) \sim z^\alpha(-\log z)^{-\beta-1}\sum_{n=0}^{\infty}\frac{(\beta+1)_n D^n[\Gamma^{-1}(\alpha+1)]}{n!(-\log z)^n}$$

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providing $z \to \infty$ in $\frac{1}{2}\pi + \Delta_1' \leqq \arg z \leqq \pi - 2\Delta_1$, a result that automatically implies the formula holds if

$$-\pi + 2\Delta_1 \leqq \arg z \leqq -\tfrac{1}{2}\pi - \Delta_1'.$$

It is of passing interest to note that, with minor modification, the same proof can be used to prove that (2.14) holds when $z \to 0$ in an unrestricted manner, a result obtained in a different way in (**3**, p. 219).

In Figure 2.3, γ will now be the portion of L_3 which is not part of L_1. It will therefore consist of the part encircling $s = z$, and the two straight lines necessary to join γ to L_1. If we replace s by $(1 - s)z$, then

$$(2.15) \quad (2\pi i)^{-1} z^{\alpha} \int_{\gamma} s^{-\alpha-1}\left[\log\left(\frac{s}{z}\right)\right]^{-\beta-1} \exp(s)\, ds$$
$$= (2\pi i)^{-1} \exp z \int_{\gamma'} (1 - s)^{-\alpha-1}[\log(1 - s)]^{-\beta-1} \exp(-sz)\, ds$$

with γ' as shown below.

The radius of the circle can still be chosen arbitrarily in $0 < R < 1$.

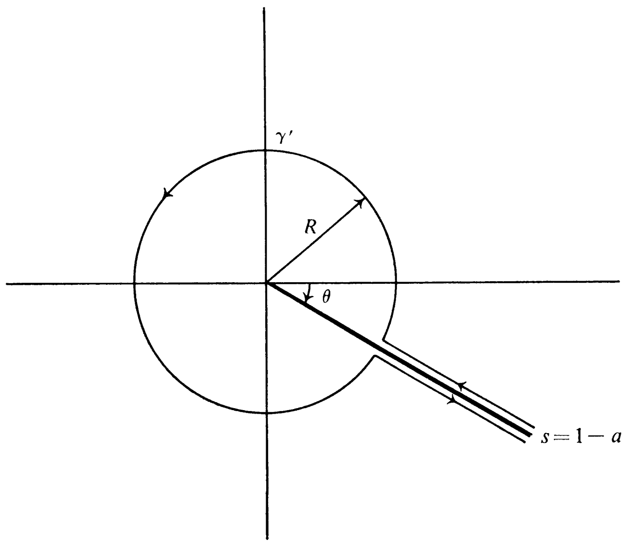

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FIGURE 2.4

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Within and on γ', $(1 - s)^{-\alpha-1}[\log(1 - s)]^{-\beta-1}$ has a convergent expansion of the form

(2.16) $$(1 - s)^{-\alpha-1}[\log(1 - s)^{-\beta-1} = (-s)^{\beta-1} \sum_{n=0}^{\infty} a_n(-s)^n, \qquad |s| < 1,$$

where

(2.17) $$a_n = \frac{(-1)^n}{n!} \frac{d^n}{ds^n} \left\{ (1 - s)^{-\alpha-1} \left[\frac{\log(1 - s)}{-s} \right]^{-\beta-1} \right\} \Bigg|_{s=0}.$$

Hence

(2.18) $$(1 - s)^{-\alpha-1}[\log(1 - s)]^{-\beta-1} = (-s)^{-\beta-1} \left[\sum_{n=0}^{N} a_n(-s)^n + R_N \right],$$

where for any fixed integer $N \geqq 0$, R_N is a regular function of s in $|s| < 1$, and

(2.19) $$|R_N| \leqq K|s|^{N+1}, \qquad |s| < 1,$$

for some fixed $K > 0$. Hence

(2.20) $$(2\pi i)^{-1} \int_{\gamma'} (1 - s)^{-\alpha-1}[\log(1 - s)]^{-\beta-1} \exp(-zs)\, ds$$
$$= \sum_{n=0}^{N} a_n (2\pi i)^{-1} \int_{\gamma'} (-s)^{n-\beta-1} \exp(-zs)\, ds$$
$$+ (2\pi i)^{-1} \int_{\gamma'} (-s)^{-\beta-1} R_N \exp(-zs)\, ds.$$

If the integer N is chosen so that $N - \operatorname{Re}(\beta - 1) > 0$, then the circular part of γ', for the remainder term only, can be shrunk to zero, leaving only straight line segments embracing the line joining $s = 0$ to $s = 1 - a$. By using (2.19), one obtains

(2.21) $$\left| \int_{\gamma'} (-s)^{-\beta-1} R_N \exp(-zs)\, ds \right| \leqq 2K \int_0^{1-a} |s^{N-\beta} \exp(-zs)\, ds|$$
$$\leqq \frac{2K}{|z|^{N+1-\beta}},$$

providing that $|\arg(ze^{-i\theta})| < \frac{1}{2}\pi$, or $-\frac{1}{2}\pi + \theta < \arg z < \frac{1}{2}\pi + \theta$, and this range will embrace the transitional region $\arg z = \frac{1}{2}\pi$.

