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A FLAT LAGUERRE PLANE OF KLEINEWILLINGHÖFER TYPE V

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Abstract

The Kleinewillinghöfer types of Laguerre planes reflect the transitivity properties of certain groups of central automorphisms. Polster and Steinke have shown that some of the conceivable types for flat Laguerre planes cannot exist and given models for most of the other types. The existence of only a few types is still in doubt. One of these is type V.A.1, whose existence we prove here. In order to construct our model, we make systematic use of the restrictions imposed by the group. We conjecture that our example belongs to a one-parameter family of planes all of type V.A.1.

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1. Introduction

Recall that a Laguerre plane is an incidence structure that is defined in terms of points and circles and an incidence relationship between them. Kleinewillinghöfer [5] classified Laguerre planes with respect to linearly transitive groups of central automorphisms. We give the definitions in Section 3. Polster and Steinke [11] and Steinke [16] considered flat Laguerre planes and investigated their so-called Kleinewillinghöfer types, that is, the Kleinewillinghöfer types of their full automorphism groups. In particular, all possible types of flat Laguerre planes with respect to Laguerre translations (automorphisms of Laguerre planes that either are the identity or fix the points of precisely one parallel class and induce a translation in the derived affine plane at one of its fixed points) were completely determined in [11]. The case of Laguerre homotheties (automorphisms of Laguerre planes that either are the identity or fix precisely two nonparallel points and induce a homothety in the derived affine plane at each of these two fixed points) was dealt with in [16]. Examples of some of the feasible Kleinewillinghöfer types of flat Laguerre planes can be found in [11, Section 6] and [8].

In this paper, we provide an example of a flat Laguerre plane of Kleinewillinghöfer type V.A.1. With this model, the number of open cases of Kleinewillinghöfer types (with respect to Laguerre homologies, Laguerre translations and Laguerre homotheties combined) is reduced to four. Moreover, numerical evidence suggests that this example belongs to an infinite one-parameter family of flat Laguerre planes all of Kleinewillinghöfer type V.A.1.

2. Flat Laguerre planes

A *flat Laguerre plane* or *topological, locally compact, two-dimensional Laguerre plane* $\mathcal{L}$ is an incidence structure of points and circles whose point set Z is the cylinder $\mathbb{S}^1 \times \mathbb{R}$ and whose circles $C \in \mathcal{C}$ are graphs of continuous functions from $\mathbb{S}^1$ to $\mathbb{R}$ such that any three points, no two of which lie on the same generator $\{c\} \times \mathbb{R}$ of the cylinder, can be joined by a unique circle and also such that the circles that touch a fixed circle K at $p \in K$ partition the complement in Z of the generator that contains p. For more information on flat Laguerre planes, we refer to [2, 3] or [10, Ch. 5]. There are also topological Laguerre planes whose point spaces are locally compact and topologically zero or four dimensional, but we deal exclusively with flat Laguerre planes in this paper.

The generators of Z are usually referred to as the parallel classes of $\mathcal{L}$. We denote the parallel class of a point p by $|p|$. We further say that two points are parallel if they belong to the same parallel class.

For each point p of $\mathcal{L}$, we form the incidence structure $\mathcal{A}_p = (A_p, \mathcal{L}_p)$, whose point set A_p consists of all points of $\mathcal{L}$ that are not parallel to p and whose line set $\mathcal{L}_p$ consists of all restrictions to A_p of circles of $\mathcal{L}$ passing through p and of all parallel classes not passing through p. It is easy to see that $\mathcal{A}_p$ is an affine plane. We call $\mathcal{A}_p$ the *derived affine plane at* p.

Each derived affine plane $\mathcal{A}_p$ of a flat Laguerre plane is a topological, locally compact, two-dimensional affine plane, and extends to a topological, compact, two-dimensional projective plane $\mathcal{P}_p$ that we call the *derived projective plane at* p. Circles not passing through the distinguished point p induce closed ovals in $\mathcal{P}_p$ by removing the point parallel to p and adding in $\mathcal{P}_p$ the point ω at infinity of the lines that come from parallel classes of $\mathcal{L}$. The line at infinity of $\mathcal{P}_p$ (relative to $\mathcal{A}_p$) is a tangent to this oval.

The *classical real Laguerre plane* is obtained as the geometry of nontrivial plane sections of a cylinder in $\mathbb{R}^3$ with an ellipse in $\mathbb{R}^2$ as base or, equivalently, as the geometry of nontrivial plane sections of an elliptical cone in real three-dimensional projective space with its vertex removed. The parallel classes are the generators of the cylinder or cone. By replacing the ellipse in the construction of the classical flat Laguerre plane by arbitrary closed ovals in $\mathbb{R}^2$, that is, convex, differentiable, simply closed curves, we also obtain flat Laguerre planes. These are the so-called *flat ovoidal Laguerre planes*.

An automorphism of a flat Laguerre plane is a permutation of its point set that maps parallel classes to parallel classes and circles to circles. Such an automorphism is continuous and thus is a homeomorphism of Z.

The collection of all automorphisms of a flat Laguerre plane $\mathcal{L}$ forms a group with respect to composition. This group is called the automorphism group Γ of $\mathcal{L}$. When equipped with the compact-open topology, it is a Lie group (see [14]). We call the dimension of Γ the *group dimension* of $\mathcal{L}$. In general, Γ is not connected. Note that $\dim \Gamma = \dim \Gamma^1$, where Γ^1 is the connected component of Γ that contains the identity element. The maximum group dimension is seven and is attained precisely for the classical real Laguerre plane. Group dimension six does not occur. Flat Laguerre planes of group dimension five must be special ovoidal Laguerre planes (see [7, Theorem 1]).

3. Kleinewillinghöfer types of flat Laguerre planes

Kleinewillinghöfer considered four kinds of central automorphisms of Laguerre planes: C-homologies, G-translations, $(G, B(q, C))$-translations and $\{p, q\}$-homotheties. See below for definitions. *Central automorphisms* are automorphisms that have at least one fixed point at which they induce central collineations in the derived projective plane. The four different kinds of central automorphisms given above are distinguished according to the relative position of the centre and the axis and whether or not the axis is the line at infinity of the derived affine plane at the fixed point at which we derive. The notions of translation, homothety and homology describe the sort of central collineation one sees in this derived affine plane.

A subgroup of central automorphisms that have the same 'centre' and 'axis' is *linearly transitive* if the induced group of central collineations in a derived projective plane at the distinguished fixed point is transitive on each central line except for the obvious fixed points, namely, the centre and the point of intersection with the axis. Kleinewillinghöfer considered groups of automorphisms and determined their types according to linearly transitive subgroups of central automorphisms contained in them. A group of automorphisms is said to be linearly transitive if it contains a linearly transitive subgroup of central automorphisms. The Kleinewillinghöfer type of a Laguerre plane is defined to be the type of its full automorphism group. Many of the types that Kleinewillinghöfer found can be realized as types of certain subgroups, but it is much more difficult to find Laguerre planes with these types.

