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SEMICONVEX GEOMETRY

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Abstract

Semiconvex sets are objects in the algebraic variety generated by convex subsets of real linear spaces. It is shown that the fundamental notions of convex geometry may be derived from an entirely algebraic approach, and that conceptual advantages result from applying notions derived from algebra, such as ideals, to convex sets. Some structural decomposition results for semiconvex sets are obtained. An algebraic proof of the algebraic Hahn-Banach theorem is presented.

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Introduction

Concepts arising from category theory, enabling the construction of varieties of equational algebras from their free objects, have led to the discovery of important new classes. Swirszck (1973) demonstrated that the class of convex subsets of linear spaces may be given an intrinsic definition as a set provided with a family of binary operations satisfying three identities, together with a cancellative axiom. If the restriction implied by the cancellative law is removed, the resultant algebraic variety has been christened the variety of semiconvex sets.

The semiconvex sets are those algebras arising from congruence equivalence relations on ordinary convex sets. In the same way that the study of fields and integral domains leads naturally to rings, semiconvex sets are the natural domain for an entirely algebraic approach to the geometry of convex sets. Previous approaches to convexity from an algebraic or categorical viewpoint have been hampered by the necessity to embed in a linear space and by an inability to form such basic constructions as quotients or tensors. Virtually all the major notions and theorems of convex geometry have their counterparts in the extended variety. This concerted approach, containing as it does the semilattices, permits the introduction of logical and choice variables into all the standard applications of convex geometry, such as linear programming and economic choice sets.

The critical notion of convexity is "betweenness", or of the segment joining a pair of points. In our relaxed situation, the segment is shown to be either a real line segment or a single point, the "infimum" of the pair. In Section 2, the introduction of the algebraic concept of an ideal, to replace the standard geometric ideas of interior and support manifold, pays simplifying dividends in the study of boundaries of convex sets. The affine hull is found to be better defined as an algebraic equivalence class.

It is shown that semiconvex sets may be embedded in several rather more comprehensible mathematical objects: in certain semigroups called semicones, and as subobjects of products of copies of a particular elementary semicone $(-\infty, \infty]$. An analysis of the nearest analogue to a line is performed in Section 3, which provides a criterion for cancellativeness in terms of lines through a pair of points. Finally the Hahn-Banach theorem for semiconvex sets is proven, using a new explicit construction of the separation function.

1. Semicones and semiconvex sets

The first definition is fairly natural.

1.1 DEFINITION. A (convex) *semicone* is a semimodule over the semiring of non-negative reals. That is, a semicone is an abelian semigroup X with identity $\mathbf{0}$, together with a monoidal (left) action of the non-negative reals, such that both distributive laws are observed, $\mathbf{0}$ is invariant, and $0x = \mathbf{0}$ for all x in X.

A semicone is a group if and only if it a real vector space. As is well known for semigroups, a semicone can be embedded in a vector space if and only if it is *cancellative*; that is, if $x + y = x + z$, then $y = z$, for any x, y, z in X.

Examples of semicones are any convex cone in a vector space with zero as vertex, or any abelian band with the trivial multiplication $rx = x$ for all $r \neq 0$. Two of the more pathological examples of semicones are the reals with ordinary positive multiplication but with the addition changed;

a) the *blocked semicone*, with $x + y = x$ for x positive and y negative, but otherwise addition as normal,

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b) the *reversal semicone*, with $x + (-y) = x + y$, for $x, y > 0$, but otherwise addition as normal.

1.2 DEFINITION (Swirszck). A semiconvex set is a set X together with a family of binary operations P_λ, one for every λ in the real interval (0, 1), which satisfy the identities, for x, y, z in X and $0 < \lambda, \mu < 1$

(1) (reflexivity) $P_\lambda(x, x) = x$,

(2) (symmetry) $P_\lambda(x, y) = P_{(1-\lambda)}(y, x)$,

(3) (associativity) $P_r(P_\lambda(x, y), z) = P_{r\lambda}(x, P_\mu(y, z))$

for $r = \mu/(1 - \lambda + \lambda\mu)$.