Again by an argument used before,

(2.22) $$(2\pi i)^{-1} \int_{\gamma'} (-s)^{n-\beta-1} \exp(-zs)\, ds = (2\pi i)^{-1} \int_L (-s)^{n-\beta-1} \exp(-zs)\, ds$$
$$+ O(\exp(-\epsilon|z|)), \quad \text{as } |z| \to \infty,$$

where L consists of γ' and the extension of the straight line portions to infinity in the direction $\arg s = -\theta$. This yields:

$$(2.23)\quad (2\pi i)^{-1} \int_{\gamma'} (-s)^{n-\beta-1} \exp(-zs)\, ds = \frac{z^{\beta-n}}{\Gamma(\beta+1-n)} + O(\exp(-\epsilon|z|)), \quad \text{as } z \to \infty.$$

Coupling these results together yields:

$$(2.24)\quad (2\pi i)^{-1} \int_{\gamma'} (1-s)^{-\alpha-1}[\log(1-s)]^{-\beta-1} \exp(-zs)\, ds \sim z^{\beta} \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\beta+1-n) z^n}.$$

Hence

$$(2.25)\quad (2\pi i)^{-1} z^{\alpha} \int_{\gamma} s^{-\alpha-1}\left[\log\left(\frac{s}{z}\right)\right]^{-\beta-1} \exp(s)\, ds \sim z^{\beta} \exp(z)\left[\sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\beta+1-n) z^n}\right],$$

as $z \to \infty$ in $0 \leqq \arg z < \frac{1}{2}\pi + \theta$. The exponential nature of this behaviour allows one then to write

$$(2.26)\quad \mu(z, \beta, \alpha) \sim z^{\beta} \exp(z)\left[\sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\beta+1-n) z^n}\right] + z^{\alpha}(-\log z)^{-\beta-1}\left[\sum_{n=0}^{\infty} \frac{(\beta+1)_n}{n!} \frac{D^n[\Gamma^{-1}(\alpha+1)]}{(-\log z)^n}\right],$$

as $z \to \infty$ in $|\arg z| \leqq \pi - 2\Delta$, for any fixed Δ in $0 < \Delta < \frac{1}{2}\pi$, a result that is uniform in the approach of $z \to \infty$. The meaning of this result is of course that for any fixed integer $M \geqq 0$ and $N \geqq 0$,

$$(2.27)\quad \mu(z, \beta, \alpha) = z^{\beta} \exp(z)\left[\sum_{n=0}^{N} \frac{a_n}{\Gamma(\beta+1-n) z^n} + O\left(\frac{1}{z^{N+1}}\right)\right] + z^{\alpha}(-\log z)^{-\beta-1}\left[\sum_{n=0}^{M} \frac{(\beta+1)_n D^n[\Gamma^{-1}(\alpha+1)]}{n!(-\log z)^n} + O\left(\frac{1}{(\log z)^{M+1}}\right)\right],$$

as $z \to \infty$. Clearly, when $|\arg z| \leqq \frac{1}{2}\pi - \Delta$, the exponential dominates, and every term of the first series dominates every term of the second series. If $|\arg z| \geqq \frac{1}{2}\pi + \Delta$, the converse situation holds. When $\arg z \to \pm\frac{1}{2}\pi$, no clear-cut dominance exists except under special circumstances. If $\operatorname{Re} \alpha > \operatorname{Re} \beta$, then the first series can be dropped.

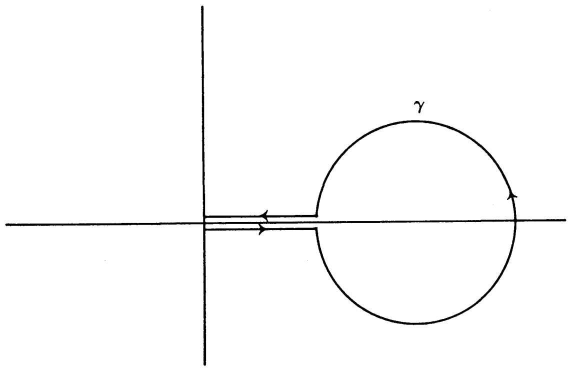
3. Two special cases. When $\beta = -m - 1$, $m = 0, 1, 2, 3, \ldots$, then all of the terms of the first series of (2.26) are zero, the second series has only a finite number of terms. In this case, there is no singularity at $s = 1$, and

$$\begin{aligned}(3.1)\qquad \mu(z, \beta, \alpha) &= (2\pi i)^{-1} \int_{L_1} s^{-\alpha-1}(\log s)^m \exp(sz)\, ds \\ &= (-1)^m D^m\left[(2\pi i)^{-1} \int_{L_1} s^{-\alpha-1}\exp(sz)\, ds\right] \\ &= (-1)^m D^m\left\{\frac{z^\alpha}{\Gamma(\alpha+1)}\right\} \\ &= (-1)^m \cdot z^\alpha \cdot (\log z)^m \sum_{n=0}^{m} \binom{m}{n} \frac{D^n\{\Gamma^{-1}(\alpha+1)\}}{(\log z)^n},\end{aligned}$$

and this exact well-known expansion adequately describes the asymptotic behaviour of $\mu(z, \beta, \alpha)$ as $z \to \infty$ or $z \to 0$ in an unrestricted manner.