A *Laguerre homology* of a Laguerre plane $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that either is the identity or fixes precisely the points of one circle. One speaks of a *C-homology* if C is the circle that is fixed pointwise. For each point $q \in C$, a C-homology induces a homology of the derived projective plane $\mathcal{P}_q$. The centre of the induced homology is the point ω at infinity corresponding to the parallel classes of $\mathcal{L}$. Kleinewillinghöfer [5, Satz 3.1] found seven types of groups of automorphisms of Laguerre planes, labelled I, II, III, IV, V, VI and VII with respect to Laguerre homologies. It is known that type VI cannot occur as the type of a flat Laguerre plane (see [11, Proposition 3.4]). The

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examples from [11, Section 6], [8, 13] and the model given in this paper show that all remaining types with respect to Laguerre homologies occur, except perhaps type IV.

A *Laguerre translation* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that either is the identity or fixes precisely the points of one parallel class and induces a translation in the derived affine plane at one of its fixed points. Laguerre translations come in two different varieties. The first variety is a nonidentity *G-translation* of $\mathcal{L}$, which is a Laguerre translation that fixes precisely the points of the parallel class G, and further fixes each parallel class globally.

In order to describe the second variety of Laguerre translations, we consider a tangent bundle $B(p, C)$, that is, all circles that touch the circle C at the point p. In the derived affine plane at p, the tangent bundle represents a bundle of parallel lines, and we can look at translations in this direction. A $(G, B(p, C))$-*translation* of $\mathcal{L}$ is a Laguerre translation that fixes C and each circle in $B(p, C)$ globally.

Kleinewillinghöfer obtained 11 types of groups of automorphisms of Laguerre planes that can occur with respect to Laguerre translations and labelled them A to K (see [5, Satz 3.3] or [6, Satz 2]). Out of these 11 types, only types F, I and J cannot occur as the types of flat Laguerre planes (see [11, Proposition 4.8]). There are examples of flat Laguerre planes for each of the eight remaining types with respect to Laguerre translations (see [11, Section 6]).

Finally, a *Laguerre homothety* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that either is the identity or fixes precisely two nonparallel points and induces a homothety in the derived affine plane at each of these two fixed points. One speaks of a $\{p, q\}$-*homothety* if p and q are the two fixed points.

Kleinewillinghöfer (see [5, Satz 3.2] or [6, Satz 1]) obtained 13 types of groups of automorphisms of Laguerre planes with respect to Laguerre homotheties and labelled them from 1 to 13. Types 5, 6, 7, 9, 10 and 12 cannot occur as the types of flat Laguerre planes (see [11, Proposition 5.6] and [16]). There are examples of flat Laguerre planes for each of the remaining types with respect to Laguerre homotheties except for, possibly, type 2 (see [11, Section 6]).

Combining all three classifications, Kleinewillinghöfer obtained a total of 46 types. Of these 46 types, 21 cannot occur for flat Laguerre planes. There are models of flat Laguerre planes of types I.A.1, I.B.1, I.B.3, I.C.1, I.E.1, I.E.4, I.G.1, I.H.1, I.H.11, II.A.1, II.E.1, II.E.4, II.G.1, III.B.1, III.B.3, III.H.1, III.H.11, VII.D.1, VII.D.8 and VII.K.13 (see [11, Section 6] and [8, 13, 15, 16]). Here a combined type just refers to the constituent simple types. The examples from [11, Section 6] and [8, 13], together with the model given in this paper, leave the question of existence for flat Laguerre planes open for Kleinewillinghöfer types I.A.2, II.A.2, IV.A.1 and IV.A.2 only.

4. The general setting for a flat Laguerre plane of Kleinewillinghöfer type V

In this section, we consider a flat Laguerre plane $\mathcal{L}$ of type V.A.1. This means that the set $\mathcal{Z}$ of all circles C for which the automorphism group of $\mathcal{L}$ is linearly transitive with respect to C-homologies consists of a flock of $\mathcal{L}$, that is, the circles in $\mathcal{Z}$ partition

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the point set of $\mathcal{L}$ (type V), that there is neither a tangent bundle nor a parallel class for which the group of Laguerre translations is linearly transitive (type A) and that there is no group of Laguerre homotheties that is linearly transitive (type 1).

Theorem 4.1. *A flat Laguerre plane $\mathcal{L}$ of Kleinewillinghöfer type V with respect to Laguerre homologies has group dimension two or three. In the latter case, the automorphism group of $\mathcal{L}$ has precisely two orbits on the circle space C. One orbit consists of all the circles in the flock $\mathcal{F}$ as in type V, and the other orbit is $C \setminus \mathcal{F}$.*

Proof. Let $\mathcal{L}$ be a flat Laguerre plane of Kleinewillinghöfer type V and let $\mathcal{F}$ be the flock of $\mathcal{L}$ as in type V. Let G be the group generated by all Laguerre homologies at circles in $\mathcal{F}$. By the definition of Laguerre homologies, the group G fixes each parallel class. Moreover, G acts two-transitively and effectively on $\mathcal{F}$ and on each parallel class. Furthermore, G is isomorphic to the two-dimensional affine group consisting of all transformations of the form $x \mapsto ax + b$, where $a, b \in \mathbb{R}$ and $a \neq 0$.

Every automorphism of $\mathcal{L}$ leaves $\mathcal{F}$ invariant. In particular, each automorphism in the connected component Γ^1 of the automorphism group Γ of $\mathcal{L}$ that contains the identity fixes $\mathcal{F}$. Furthermore, Γ^1 is transitive on $\mathcal{F}$ and on each parallel class.

Let C_0 and C_1 be two circles in $\mathcal{F}$ and let $\Sigma = (\Gamma_{C_0,C_1})^1$ be the connected component of the stabilizer Γ_{C_0,C_1} of the two circles. Note that Σ is a subgroup of Γ^1, by connectedness. Since C_0 and C_1 have one-dimensional orbits under Γ (namely $\mathcal{F}$),

$$\dim \Gamma = 2 + \dim \Sigma$$

by the dimension formula for Lie transformation groups, which relates the dimensions of the group, the orbits and the stabilizers. Moreover, Σ acts effectively on C_0.

Let p_0 be a point on C_0 and consider the stabilizer $\Sigma_0 = \Sigma_{p_0}$. Then Σ_0 induces a group $\tilde{\Sigma}_0$ of collineations of the derived projective plane $\mathcal{P}_{p_0}$ of $\mathcal{L}$ at p_0. This group fixes the line W at infinity, the oval $\tilde{C}_1$ induced by the circle C_1 and the point w_0 at infinity of the line that comes from C_0. Since $\tilde{C}_1$ is a topological oval in $\mathcal{P}_{p_0}$, there are precisely two tangents to $\tilde{C}_1$ through w_0 (see [4, Statement 2.5.b], [1, Satz 3.7.a] or [12, Proof of Proposition 55.17]). One of these tangents is W. Let L be the other tangent and let $q_1 = L \cap \tilde{C}_1$. Then $\tilde{\Sigma}_0$ fixes q_1, and so does Σ_0 because q_1 is a point of $\mathcal{L}$. Let $p_1 = C_1 \cap |p_0|$ and $q_0 = C_0 \cap |q_1|$. Then Σ_0 fixes the four points p_0, p_1, q_0 and q_1. Note that the circle D_0 that induces L in $\mathcal{P}_p$ touches C_0 at p_0 and C_1 at q_1.

