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ON PRODUCTS OF CONDITIONAL EXPECTATION OPERATORS

BY
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ABSTRACT. Let (X, Σ, μ) be a probability space, let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ be k σ-subalgebras of Σ, and let $p \in \mathbb{R}$ be such that $1 < p < +\infty$. Let $P_i: L^p(X, \Sigma, \mu) \longrightarrow L^p(X, \Sigma, \mu)$ be the conditional expectation operator corresponding to $\mathcal{F}_i$ for every $i = 1, 2, \ldots, k$, and set $T = P_1 \ldots P_k$. Our goal in the note is to give a new and simpler proof of the fact that for every $f \in L^p(X, \Sigma, \mu)$, the sequence $(T^n f)_{n \in \mathbb{N}}$ converges in the norm topology of $L^p(X, \Sigma, \mu)$, and that its limit is the conditional expectation of f with respect to $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \ldots \cap \mathcal{F}_k$.

1. **Introduction.** Let (X, Σ, μ) be a probability space, let $p \in \mathbb{R}$ be such that $1 < p < +\infty$, and let $P_i: L^p(X, \Sigma, \mu) \longrightarrow L^p(X, \Sigma, \mu)$, $i = 1, 2, \ldots, k$ be k conditional expectation operators. Let $\mathcal{F}_i$ be the σ-subalgebra of Σ which defines P_i for every $i = 1, 2, \ldots, k$. Set $T = P_1 P_2 \ldots P_k$.

In [5] Halperin proved that if $p = 2$, then for every $f \in L^2(X, \Sigma, \mu)$, the sequence $(T^n f)_{n \in \mathbb{N}}$ converges in the norm topology of $L^2(X, \Sigma, \mu)$ to the conditional expectation of f with respect to $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \ldots \cap \mathcal{F}_k$. Our purpose in the present note is to give a simpler exposition of Halperin's proof, applicable for all p. The case $k = 2$ has been known for a long time (see [5]), and has been extended to a more general situation by Akcoglu and Sucheston [1]. Extensions and applications of Halperin's result can be found in the papers of Amemiya and Ando [3], Hildebrandt [6], [7], and Hildebrandt and Schmidt [8].

As expected, the question whether or not $(T^n f)_{n \in \mathbb{N}}$ converges a.e. is more difficult. If $k = 2$, and $p = 2$, then Burkholder and Chow [4] proved that $(T^n f)_{n \in \mathbb{N}}$ converges a.e. for every $f \in L^2(X, \Sigma, \mu)$. Using a beautiful construction of Rota [11] or a delicate argument of Stein [12], it follows that if $k = 2$, and $p > 1$, then $(T^n f)_{n \in \mathbb{N}}$ converges a.e. for every $f \in L^p(X, \Sigma, \mu)$. To everyone's Ornstein [10] was able to construct an example proving that it is not true, in general, that if $k = 2$, then $(T^n f)_{n \in \mathbb{N}}$ converges

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a.e. for every $f \in L^1(X, \Sigma, \mu)$. If $k \geqq 3$, it is still an open problem whether or not $(T^n f)_{n\in\mathbb{N}}$ converges a.e. for every $f \in L^p(X, \Sigma, \mu)$, $1 < p < +\infty$.

2. **Preliminaries.** The following lemma has been obtained independently by Zbăganu [14] and by Akcoglu and Sucheston [2].

LEMMA 1. *Let (X, Σ, μ) be a probability space, let $p \in \mathbb{R}$ be such that $1 < p < +\infty$, let $P: L^p(X, \Sigma, \mu) \longrightarrow L^p(X, \Sigma, \mu)$ be a conditional expectation operator, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of $L^p(X, \Sigma, \mu)$ such that $\|f_n\| = 1$ for every $n \in \mathbb{N}$. If $\lim_{n\to+\infty} \|Pf_n\| = 1$, then $\lim_{n\to+\infty} \|f_n - Pf_n\| = 0$.*

Let (X, Σ, μ) be a probability space, let $p \in \mathbb{R}$ be such that $1 < p < +\infty$, and let S be a contraction of $L^p(X, \Sigma, \mu)$.

Set $M_n = (1/n)\sum_{i=0}^{n-1} S^i$ for every $n \in \mathbb{N}$.

The Banach space $L^p(X, \Sigma, \mu)$ is reflexive. Hence, by the mean ergodic theorem (see Chap. 2 of Krengel's book [9]), the sequence $(M_n f)_{n\in\mathbb{N}}$ is convergent in the norm topology of $L^p(X, \Sigma, \mu)$ for every $f \in L^p(X, \Sigma, \mu)$.

The next lemma has been noticed independently by Wittmann [13]. We state it here in Wittmann's formulation (our form is slightly weaker).

LEMMA 2. *Assume that for every $f \in L^p(X, \Sigma, \mu)$, one has that $\lim_{n\to+\infty} \|S^{n+1}f - S^n f\| = 0$. Then, for every $f \in L^p(X, \Sigma, \mu)$, the sequence $(S^n f)_{n\in\mathbb{N}}$ converges in the norm topology of $L^p(X, \Sigma, \mu)$, and $\lim_{n\to+\infty} S^n f = \lim_{n\to+\infty} M_n f$.*

PROOF. Let $f \in L^p(X, \Sigma, \mu)$, and set $g = \lim_{n\to+\infty} M_n f$, the limit being taken in the norm topology of $L^p(X, \Sigma, \mu)$.