The case $\beta = m, m = 0, 1, 2, \ldots$, is also a special case in that the singularity at $s = 1$ is a pole of order $m + 1$, and is no longer a branch point. The path of integration to determine μ can be broken into two parts: a path of the form L_1 and a complete circle around $s = 1$. There exists a rather elegant evaluation of the integral that encircles $s = 1$, the value is of course the residue of the integrand at this pole.

The circle around $s = 1$ is deformed into the path of integration γ shown below:



s-plane

FIGURE 3.1

We consider

$$(3.2)\qquad I = (2\pi i)^{-1} \int_\gamma s^{-\alpha-1}(\log s)^{-m-1}\exp(sz)\, ds, \quad m = 0, 1, 2, \ldots, \quad \operatorname{Re}\alpha < 0.$$

Since the integral is now regular on the real axis $0 < s < 1$, the integral along the straight line portions vanishes. The substitution $s = e^{-w}$ yields:

$$(3.3)\qquad I = (2\pi i)^{-1} \int_L (-w)^{-m-1}\exp(ze^{-w} + \alpha w)\, dw,$$

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where L begins at $w = \infty$, loops the origin in the counter-clockwise manner, and ends at $w = \infty$. Since

$$\exp(ze^{-w}) \cdot \exp(\alpha w) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \exp[-(n-\alpha)w], \tag{3.4}$$

$$\begin{aligned} I &= \frac{1}{m!} \sum_{n=0}^{\infty} \frac{z^n}{n!} (n-\alpha)^m \\ &= \frac{1}{m!} \sum_{r=0}^{m} \binom{m}{r} (-\alpha)^{m-r} \sum_{n=0}^{\infty} \frac{z^n \cdot n^r}{n!} \\ &= \frac{1}{m!} \sum_{r=0}^{m} \binom{m}{r} (-\alpha)^{m-r} \theta^r e^z, \quad \text{where } \theta = z\frac{d}{dz}. \end{aligned} \tag{3.5}$$

By the Leibniz rule, one obtains

$$I = z^\alpha \theta^m (z^{-\alpha} e^z)/m!, \tag{3.6}$$

with this exact evaluation, the condition on α can now be removed. This result implies:

$$\mu(z, m, \alpha) \sim \frac{z^\alpha}{m!} \theta^m (z^{-\alpha} e^z) + \frac{(-1)^{m+1} z^\alpha}{(\log z)^{m+1}} \sum_{n=0}^{\infty} \binom{-m-1}{n} \frac{D^n[\Gamma^{-1}(\alpha+1)]}{(\log z)^n}, \tag{3.7}$$

where $m = 0, 1, 2, \ldots$.

This result can be used to correct a result given in (**3**), where a statement equivalent to

$$\nu(z, \alpha) = \mu(z, 0, \alpha) = e^z + O(z^{\alpha - N}), \quad \text{as } z \to \infty \tag{3.8}$$

in $|\arg z| \leqq \pi$ is given, for every integer $N \geqq 0$. For $m = 0$, (3.7) yields:

$$\nu(z, \alpha) = e^z - \frac{z^\alpha}{\log z} \left\{ \frac{1}{\Gamma(\alpha+1)} + O\left(\frac{1}{\log z}\right) \right\},$$

and the result given in (3.8) is incorrect since $z^\alpha/\log z \neq O(z^{\alpha-N})$, for any non-negative integer N. Although it is not difficult to pick up the error in the outlined proof that leads to (3.8), this hardly seems worth the effort in view of the rather simple derivation of the correct result for $\nu(z, \alpha)$, which one could obtain by the method used in the present paper.

4. Conclusion. In the earlier work of Barnes (**1**) and the later work of Watson (**4**), the success of finding the asymptotic behaviour of a Laplace integral $\int_L f(t) \exp(zt)\, dt$ depended on $f(t)$ having at most a branch-point singularity at $t = 0$ of the form t^α. It is now clear from the asymptotic behaviour of $\mu(z, \beta, \alpha)$ that this work can be extended to allow $f(t)$ to have combinations of branch-point singularities and logarithmic singularities of the form $t^\alpha (\log t)^\beta$. Although some work along these lines has already been accomplished by Erdélyi (**2**), further generalization is possible.

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