Let D_1 be the circle through p_1 that touches C_0 at q_0. If, for example, C_1 is above C_0 in Z, then we see that $p_1 \in D_1$ is above $p_0 \in D_0$ in $|p_0|$ and $q_0 \in D_1$ is below $q_1 \in D_0$ in $|q_1|$. Hence, D_0 and D_1 must intersect in two points r_1 and r_2 and Σ_0 fixes $\{r_1, r_2\}$. However, r_1 and r_2 lie in different connected components of $Z \setminus \{|p_0|, |q_1|\}$, so Σ_0 must fix each of r_1 and r_2 by the connectedness of Σ.

Now, in a flat Laguerre plane, given three points on a circle and a fourth point off this circle, the stabilizer of these four points is trivial (see [14]). We deduce that $\Sigma_0 = \{\mathrm{id}\}$. In particular, $\dim \Sigma_0 = 0$, so $\dim \Sigma \leq 1$, by the dimension formula, and thus Γ is at most three dimensional.

We now assume that Γ is three dimensional. From the arguments above, we infer that Σ must be one dimensional. Furthermore, Σ is connected, transitive and effective on C_0.

Given a circle C not in $\mathcal{F}$, there is a circle B in $\mathcal{F}$ that touches C from below. Using the group Γ, the circle B and the point of touching $C \cap B$ can be mapped to the circle C_0 and any point p_0 on C_0. The group of C_0-homologies is transitive on the set of circles touching C_0 at p_0 other than C_0, and so B can be taken to any circle that touches C_0 at p_0 other than C_0. For example, B can be taken to the circle D_0 from above. This shows that the circles in $\mathcal{C} \setminus \mathcal{F}$ form an orbit under Γ. □

We keep the notation of the proof of Theorem 4.1 and assume that Γ is three dimensional. We represent the cylinder Z as $\mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. The coordinates obtained in this way differ from those usually used but are more convenient for us. We may choose any interval $[u, u + 2\pi)$ of length 2π with its end points identified to represent $\mathbb{S}^1$. In particular, circles are represented by graphs of continuous periodic functions with period 2π.

As we have seen, Σ is one dimensional and connected, and acts transitively and effectively on C_0. Hence, Σ is isomorphic to the rotation group $SO_2(\mathbb{R})$ and, in fact, acts equivalently to the standard group of rotations on Z by Brouwer's theorem (see [12, Theorem 96.30]). We may therefore assume that the transformations in Σ are given by

$$(x, y) \mapsto (x + t, y),$$

where $t \in \mathbb{R}/2\pi\mathbb{Z}$. The circles in the flock $\mathcal{F}$ are orbits under this group, so we obtain the circles $\{(x, a) \mid x \in \mathbb{S}^1\}$ with $a \in \mathbb{R}$.

As in the proof of Theorem 4.1, let G be the group generated by all Laguerre homologies at circles in $\mathcal{F}$. Since $\gamma \in G$ fixes each parallel class and $\sigma \in \Sigma$ fixes each circle in $\mathcal{F}$, the commutator $\gamma^{-1}\sigma^{-1}\gamma\sigma$ fixes each parallel class and each circle in $\mathcal{F}$, and thus must be the identity. This shows that G and Σ commute. We may therefore assume that the maps

$$(x, y) \mapsto (x + t, sy + a),$$

where $a, s \in \mathbb{R}$ with $s \neq 0$ and $t \in \mathbb{R}/2\pi\mathbb{Z}$, are automorphisms of $\mathcal{L}$.

Let $\{(x, f(x)) \mid x \in \mathbb{S}^1\}$ be a circle through $(\pi, 0)$ and $(0, 0)$, where $f : \mathbb{S}^1 \to \mathbb{R}$ is continuous but not identically 0. Then the circles of $\mathcal{L}$ are of the form

$$\{(x, sf(x + t) + a) \mid x \in \mathbb{R}/2\pi\mathbb{Z}\},$$

where $a, s \in \mathbb{R}$ and $t \in \mathbb{R}/2\pi\mathbb{Z}$. This shows that $\mathcal{L}$ is completely determined by the single function f in this case.

In particular, when $a = 0$, $s = 1$ and $t = \pi$, one has the circle

$$\{(x, f(x + \pi)) \mid x \in \mathbb{R}/2\pi\mathbb{Z}\}$$

that passes through $(\pi, 0)$ and $(0, 0)$. Thus, it must be of the form $y = sf(x)$ for a suitable nonzero s, that is,

$$f(x + \pi) = sf(x)$$

for all $x \in \mathbb{R}/2\pi\mathbb{Z}$. Applying this identity again to $f(x+\pi)$, one finds that

$$f(x) = f(x+2\pi) = sf(x+\pi) = s^2 f(x)$$

for all $x \in \mathbb{R}/2\pi\mathbb{Z}$. Hence, $s^2 = 1$ and then $s = -1$, because a rotation through π takes the positive half of $y = f(x)$ to the negative half of $y = sf(x)$. Therefore,

$$f(x+\pi) = -f(x)$$

for all $x \in \mathbb{R}/2\pi\mathbb{Z}$.

Since Γ is transitive on the points of $\mathcal{L}$, it follows that $\mathcal{L}$ is a Laguerre plane if and only if the derived incidence structure $\mathcal{A}$ of $\mathcal{L}$ at $(\pi, 0)$ is an affine plane. From the description of circles above, one finds that the nonvertical lines of $\mathcal{A}$ are given by

$$y = s(f(x+u) + f(u))$$

for all $s \in \mathbb{R}$ and $u \in \mathbb{R}/2\pi\mathbb{Z}$.

Note that the classical flat Laguerre plane admits a three-dimensional group of automorphisms as described above. The usual parabola model of the classical flat Laguerre plane has point set $(\mathbb{R} \cup \{\infty\}) \times \mathbb{R}$ and circles

$$\{(x, ax^2 + bx + c) \mid x \in \mathbb{R}\} \cup \{(\infty, a)\},$$

where $a, b, c \in \mathbb{R}$.

According to [9, Proposition 2], there is a unique topology extending the natural topology of $\mathbb{R}^2$ such that one obtains a flat Laguerre plane. In this model, an example of the group of transformations is given by

$$\left\{(x, y) \mapsto \left(\frac{x\cos t - \sin t}{x \sin t + \cos t}, \frac{sy + a(x^2+1)}{(x\sin t + \cos t)^2}\right) \,\middle|\, a, s, t \in \mathbb{R}, s \neq 0\right\}.$$

Here, in the first coordinate, the usual conventions for linear fractional maps on $\mathbb{R} \cup \{\infty\}$ apply when dealing with the symbol ∞ or when dividing by 0.