Let $\epsilon > 0$. Obviously, there exists $m \in \mathbb{N}$ such that $\|M_m f - g\| < \epsilon/2$.

It is easy to see that for every $n \in \mathbb{N}$, $S^{n-1} - M_n = (I - S)R_n$ for some linear bounded operator R_n.

It follows that

$$\lim_{n\to+\infty} \|S^n(S^{m-1} - M_m)f\| = \lim_{n\to+\infty} \|S^n(I - S)R_m f\| = 0.$$

Accordingly, there exists $\ell \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geqq \ell$ $\|S^n(S^{m-1} - M_m)f\| < \epsilon/2$.

Set $n_\epsilon = m + \ell$.

Since $Sg = g$, we obtain that for every $n \geqq n_\epsilon$

$$\begin{aligned} \|S^n f - g\| &\leqq \|S^n f - S^{n-m+1} M_m f\| \\ &\quad + \|S^{n-m+1} M_m f - g\| \\ &= \|S^{n-m+1}(S^{m-1} - M_m)f\| \\ &\quad + \|S^{n-m+1}(M_m f - g)\| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

We have therefore proved that for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for every $n \geqq n_\epsilon$ one has that $\|S^n f - g\| < \epsilon$. Q.E.D.

REMARK. Lemma 2 can be stated in a much more general form; that is, the space $L^p(X, \Sigma, \mu)$ can be replaced by any reflexive Banach space.

3. **The convergence of the iterates of a product of conditional expectation operators.** Let (X, Σ, μ) be a probability space, let $p \in \mathbb{R}$ be such that $1 < p < +\infty$, and let $P_1, P_2, \ldots, P_k : L^p(X, \Sigma, \mu) \longrightarrow L^p(X, \Sigma, \mu)$ be k conditional expectation operators. Set $T = P_1 P_2 \ldots P_k$.

LEMMA 3. *For every $f \in L^p(X, \Sigma, \mu)$, one has that* $\lim_{n \to +\infty} \|(T^{n+1} - T^n)f\| = 0$.

PROOF. Let $f \in L^p(X, \Sigma, \mu)$. Obviously, we may assume that $\lim_{n \to +\infty} \|T^n f\| \neq 0$.

Set $\alpha = \lim_{n \to +\infty} \|T^n f\|$.

It follows that $\|T^{n+1} f\| \leqq \|P_i P_{i+1} \ldots P_k T^n f\| \leqq \|T^n f\|$ for every $n \in \mathbb{N} \cup \{0\}$, and $i = 1, 2, \ldots, k$.

Accordingly, $\lim_{n \to +\infty} \|P_i P_{i+1} \ldots P_k T^n f\| = \alpha$ for every $i = 1, 2, \ldots, k$.

Using Lemma 1, we obtain that

$$\begin{aligned}\lim_{n \to +\infty} \|T^{n+1} f - T^n f\| \leqq \sum_{i=1}^{k-1} \lim_{n \to +\infty} &\|P_i P_{i+1} \ldots P_k T^n f \\ &- P_{i+1} P_{i+2} \ldots P_k T^n f\| \\ &+ \lim_{n \to +\infty} \|P_k T^n f - T^n f\| = 0.\end{aligned}$$

Q.E.D.

Let $\mathcal{F}_i$ be the σ-subalgebra of Σ which defines P_i for every $i = 1, 2, \ldots, k$.

THEOREM 4. *For every $f \in L^p(X, \Sigma, \mu)$, the sequence $(T^n f)_{n \in \mathbb{N}}$ converges in the norm topology of $L^p(X, \Sigma, \mu)$ to the conditional expectation of f with respect to $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \ldots \cap \mathcal{F}_k$.*

PROOF. Using Lemma 3 and Lemma 2, we obtain that for every $f \in L^p(X, \Sigma, \mu)$, the sequence $(T^n f)_{n \in \mathbb{N}}$ converges in the norm topology of $L^p(X, \Sigma, \mu)$, and that $\lim_{n \to +\infty} T^n f = \lim_{n \to +\infty} (1/n) \sum_{i=0}^{n-1} T^i f$.

Set $g = \lim_{n \to +\infty} T^n f$.

Notice that g is measurable with respect to $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \ldots \cap \mathcal{F}_k$. Clearly, we may and do assume that $g \neq 0$. Taking into consideration that $\|T^{n+1} f\| \leqq \|P_i P_{i+1} \ldots P_k T^n f\| \leqq \|T^n f\|$ for every $i = 1, 2, \ldots, k$, and every $n \in \mathbb{N} \cup \{0\}$, it follows (using Lemma 1) that $g = P_k g = P_{k-1} P_k g = \cdots = P_1 P_2 \cdots P_k g$; hence, $g = P_1 g = P_2 g = \cdots = P_k g$.

To complete the proof of the theorem, it is enough to prove that for every $A \in \mathcal{F}_1 \cap \ldots \cap \mathcal{F}_k$, one has that $\int_A f d\mu = \int_A g d\mu$.

To this end, let $A \in \mathcal{F}_1 \cap \mathcal{F}_2 \cap \ldots \cap \mathcal{F}_k$. Then,

$$\begin{aligned}\int_A g\,d\mu &= \int_A \left(\lim_{n \to +\infty} (1/n) \sum_{i=0}^{n-1} T^i f \right) d\mu \\ &= \lim_{n \to +\infty} \int_A \left((1/n) \sum_{i=0}^{n-1} T^i f \right) d\mu \\ &= \int_A f\,d\mu.\end{aligned}$$

Q.E.D.

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