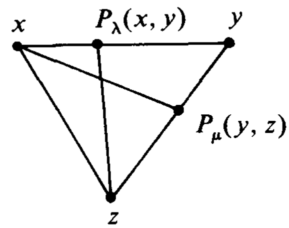

FIGURE 1

Equation (3), in geometric terms, says that two lines from vertices to opposite sides of a triangle must meet, in a prescribed ratio.

These three axioms are sufficient for our geometry. Sometimes for completeness we also include the binary identity functions P_1 and P_0, defined as $P_1(x, y) = x$ and $P_0(x, y) = y$. Then (1) and (2) hold for λ, μ in [0, 1], and (3) if $\lambda(1 - \mu) \neq 1$.

Any convex subset of a vector space, or any semicone, is a semiconvex set under the operations $P_\lambda(x, y) = \lambda x + (1 - \lambda)y$ and will be given this canonical structure unless otherwise stated. Swirszck (1973) has shown: (i) a semiconvex set may be embedded as a convex subset of a vector space if and only if it is *cancellative*; that is, $P_r(x, y) = P_r(x, z)$ for any x, y, z in X impliesx that $y = z$;

(ii) the free semiconvex sets are the simplexes, where a simplex on the set I is the set of [0, 1]-valued functions f of finite support such that $\sum_{i \in I} f(i) = 1$.

Identities (1) to (3) are the algebraic theory of convexity: adding the cancellative axiom gives the geometric theory of convexity.

1.3 Examples of non-cancellative semiconvex sets are any meet-semilattice, with $P_r(x, y) = x \wedge y$ for r in (0, 1), and the *blocked space* Ξ, which is the interval $[-1, 1]$ in the blocked semicone, so that $P_r(x, z) = P_r(y, z)$ for any r in (0, 1), negative x and y, and positive z.

1.4 A function f between semiconvex sets is *affine* if it is an algebra morphism; that is, $P_r(f(x), f(x')) = f(P_r(x, x'))$ for all r in $(0, 1)$ and x, x' in the domain of f.

Any semiconvex set X can be embedded in a semicone $K(X)$; let $K(X)$ be the set $X \times \mathbf{R}^+$, where $\mathbf{R}^+$ is the non-negative reals, with all points $\langle x, 0\rangle$ identified; define $r\langle x, s\rangle = \langle x, rs\rangle$ and

$$\langle x, s\rangle + \langle y, t\rangle = \langle P_{s/(s+t)}(x, y), s + t\rangle$$

for any r, s, t in $(0, 1)$ and x, y in X.

The embedding $x \to \langle x, 1\rangle$ is affine and universal (see Semadeni, p. 415 for the cancellative version). $K(X)$ will be cancellative if X is.

1.5 Some of the following important derivative identities of identities (1) to (3) may be obtained directly; however, they are more easily verified in semicone.

For any x, y, z, a, b in semiconvex X and r, s, t in $(0, 1)$

(4) $$P_r(P_s(x, y), y) = P_{rs}(x, y)$$

(5) $$P_r(P_s(x, y), z) = P_{rs}(x, P_{(r-rs)/(1-rs)}(y, z))$$

(6) $$P_r(P_s(x, z), P_t(y, z)) = P_\mu(P_\nu(x, y), z) \quad \text{for } r, s, t \text{ in } [0, 1]$$

where $\mu = P_r(s, t)$ in $\mathbf{R}$ (with the standard semiconvex structure), and $\nu = rs/\mu$. Furthermore, $\mu^{-1} = P_\nu(s^{-1}, t^{-1})$.

Putting $x = y$, we see that for x, z in X, the function $\psi_{x,z}\colon [0, 1] \to X$ defined by $\psi_{x,z}(r) = P_r(x, z)$ is affine. Putting $s = t$, the function $\varphi_z\colon X \to X$, $\varphi_z(x) = P_s(x, z)$ is affine.