Because of the way the infinite parallel class $\{\infty\} \times \mathbb{R}$ fits topologically into the cylinder in the parabola model, the behaviour in the second coordinate is less straightforward. For example, (∞, y) is mapped to $(\cot t, (sy + a)/\sin^2 t)$ if $\sin t \neq 0$. Also, an affine point (u, v) is close to (∞, y) if and only if u is close to ∞, that is, if and only if $|u|$ is large and also $v/(u^2+1)$ is close to y.

However, of course, the classical Laguerre plane has type VII. The coordinate transformation

$$(\mathbb{R} \cup \{\infty\}) \times \mathbb{R} \to (-\pi, \pi] \times \mathbb{R} : (x, y) \mapsto \begin{cases} \left(2\tan^{-1}(x), \dfrac{y}{x^2+1}\right) & \text{when } x \in \mathbb{R}, \\ (\pi, y) & \text{when } x = \infty \end{cases}$$

takes a circle

$$\{(x, ax^2 + bx + c) \mid x \in \mathbb{R}\} \cup \{(\infty, a)\}$$

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to

$$\left\{\left(u, \frac{b}{2}\sin u + \frac{c-a}{2}\cos u + \frac{c+a}{2}\right) \,\middle|\, u \in \mathbb{R}/2\pi\mathbb{Z}\right\}$$

and brings the group to the form that we used above. Consequently, the function $f(x) = \sin x$ yields a Laguerre plane in the setting above, albeit the classical plane.

5. A model for a flat Laguerre plane of type V.A.1

In this section, we construct a flat Laguerre plane of Kleinewillinghöfer type V.A.1. We build on the results found in the previous section, assuming that there is a three-dimensional automorphism group. In order to obtain our model, we modify the describing function of the classical real Laguerre plane. More precisely, we use the describing function

$$f(x) = \frac{\sin x}{1 + \sin^2 x}$$

for all $x \in \mathbb{R}/2\pi\mathbb{Z}$.

As in Section 4, our model for a Laguerre plane has point set $Z = \mathbb{S}^1 \times \mathbb{R}$ and circles

$$C_{a,t,b} = \{(x, af(x+t) + b) \mid x \in \mathbb{R}/2\pi\mathbb{Z}\},$$

where $a, b \in \mathbb{R}$ and $t \in \mathbb{R}/\pi\mathbb{Z}$. We claim that the collection $\mathcal{C}$ of the above sets forms the circle set of a flat Laguerre plane $\mathcal{L}$.

Note that the parameters a and t are not uniquely determined by a circle. Indeed, $C_{-a,t,b} = C_{a,t+\pi,b}$ and $C_{0,0,b} = C_{0,t,b}$ for all t. The first of these coincidences is avoided by taking t modulo π rather than 2π. The second of the coincidences cannot be avoided. We often use $[-\frac{1}{2}\pi, \frac{3}{2}\pi)$ or $(-\pi, \pi]$ or any other convenient interval of length 2π with its end points identified to represent $\mathbb{S}^1$, the set of first coordinates of Z. Moreover, note that each circle is the graph of a continuous function from $\mathbb{S}^1 \approx \mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$.

The circles $C_{0,0,b}$, where $b \in \mathbb{R}$, form a partition of the cylinder Z, that is,

$$\mathcal{F} = \{C_{0,0,b} \mid b \in \mathbb{R}\}$$

is a flock. Note that the restrictions on t made above ensure that each circle not in $\mathcal{F}$ uniquely determines its parameters a, t and b.

It is readily verified that the permutations

$$\gamma_{r,c,s} : (x, y) \mapsto (x + s, ry + c),$$

where $x \in \mathbb{R}/2\pi\mathbb{Z}$, $c \in \mathbb{R}$, $r \in \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}/2\pi\mathbb{Z}$, are automorphisms of $\mathcal{L}$ and that $\gamma_{r,c,s}(C_{a,t,b}) = C_{ra,t-s,rb+c}$. Moreover, the permutation

$$\sigma : (x, y) \mapsto (-x, y)$$

is an automorphism of $\mathcal{L}$ and $\sigma(C_{a,t,b}) = C_{-a,-t,b}$. The group Δ generated by these permutations of Z is transitive on Z and has two orbits on the set of circles, namely, $\mathcal{F}$ and $\mathcal{C} \setminus \mathcal{F}$.

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Before we come to the verification of the geometric axioms of a Laguerre plane, we list some useful properties of the function f that are straightforward to check. The function f is periodic with period 2π and $f(x+\pi) = -f(x)$ for all $x \in \mathbb{R}/2\pi\mathbb{Z}$. Furthermore, f is infinitely often differentiable, and

$$f'(x) = \frac{\cos^3 x}{(1+\sin^2 x)^2}$$

and

$$f''(x) = -\frac{\sin x \cos^2 x(6+\cos^2 x)}{(1+\sin^2 x)^3}.$$

Proposition 5.1. *Two distinct circles intersect in at most two points.*

Proof. Let C_1 and C_2 be two distinct circles. Since the automorphism group has two orbits on $\mathcal{C}$, it suffices to look at the following three cases.

In the first case, when $C_1, C_2 \in \mathcal{F}$, the circles are disjoint because the circles in $\mathcal{F}$ form a partition of Z.

In the second case, when $C_1 \in \mathcal{F}$ and $C_2 \in \mathcal{C} \setminus \mathcal{F}$, then, by using the group Δ, we may assume that $C_1 = C_{0,0,b}$ and $C_2 = C_{1,0,0}$. We then have to solve the equation

$$f(x) = \frac{\sin x}{1+\sin^2 x} = b$$

for x. When $b = 0$, it is obvious that $C_1 \cap C_2 = \{(0,0), (\pi, 0)\}$, and there are two points of intersection. When $b \neq 0$, we obtain the equation

$$\sin^2 x - \beta \sin x + 1 = 0,$$

where we have written $\beta = 1/b$.

The quadratic polynomial in $\sin x$ on the left-hand side has discriminant $\beta^2 - 4$. If $|\beta| < 2$, then there are no solutions. If $|\beta| = 2$, then $\sin x = \frac{1}{2}\beta = \pm 1$, and we obtain precisely one solution in $\mathbb{S}^1$. If $|\beta| > 2$, then $\sin x = \frac{1}{2}(\beta \pm \sqrt{\beta^2 - 4})$. However, if $\beta > 2$, then $\frac{1}{2}(\beta + \sqrt{\beta^2-4}) > 1$ and, if $\beta < 2$, then $\frac{1}{2}(\beta - \sqrt{\beta^2-4}) < -1$, which is not admissible. Therefore,

$$\sin x = \frac{1}{2}\Big(\beta - \operatorname{sign}(\beta)\sqrt{\beta^2 - 4}\Big),$$

and this yields precisely two solutions in $\mathbb{S}^1$. Hence, $|C_1 \cap C_2| \leq 2$.

In the third case, $C_1, C_2 \in \mathcal{C} \setminus \mathcal{F}$. By using the group Δ and the symmetry between the two circles, we may assume that $C_1 = C_{a,0,b}$ and $C_2 = C_{1,t,0}$. Note that $a \neq 0$, because $C_1 \notin \mathcal{F}$. Furthermore, we may assume that $0 < t < \pi$, because if $t = 0$ then there is no point of intersection when $a = 1$ and there are at most two points of intersection when $a \neq 1$. Indeed, if $a \neq 1$, then one is led to the equation $f(x) = b/(1-a)$, which has at most two solutions, as we saw in the second case.

We then have to solve an equation of the form

$$f(x+t) - af(x) = b.$$

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Let

$$g_{a,t}(x) = f(x+t) - af(x)$$

denote the function on the left-hand side of the equation. Now $g_{a,t}$ is a bounded, continuously differentiable function and thus has a maximum and a minimum. At the extremal points, the derivative of $g_{a,t}$ must be zero. Since $f(x+\pi) = -f(x)$, one sees that $g'_{a,t}(x+\pi) = -g'_{a,t}(x)$. Hence, the roots of the derivative come in pairs, one in $[-\frac{1}{2}\pi, \frac{1}{2}\pi)$ and one in $[\frac{1}{2}\pi, \frac{3}{2}\pi)$.

We want to show that the equation $g'_{a,t}(x) = 0$ has exactly one solution x_0 in $[-\frac{1}{2}\pi, \frac{1}{2}\pi)$. Once this is verified and, say $g'_{a,t}(-\frac{1}{2}\pi) > 0$, we may deduce that $g_{a,t}(x)$ is strictly increasing between $x_0 - \pi$ and x_0 and strictly decreasing from x_0 to $x_0 + \pi$. Thus, $g_{a,t}(x) = b$ has at most two solutions. Hence, the two circles C_1 and C_2 have at most two points in common.

To verify our claim about the zeros of $g'_{a,t}(x)$ in $[-\frac{1}{2}\pi, \frac{1}{2}\pi)$, note that $f'(\pm\frac{1}{2}\pi) = 0$ and $f'(x) > 0$ when $x \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$. This implies that $g'_{a,t}(-\frac{1}{2}\pi) \neq 0$. Then $g'_{a,t}(x) = 0$ if and only if $f'(x+t)/f'(x) = a$. It therefore suffices to show that the function

$$h_t(x) = \frac{f'(x+t)}{f'(x)}$$

is a bijection from $I = (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ onto $\mathbb{R}$. Now $f'(-\frac{1}{2}\pi + t) > 0$ and $f'(\frac{1}{2}\pi + t) < 0$ (note that $0 < t < \pi$), and so $\lim_{x \to -\frac{1}{2}\pi^+} h_t(x) = +\infty$ and $\lim_{x \to \frac{1}{2}\pi^-} h_t(x) = -\infty$. Hence, $h_t(x)$ is surjective.

For the injectivity of $h_t(x)$, consider

$$h'_t(x) = \frac{f''(x+t)f'(x) - f'(x+t)f''(x)}{(f'(x))^2}.$$

Clearly, h'_t has a zero at $\frac{1}{2}\pi - t$ in I. In order to find other possible zeros of h'_t in I, we follow an idea of one of the referees, which shortens our original proof. We consider

$$F(x) = -\frac{f''(x)}{f'(x)} = \frac{(6 + \cos^2 x)\sin x}{(1 + \sin^2 x)\cos x} = G(\tan x),$$

where G is the rational function

$$G(u) = \frac{(7 + 6u^2)u}{1 + 2u^2} = 3u + \frac{4u}{1 + 2u^2}.$$

Now

$$G'(u) = \frac{7 + 4u^2 + 12u^4}{(1 + 2u^2)^2} > 0$$

and thus

$$F'(x) = G'(\tan x)(1 + \tan^2 x) > 0.$$

This shows that F is injective.

We now assume that $x_0 \neq \frac{1}{2}\pi - t$ is a zero of h_t' in I. Then $f'(x_0)f'(x_0 + t) \neq 0$, and we see that $F(x_0 + t) = F(x_0)$. However, as seen before, F is injective on I, so this identity cannot hold on I. If $x_0 + t > \frac{1}{2}\pi$, then $x_0 + t - \pi \in I$. But F is periodic with period π, so

$$F(x_0 + t - \pi) = F(x_0 + t) = F(x_0),$$

which again contradicts the injectivity of F.

This shows that $\frac{1}{2}\pi - t$ is the only zero of h_t' in the interval I. Hence, $h_t'(x) < 0$ for all $x \in I \setminus \{\frac{1}{2}\pi - t\}$, and we have shown that $h_t(x)$ is injective and thus a bijection. □

To verify that $\mathcal{L}$ is a Laguerre plane, it suffices to show that the derived incidence structure $\mathcal{A} = \mathcal{A}_{(\pi,0)}$ at the point $(\pi, 0)$ is an affine plane, as a consequence of the transitivity of the group Δ on the point set. The nonvertical lines of $\mathcal{A}$ are the circles $C_{a,t,af(t)}$ minus the point $(\pi, 0)$. Thus, the nonvertical lines $L_{a,t}$ of $\mathcal{A}$ are given by

$$y = a(f(x + t) + f(t)),$$

where $a \in \mathbb{R}$, $a \neq 0$ and $t \in \mathbb{R}/\pi\mathbb{Z}$.

Proposition 5.2. *Two distinct points in $\mathcal{A}$ are joined by a unique line.*

Proof. Let (x_1, y_1) and (x_2, y_2) be two given points of $\mathcal{A}$. If $x_1 = x_2$, then the vertical line $x = x_1$ is the only line of $\mathcal{A}$ joining the two points.

We now assume that $x_1 \neq x_2$. We then have to solve the system of equations

$$\begin{aligned} y_1 &= a(f(x_1 + t) + f(t)),\\ y_2 &= a(f(x_2 + t) + f(t)) \end{aligned}$$

for a and t.

If $y_1 = y_2 = 0$, then $a = 0$ is a solution. Otherwise, $a \neq 0$, so t is a solution of

$$y_2(f(x_1 + t) + f(t)) - y_1(f(x_2 + t) + f(t)) = 0.$$

Let $h(t)$ denote the function of t on the left-hand side of the above equation. Then

$$h(0) = y_2 f(x_1) - y_1 f(x_2),$$

$$\begin{aligned} h(\pi) &= y_2 f(x_1 + \pi) - y_1 f(x_2 + \pi)\\ &= y_1 f(x_2) - y_2 f(x_1) = -h(0). \end{aligned}$$

Hence, when $h(0) \neq 0$, there is at least one solution t in the interval $(0, \pi)$ of the above equation by the continuity of h. If $h(0) = 0$, then of course $t = 0$ is a solution.