(7) $$P_r(P_s(a, b), P_s(x, y)) = P_s(P_r(a, x), P_r(b, y)).$$

2. Geometry and congruences

2.1 If $P_r(x, y) = z$ in semiconvex X for some r in $(0, 1)$, say *z lies between x and y*. Write (x, y) for the set of points between x and y, and $[x, y]$ where the endpoints x and y of the segment are included.

A subset G of X is called *semiconvex* or a *subalgebra* of x if it is closed under convex combinations P_r.

A *congruence* on a semiconvex set is an equivalence relation $\sim$ on X which is a subalgebra of $X \times X$ (it is sufficient that for $x \sim y$ and z in x, $P_r(x, z) \sim P_r(y, z)$ for any r in $(0, 1)$). The quotient of any semiconvex set by a congruence is again a semiconvex set.

Any equivalence class for a congruence (or more generally any equalizer) is called a *manifold*; a manifold is always a subalgebra.

2.2 EXAMPLE. A subset I of X is an *ideal* if for x in X and y in I, $(x, y) \subseteq I$. Then the union of the diagonal and $(I \times I) \subseteq (X \times X)$ is a congruence, so that I may be shrunk to a single point in the quotient without violating semiconvexity. An ideal is *prime* if it has semiconvex complement; then the complement is called a *support manifold*. If a single point is a support manifold, it is an *extreme point*. Each prime ideal in X corresponds to an affine function into the two-point semilattice $\{0, 1\}$.

Any intersection or union of ideals is again an ideal. For $A \subset X$, the *ideal* $\langle A \rangle$ *generated by* A is $\bigcup\{(x, a)\colon a \in A, x \in X\}$, and is the smallest ideal containing A.

2.3 EXAMPLE. The map $x \to \langle x \rangle$ taking semiconvex X into its semilattice of ideals under intersection is affine, because $\langle P_r(x, y)\rangle = \langle x \rangle \cap \langle y \rangle$ for any x, y in X and r in (0, 1). In fact, if $z \in \langle x \rangle \cap \langle y \rangle$, choose a, b in X and s in (0, 1) using identity (4), so that $P_s(x, a) = P_s(y, b) = z$. Then $P_s(P_r(x, y), P_r(a, b)) = P_r(P_s(x, a), P_s(y, b)) = z$ by identities (7) and (1), so that z is in $\langle P_r(x, y)\rangle$, with the other inclusion obvious. Accordingly, ($x \sim y$ if $\langle x \rangle = \langle y \rangle$) defines a congruence on X, and the quotient is a reflection of X into the category of semilattices.

2.4 EXAMPLE. If G is a subalgebra of semiconvex X, say $x \sim_G y$ if $P_r(x, g_1) = P_r(y, g_2)$ for some r and g_1, g_2 in G. Then $\sim_G$ is a congruence, called the *affine congruence generated by* G. The equivalence class of G is the *affine hull of* G, and is denoted $H(G)$. So $H(G) = \{x \in X | (x, g) \text{ meets } G \text{ for some } g \text{ in } G\}$.

2.5 EXAMPLE. The ideals in a semicone are the semigroup ideals closed under multiplication. Any (semiconvex) congruence on a semicone is a congruence for the variety of semicones. If C is any cone, the affine congruence generated by the diagonal in $C \times C$ has as quotient a vector space $L(C)$, and the embedding

$$C \to \langle C, 0 \rangle \to L(C)$$

is universal (see Semadeni (1971), p.415. This is the standard construction of a group reflection for abelian semigroups). The embedding is injective if and only if C is cancellative.

If X is semiconvex, the composite $X \to C(X) \to L(C(X))$ will be a universal arrow from X to the category of real linear spaces. The composite functor LC is a reflection from the variety of semiconvex sets to the variety of real linear spaces.

Being able to partition X into disjoint semiconvex sets is important.

2.6 PROPOSITION. (i) *If G is a subalgebra of semiconvex X, and H is a subalgebra of X maximal with respect to exclusion of G, the complement of H is semiconvex;*

(ii) *the ideal maximal with respect to exclusion of G is prime;*

(iii) *any ideal is an intersection of prime ideals.*

PROOF. (i) If x is not in H, the set $\bigcup_{h\in H}[x, h]$ is a subalgebra containing H which must meet G. So for x, y not in H, there are h_1, h_2 in H such that $P_s(x, h_1)$ and $P_t(y, h_2)$ are in G, and $0 < s, t < 1$.