Both y_1 and y_2 cannot be equal to 0, and so at least one of $f(x_1 + t) + f(t)$ and $f(x_2 + t) + f(t)$ is nonzero, and $a = y_i/(f(x_i + t) + f(t))$ for the appropriate i. The uniqueness of the resulting line now follows by Proposition 5.1. □

PROPOSITION 5.3. *Two nonvertical lines L_{a_1,t_1} and L_{a_2,t_2} in $\mathcal{A}$ are parallel if and only if*

$$a_1 f'(t_1) = a_2 f'(t_2).$$

Moreover, the parallel axiom is satisfied in $\mathcal{A}$.

PROOF. We first assume that L_{a_1,t_1} and L_{a_2,t_2} are two nonvertical lines in $\mathcal{A}$ such that $a_1 f'(t_1) \neq a_2 f'(t_2)$. These lines come from the two circles $C_1 = C_{a_1,t_1,a_1 f(t_1)}$ and $C_2 = C_{a_2,t_2,a_2 f(t_2)}$. By assumption, the functions describing these circles have different derivatives at $x = \pi$, so the two curves locally intersect transversally, that is, there are points on C_1 near $(\pi, 0)$ that are in different connected components of $Z \setminus C_2$. But $C_1 \setminus \{(\pi, 0)\} \approx \mathbb{R}$, which is connected, so C_1 and C_2 must intersect in at least another point different from $(\pi, 0)$. This means that the lines in $\mathcal{A}$ must also intersect in $\mathcal{A}$ and therefore are not parallel.

We now consider a real number m and a point $(x, y) \in P = (-\pi, \pi) \times \mathbb{R}$. We claim that there is a unique line $L_{a,t}$ passing through (x, y) such that $m = -af'(t)$. Once this claim is verified, it will follow that the lines $L_{a,t}$ with the same values of $af'(t)$ partition P. In particular, two such lines are parallel and the parallel axiom is satisfied in $\mathcal{A}$.

We begin with the case when $m = 0$. In this case, either $a = 0$ or $t = \frac{1}{2}\pi$. If $a = 0$, then our line is $L_{0,0}$, and otherwise

$$a = \frac{y}{f(x + \frac{1}{2}\pi) + f(\frac{1}{2}\pi)}.$$

Note that

$$f\left(x + \frac{1}{2}\pi\right) + f\left(\frac{1}{2}\pi\right) = \frac{\cos x}{1 + \cos^2 x} + \frac{1}{2} = \frac{(1 + \cos x)^2}{2(1 + \cos^2 x)} > 0$$

for all $x \in (-\pi, \pi)$, so a is well defined. Furthermore, the fact that

$$f(x + \tfrac{1}{2}\pi) + f(\tfrac{1}{2}\pi) > 0 \quad \forall x \in (-\pi, \pi)$$

implies that the lines $L_{a,\frac{1}{2}\pi}$, where $a \in \mathbb{R}$, form a partition of P.

We now assume that $m \neq 0$ and thus that $t \neq \frac{1}{2}\pi$. In this case, $a = -m/f'(t)$ and we have to find $t \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ such that

$$y_m = -\frac{y}{m} = \frac{f(x + t) + f(t)}{f'(t)}.$$

Let $h_x(t)$ be the function on the right-hand side. Since $f(x + \frac{1}{2}\pi) + f(\frac{1}{2}\pi) > 0$, it follows that $\lim_{t \to +\frac{1}{2}\pi^-} h_x(t) = +\infty$ and $\lim_{t \to -\frac{1}{2}\pi^+} h_x(t) = -\infty$. Hence, by the continuity of h_x on the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, we see that for each y_m there is at least one $t \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ such that $y_m = h_x(t)$.

It remains to show uniqueness. We do this by verifying that h_x is injective. In fact, we want to show that h_x is strictly increasing. Explicitly,

$$\begin{aligned} h_x(t) &= \frac{f(x+t)+f(t)}{f'(t)} \\ &= \frac{(\sin(x+t)+\sin t)(1+\sin t\sin(x+t))(1+\sin^2 t)}{(1+\sin^2(x+t))\cos^3 t}. \end{aligned}$$

We make the substitutions

$$u = \tan t \quad \text{and} \quad v = \tan\left(\frac{x}{2}\right),$$

so $\sin x = 2v/(1+v^2)$ and $\cos x = (1-v^2)/(1+v^2)$, and use the addition formula for sines to obtain

$$H_v(u) = 2\frac{(u+v)((u+v)^2+1+u^2)(1+2u^2)}{(1+v^2)^2(1+u^2)+(2v+(1-v^2)u)^2}.$$

The function H_v has derivative

$$H_v'(u) = \frac{2p(u,v)}{((1+v^2)^2(1+u^2)+(2v+(1-v^2)u)^2)^2},$$

where

$$\begin{aligned} p(u,v) = {} & 24(v^4+1)u^6 + 32(v^5-2v^3+3v)u^5 \\ & + (12v^6-68v^4+228v^2+28)u^4 + (-16v^5+208v^3+64v)u^3 \\ & + (4v^6+102v^4+108v^2+10)u^2 + (32v^5+72v^3+8v)u \\ & + 7v^6+19v^4+5v^2+1. \end{aligned}$$

If we can show that $p(u,v) \geq 1$ for all $u, v \in \mathbb{R}$, then $H_v'(u) > 0$ and thus $h_x'(t) > 0$ and our claim about the monotonicity of h_x is verified.

Note that

$$\begin{aligned} p(u,0) &= 24u^6+28u^4+10u^2+1 \geq 1, \\ p(0,v) &= 7v^6+19v^4+5v^2+1 \geq 1, \end{aligned}$$

and that these polynomials take the value 1 if and only $u = 0$ and $v = 0$. Further, there are no linear terms in $p(u,v)$, so the graph of p has a horizontal tangent plane at $(0,0,1)$. The quadratic terms of $p(u,v)$ are $10u^2+8uv+5v^2$. Since this quadratic form is nondegenerate and positive definite, $p(u,v)$ has a local minimum at $(0,0)$.

We consider the function

$$r(u,v) = p(u,v) - (1+10u^2+8uv+5v^2).$$

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When $u, v \neq 0$, we let $z = v/u$. Then

$$\begin{aligned}\frac{r(u, zu)}{u^4} = (24z^4 + 32z^5 + 12z^6)u^6 + (-16z^5 - 68z^4 - 64z^3 + 4z^6)u^4 \\ + (24 + 7z^6 + 102z^4 + 228z^2 + 208z^3 + 96z + 32z^5)u^2 \\ + 28 + 19z^4 + 108z^2 + 64z + 72z^3.\end{aligned}$$

By substituting w for u^2 in the above equation, we obtain the cubic polynomial

$$\begin{aligned}r_z(w) = 4z^4(3z^2 + 8z + 6)w^3 + 4z^3(z^3 - 4z^2 - 17z - 16)w^2 \\ + (7z^6 + 32z^5 + 102z^4 + 208z^3 + 228z^2 + 96z + 24)w \\ + 19z^4 + 72z^3 + 108z^2 + 64z + 28,\end{aligned}$$

and $r(u, zu) = u^4 r_z(u^2)$. We calculate that

$$\begin{aligned}r'_z(w) = 12z^4(3z^2 + 8z + 6)w^2 + 8z^3(z^3 - 4z^2 - 17z - 16)w \\ + 7z^6 + 32z^5 + 102z^4 + 208z^3 + 228z^2 + 96z + 24,\end{aligned}$$

and this quadratic polynomial in w has discriminant $-16z^4 D(z)$, where

$$\begin{aligned}D(z) = 59z^8 + 488z^7 + 1884z^6 + 4480z^5 + 7212z^4 \\ + 7904z^3 + 5600z^2 + 2304z + 432.\end{aligned}$$

We make the substitution $z = z_1 - 1$, in order to reduce the magnitude of the coefficients in $D(z)$. This yields

$$\begin{aligned}d(z_1) &= D(z_1 - 1) \\ &= 59z_1^8 + 16z_1^7 + 120z_1^6 + 120z_1^5 + 122z_1^4 - 48z_1^3 + 24z_1^2 + 8z_1 + 11.\end{aligned}$$

Since we can write

$$d(z_1) = (z_1 + 1)^2(8z_1^6 + 60z_1^4 + 4) + 12z_1^2(2z_1 - 1)^2 + 51z_1^8 + 52z_1^6 + 14z_1^4 + 8z_1^2 + 7,$$

we see that $d(z_1) \geq 7$ and thus $D(z) \geq 7$. Therefore, the discriminant of $r'_z(w)$ is also always negative if $z \neq 0$. The coefficient of w^2 in $r'_z(w)$ is $12z^4(3z^2 + 8z + 6)$, which is positive when $z \neq 0$, and so $r'_z(w) \geq 0$ for all w and all $z \neq 0$. Hence, $r_z(w)$ is strictly increasing in w. But

$$\begin{aligned}r_z(0) &= 19z^4 + 72z^3 + 108z^2 + 64z + 28 \\ &= \frac{1}{19}(19z + 36)^2 z^2 + \frac{428}{133}z^2 + \frac{4}{7}(8z + 7)^2 > 0,\end{aligned}$$

so $r_z(w) > 0$ for all $w \geq 0$. Therefore, $r(u, zu) \geq 0$ for all u and z. This implies that $r(u, v) \geq 0$ and thus

$$p(u, v) \geq 1 + 10u^2 + 8uv + 5v^2 \geq 1$$

when $uv \neq 0$. Using this together with the previous result for $p(u, 0)$ and $p(0, v)$, we deduce that $p(u, v) \geq 1$ for all u and v. □

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Proposition 5.4. *The derived incidence structure $\mathcal{A} = \mathcal{A}_{(\pi,0)}$ of $\mathcal{L}$ at $(\pi, 0)$ is a non-Desarguesian affine plane.*

Proof. By Propositions 5.1 and 5.2, $\mathcal{A}$ is a linear space. Proposition 5.3 shows that the parallel axiom holds in $\mathcal{A}$. Hence, $\mathcal{A}$ is an affine plane.

It remains to show that $\mathcal{A}$ is not Desarguesian. To see this, we consider the two triangles with vertices

$$\begin{aligned} p_1 &= (0, -20\sqrt{3}), & p_2 &= (\tfrac{2}{3}\pi, 0), & p_3 &= (-\tfrac{1}{3}\pi, 0),\\ p'_1 &= (0, 0), & p'_2 &= (\tfrac{2}{3}\pi, \tfrac{20}{49}\sqrt{3}), & p'_3 &= (-\tfrac{1}{3}\pi, \tfrac{36}{7}\sqrt{3}), \end{aligned}$$

respectively. The sides of these triangles are given by

$$\begin{aligned} p_1p_2 &= L_{35,-\frac{1}{3}\pi},\\ p_1p_3 &= L_{-25\sqrt{3},\frac{\pi}{6}},\\ p'_1p'_2 &= L_{\frac{10}{7},0},\\ p'_1p'_3 &= L_{-18,0}. \end{aligned}$$

Furthermore, the lines $p_ip'_i$ are parallel. Indeed, we see that these lines are the parallel classes: $|p_1| = \{0\} \times \mathbb{R}$, $|p_2| = \{\frac{2}{3}\pi\} \times \mathbb{R}$ and $|p_3| = \{-\frac{1}{3}\pi\} \times \mathbb{R}$.

It is easily checked that the corresponding sides p_1p_j and $p'_1p'_j$ of the triangles are parallel when $j = 2, 3$, by Proposition 5.3. However, the third pair of sides $p_2p_3 = L_{0,0}$ and $p'_2p'_3$ is not parallel, by Proposition 5.3. Indeed, lines parallel to $L_{0,0}$ are of the form $L_{a,\frac{1}{2}\pi}$, and intersect $|p_2|$ and $|p_3|$ in $(\frac{2}{3}\pi, \frac{1}{10}a)$ and $(-\frac{1}{3}\pi, \frac{9}{10}a)$, respectively. But the second coordinate of p'_3 is not 9 times the second coordinate of p'_2.

This shows that $\mathcal{A}$ is not Desarguesian. □

Theorem 5.5. *The incidence structure $\mathcal{L}$ is a flat Laguerre plane of Kleinewillinghöfer type V.A.1.*

Proof. We know from the transitivity properties of $\mathcal{L}$ and by Proposition 5.4 that $\mathcal{L}$ is a Laguerre plane and, hence, a flat Laguerre plane by the continuity of f.

It is easy to check that

$$\{\gamma_{s,(1-s)b,0} \mid s \in \mathbb{R} \setminus \{0\}\}$$

is a linearly transitive group of $C_{0,0,b}$-homologies. Hence, the set $\mathcal{Z}$ of all circles C for which the automorphism group of $\mathcal{L}$ is linearly transitive with respect to C-homologies contains a flock of circles.