From the identity

$$P_l(P_r(x, y), P_m(h_1, h_2)) = P_n(P_s(x, h_1), P_t(y, h_2)) \tag{8}$$

where

$$l = st/(s - sr + rt), \qquad m = rt(1 - s)/(s - sr + rt - st),$$
$$n = rt/(s - sr + rt)$$

we see that for any r there is $h_3 = P_m(h_1, h_2)$ such that $[P_r(x, y), h_3]$ meets G (since G is semiconvex). Then $P_r(x, y)$ is in the complement of H, which must be semiconvex.

(ii) The ideal P maximal with respect to exclusion of G is $\{x\colon \langle x\rangle \cap G = \varnothing\}$, so for x, y not in P, there are a_1, a_2 in X such that $[x, a_1]$ and $[y, a_2]$ meet G. As in (i), the complement of P is semiconvex.

(iii) If I is an ideal and x is not in I, the ideal maximal with respect to disjunction from $\{x\}$ contains I and is prime, by (ii), so that I is an intersection of primes.

Returning to 2.3, an extension theorem which contains the standard result (Semadeni (1971), Theorem 23.1.3) holds.

2.7 PROPOSITION. *Any affine function $f\colon S \to L$ from a semiconvex subset S of semiconvex X to a vector space L has a unique affine extension to the affine hull $H(S)$ of S in X.*

PROOF. If $a \in H(S)$, there is s_1, s_2 in S and r in $(0, 1)$ such that $P_r(a, s_1) = s_2$. Define

$$\varphi(a) = \frac{1}{r}f(s_2) + \left(1 - \frac{1}{r}\right)f(s_1).$$

If ψ is any affine extension of f, $f(s_2) = r\psi(a) + (1 - r)f(s_1)$, so $\psi = \varphi$. If there is v in $(0, 1)$ and s_3, s_4 in S such that $P_v(a, s_3) = s_4$, then, using identities (3) and (2),

$$(P_{r/(r+v-rv)})(s_2, s_1) = P_{v/(r+v-rv)}(s_2, s_3)$$

so that $rf(s_2) + (v - rv)f(s_1) = vf(s_2) + (r - rv)f(s_3)$, since f is affine, and dividing by rv

$$\frac{1}{r}f(s_2) + \left(1 - \frac{1}{r}\right)f(s_1) = \frac{1}{v}f(s_4) + \left(1 - \frac{1}{v}\right)f(s_3)$$

so φ is well defined.

To show φ affine, for b in $H(S)$ and $P_t(b, s_5) = s_6$, for s_5, s_6 in S, assume without loss of generality that $r < t$, then $P_r(b, s_5) = s_7$, where $s_7 = P_{r/t}(s_6, s_5)$, by identity (4). For q in (0, 1)

$$P_r(P_q(a, b), P_q(s_1, s_5)) = P_q(s_2, s_7), \quad \text{by identity (7)}$$

so

$$\begin{aligned}\varphi(P_q(a, b)) &= (P_q(f(s_1), f(s_5)))/r + (1 - (1/r))\, P_q(f(s_2), f(s_7)) \\ &= P_q(\varphi(a), \varphi(b))\end{aligned}$$

3. Lines and interiors

Of critical importance is the need to establish a counterpart to the line through a pair of points, by this means introducing a geometric flavour. The notion of the interior of a semiconvex set turns out to be tractable, from either an algebraic or a geometric viewpoint (3.7, 3.14). Each semiconvex set is represented as a subalgebra of a product of copies of an elementary algebra in 3.10.

3.1 A *line interval* is a semiconvex set L such that for any three points in L, one lies between the other two.

A line interval can have at most two extreme points, the *endpoints* of the interval, and all other points lie in the *interior* of the interval. By induction, any finite number of points in an interval may be contained in a segment $[a, b]$.

3.2 LEMMA. *All non-trivial congruences on the real interval* [0, 1], *or on any line interval, must equivalence all points of the interior of the interval.*

PROOF. Suppose $a \sim b$ for $0 \leqslant a < b$ in [0, 1]. Then

$$a = P_{a/b}(b, 0) \sim P_{a/b}(a, 0) = P_{a^2/b^2}(b, 0) \sim \cdots \sim P_{(a/b)^n}(a, 0)$$

so that $a \sim (a/b)^n a$ for all integers n. Because the manifold containing a and b is semiconvex, every point of the segment $(0, b]$ is equivalent. A similar argument shows every point of $[a, 1)$ is equivalent.

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Now if L is any line interval with $a \sim b$ in L, and $[c, d]$ is any segment containing $[a, b]$, the affine map

$$\psi_{c,d}\colon [0, 1] \to [c, d] \quad \text{defined by } \psi_{c,d}(r) = P_r(c, d)$$

in identity (6) induces a congruence on $[0, 1]$, by $r \sim s$ if $P_r(c, d) \sim P_s(c, d)$. Since $(0, 1)$ is then equivalent, every point of (c, d) is equivalent. Every point of the interior of L is contained in some (c, d), with $[c, d]$ containing a and b, and so it is equivalent to a and b.

3.3 PROPOSITION. *Any line interval L which is not cancellative has at most three points. Any cancellative line interval is isomorphic to a real interval.*

PROOF. If $P_r(x, z) = P_r(y, z)$ for $x \neq y$ and z in L and r in $(0, 1)$ then if a, b are points in L such that x, y, z are in $[a, b]$, the morphism $\psi_{a,b}\colon [0, 1] \to [a, b]$ cannot be an isomorphism. Since every point of $(0, 1)$ is equivalenced by $\psi_{a,b}$, $[a, b]$ has at most three points, and it follows that L itself has two or three points.

If L is cancellative, $\psi_{a,b}$ is an isomorphism, for distinct a, b in L, and its inverse may be extended to a map $\varphi\colon L \to R$, by Proposition 2.7, which must be one-to-one by Lemma 3.2.

There are in fact seven distinct nonisomorphic cancellative line intervals (including the one-point set), depending on whether one or two ends are bounded or unbounded, open or closed.

3.4 Suppose $P_r(x, z) = P_r(y, z) = a$, for x, y, z in semiconvex X and r in $(0, 1)$. Then if $s < r$, $P_s(x, z) = P_{s/r}(P_r(x, z), z) = P_s(y, z)$, by identity (4), and $P_{2m}(x, z) = P_m(x, P_r(x, z)) = P_m(x, P_r(y, z)) = P_m(y, P_r(x, z)) = P_{2m}(y, z)$ where $m = r/(1 + r)$, giving $2m > r$.

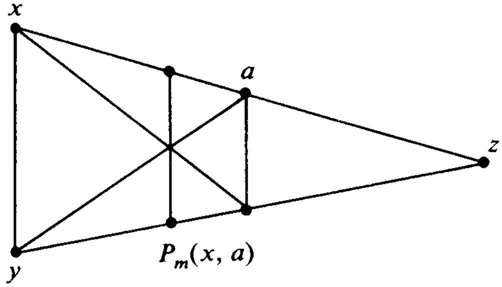

FIGURE 2

Iterating, $P_s(x, z) = P_s(y, z)$ for $1 - s$ arbitrarily small and $P_s(x, z) = P_s(y, z)$ for all s in $(0, 1)$.

Say x and y are *z-equivalent* in this case.

If $c \in \langle z \rangle$, so that $c = P_\nu(z, a)$ for a in X, ν in (0, 1), then

$$P_\mu(x, P_\nu(z, a)) = P_m(P_n(x, z), a) = P_m(P_n(y, z), a) = P_\mu(y, P_\nu(z, a))$$

for any μ and suitable m, n, so that x and y are c-equivalent.

Note that any non-cancellative space contains a non-trivial image of the blocked space of 1.3, taking -1 to x, 0 to y and 1 to z, where x and y are z-equivalent.