By [11, Proposition 3.4], the only set $\mathcal{Z}$ containing a flock of circles and one extra circle is of type VII and such a flat Laguerre plane is ovoidal by [11, Corollary 3.2]. However, the derived plane $\mathcal{A}_{(\infty,0)}$ at $(\infty, 0)$ is not Desarguesian by Proposition 5.4, and we have reached a contradiction. Hence, $\mathcal{L}$ must be of Kleinewillinghöfer type V. It follows from the list of Kleinewillinghöfer types given in [11, Theorem 6.1] that $\mathcal{L}$ is of type V.A.1. □

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THEOREM 5.6. *Each automorphism of $\mathcal{L}$ is of the form $\gamma_{r,c,s}$ or $\gamma_{r,c,s}\sigma$. Hence, the group Δ is the full automorphism group of $\mathcal{L}$.*

PROOF. Let φ be an automorphism of $\mathcal{L}$. Now $\mathcal{L}$ is a flat Laguerre plane, so φ is continuous and so is a homeomorphism of Z. Since $\mathcal{L}$ has type V with respect to Laguerre homologies, the flock $\mathcal{F}$ must be invariant. We may therefore assume that the circles $C_{0,0,0}$ and $C_{0,0,1}$ in $\mathcal{F}$ are fixed by φ, up to automorphisms of $\mathcal{L}$ of the form $\gamma_{r,c,0}$. Using an automorphism of the form $\gamma_{1,0,s}$, we may further assume that the point $(\pi, 0)$ on $C_{0,0,0}$ is fixed by φ. But then $(\pi, 1)$ is fixed as well, because φ permutes the parallel classes of $\mathcal{L}$.

To obtain more fixed points, we employ an argument similar to that in the proof of Theorem 4.1. The circle $C_{0,0,0}$ induces a line in the derived projective plane $\mathcal{P}$ at $(\pi, 0)$ and the other circle $C_{0,0,1}$ induces a closed oval O in $\mathcal{P}$. Since $\mathcal{P}$ is a (topological, compact) two-dimensional projective plane, there is exactly one line in $\mathcal{P}$ other than the line at infinity that is tangent to O and passes through the point at infinity of the line induced by $C_{0,0,0}$ (see [4, Statement 2.5], [1, Satz 3.7.a] or [12, Proof of Proposition 55.17]). In the coordinates of our Laguerre plane, this is the circle $C_{1,\pi/2,1/2}$. This circle touches $C_{0,0,0}$ at $(\pi, 0)$ and $C_{0,0,1}$ at $(0, 1)$. Hence, φ fixes the points $(0, 0)$ and $(0, 1)$. If necessary, we may use the automorphism σ to ensure that each connected component of $Z \setminus (\{0, \pi\} \times \mathbb{R})$, the complement of the parallel classes of $(0, 0)$ and $(\pi, 0)$, is left invariant.

Now the automorphism $\gamma_{-1,0,1}$ interchanges the circles $C_{0,0,0}$ and $C_{0,0,1}$, and

$$C_{-1,\pi/2,1/2} = \gamma_{-1,0,1}(C_{1,\pi/2,1/2});$$

this last circle touches $C_{0,0,0}$ at $(\pi, 1)$ and $C_{0,0,1}$ at $(0, 0)$. Hence, φ stabilizes both circles $C_{1,\pi/2,1/2}$ and $C_{-1,\pi/2,1/2}$ and hence also their intersection, given by

$$C_{1,\pi/2,1/2} \cap C_{-1,\pi/2,1/2} = \{(\tfrac{1}{2}\pi, \tfrac{1}{2}), (\tfrac{3}{2}\pi, \tfrac{1}{2})\}.$$

Since these two points lie in different connected components of $Z \setminus (\{0, \pi\} \times \mathbb{R})$, each of which is invariant under φ by assumption, we see that both of the points $(\frac{1}{2}\pi, \frac{1}{2})$ and $(\frac{3}{2}\pi, \frac{1}{2})$ are fixed by φ.

However, by [10, Lemma 5.4.2], the identity is the only automorphism of a flat Laguerre plane that fixes three points on a circle and a fourth point off this circle. Thus, by using elements in Δ, we have reduced φ to the identity. Hence, every automorphism of $\mathcal{L}$ is an element of Δ and thus is of the form $\gamma_{r,c,s}$ or $\gamma_{r,c,s}\sigma$. □

Note that

$$\{\gamma_{r,c,s} \mid r, s \in \mathbb{R}, r > 0, c \in \mathbb{R}/2\pi\mathbb{Z}\}$$

is a three-dimensional connected subgroup of Γ and thus must be the connected component Γ^1 of Γ containing the identity. It follows that Γ^1 has index 4 in Γ.

An obvious generalization of the Laguerre plane $\mathcal{L}$ is obtained in the following way. Let $0 \le r \le 1$ and define

$$f_r(x) = \frac{\sin x}{1 + r \sin^2 x}$$

when $x \in \mathbb{R}/2\pi\mathbb{Z}$. We use the function f_r to generate an incidence structure as in Section 4 using the three-dimensional group Δ. Note that we obtain the classical real Laguerre plane when $r = 0$, and we have seen that we obtain a nonclassical flat Laguerre plane of Kleinewillinghöfer type V.A.1 when $r = 1$.

A lot of additional experimentation and numerical evidence in MAPLE along with the motivation given below suggest the following conjecture.

CONJECTURE 5.7. *If* $0 < r \le 1$, *then the incidence structures* $\mathcal{L}(f_r)$ *are mutually nonisomorphic flat Laguerre planes of Kleinewillinghöfer type V.A.1.*

We sometimes work in a different setting using a coordinate transformation. The cylinder is then represented as $(\mathbb{R} \cup \{\infty\}) \times \mathbb{R}$ as in the parabola model of the classical real Laguerre plane. Consequently, circles and automorphisms are described by formulas that involve only rational functions. Apart from facilitating some of the computations, this second perspective might help to prove the conjecture.

Let $x = 2\tan^{-1}(u)$. Then, up to the scalar factor 2, the function f_r becomes

$$g_q(u) = \frac{u(u^2+1)}{u^4+qu^2+1},$$

where $2 \le q \le 6$. The admissible values $q = 2$ and $q = 6$ correspond to $r = 0$ and $r = 1$. In fact, $r = \frac{1}{4}(q-2)$. For the function g_q, the circles are the graphs of

$$u \mapsto ag_q\left(\frac{u-t}{tu+1}\right) + b,$$

where $a, b, t \in \mathbb{R}$.

In both settings, many of the steps in the verification of the axioms of a Laguerre plane or the verification of type V.A.1 go through as for $\mathcal{L}$. The exceptions are showing the injectivity of $h_t(x)$ in the last case in the proof of Proposition 5.1 and the injectivity of $h_x(t)$ in the proof of Proposition 5.3. In these latter cases, we are led to certain polynomial equations and we have to check that the corresponding polynomials do not have more than two zeros. In the cases when $q = 2$ and $q = 6$, these polynomials have a special form and an analytic solution can be found. However, we have not yet proved analytically the general case when $2 < q < 6$. The main reason for this is the impossibility of solving the general polynomial equations of degree five and higher that arise in our setting with coefficients depending on several parameters.

We have verified the conjecture for several rational numbers $q \in [2, 6]$ using the command RealRootCounting in MAPLE. The underlying algorithm relies on [17, Theorem 2.1] and is guaranteed to give the correct number of real solutions of a system of polynomial equations with rational coefficients. Of course, this method can never exhaustively check all rational numbers in our range. However, if one could show that the conjecture is true for all admissible rational numbers, then the density of $\mathbb{Q}$ in $\mathbb{R}$ coupled with continuity would prove the conjecture.

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