If $P_r(x, z) = z$ for some r, then putting $y = z$ above, or using Lemma 3.3, $(x, z] = z$. Say *x adheres to z*; if this is the case, $P_r(x, a) = P_r(z, a)$ for every a in $\langle z \rangle$, as x, z are a-equivalent.

So x adheres to itself.

EXAMPLE. In the reversal cone of 1.3, every negative number $-a$ adheres to its corresponding positive number a.

If every segment in X is cancellative, say X is *semicancellative*. The blocked space is semicancellative but not cancellative. If no segment in X is cancellative, then X is a semilattice; x adheres to y if $x \geqslant y$.

3.5 PROPOSITION. *If G is a subalgebra of semiconvex X, then $H(H(G)) = H(G)$, where $H(G)$ is the affine hull of G.*

PROOF. Let φ: $X \to X/\sim_G$ be the quotient map connected with the congruence $\sim_G$ of Example 2.4. Then $\varphi(x)$ and $\varphi(y)$ are g-equivalent by definition if and only if $x \sim_G y$, where g is the equivalence class of $H(G)$. If x is in $H(H(G))$, then $\varphi(x)$ adheres to g, so $\varphi(x) = g$ and $x \in H(G)$.

3.6 LEMMA. *Suppose $P_r(x, z) = P_s(y, z)$ for $0 < r < s < 1$ and x, y, z in semiconvex x. Then y adheres to $P_{r/s}(x, z)$.*

PROOF. $P_m(y, P_{r/s}(x, z)) = P_n(x, (y, z))$, (where $n = (r - rs)/(s - rs)$, $m = (s - r)/(1 - r)) = P_n(x, P_r(x, z)) = P_{r/s}(x, z)$.

3.7 DEFINITION. The *interior* of a semiconvex set X, denoted X^i, is the intersection of all non-empty ideals in X. The points of the interior are *internal points*. The interior may be empty, or it may be all of X; in the latter case X is *open*. The interior of X is open.

If p is any point of X^i, then $\langle p \rangle = X^i$.

3.8 PROPOSITION. *Any open semiconvex set is cancellative.*

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PROOF. By 3.4, the set of points z in semiconvex X such that x and y are z-equivalent, for any x and y in X, is an ideal, so if X is open, x and y are in the ideal, and

$$x = P_r(x, x) = P_r(y, x) = P_r(y, y) = y.$$

Therefore X is cancellative.

3.9 PROPOSITION. *If x and y are two points in semiconvex X, the interior I of the affine hull $H = H([x, y])$ is a line interval. If (x, y) is not cancellative, I has one point.*

If p is any point of I, then $I = H \cap \langle p \rangle$.

If z is any point of H not adherent to p, $H = H([z, p])$.

PROOF. If h is any point of H, the ideal $\langle h \rangle$ taken in H must contain a point i of (x, y). Since $\langle i \rangle$ in H contains (x, y), $(x, y) \subset \langle h \rangle$, and (x, y) is contained in the interior I.

If (x, y) is not cancellative, so that it is a single point c, with x and y adherent to c, then if $c \neq x$ or y, x or y is not in $\langle c \rangle$. Therefore if $z \in H$, either z is adherent to x or y, or c is in $[z, x]$ or $[z, y]$. In either case, z is adherent to c. So $I = \langle c \rangle = \{c\}$, the one-point set.

If (x, y) is cancellative, I is a cancellative semiconvex set which is the affine hull in I of the segment (x, y), and must itself be a line interval, since the inverse of $\psi_{x,y}$: $(0, 1) \to (x, y)$ may be extended to an injection from I to $\mathbf{R}$, as in Proposition 3.3.

For p in I, and $q \neq p$ in $\langle p \rangle$, there is z_1 in X such that q is in (z_1, p). So if q is in H, z_1 is in $H(H) = H$, and $q \in I$. Therefore $I = \langle p \rangle \cap H$.

If z in H does not adhere to p, so that (x, y) is cancellative, $(z, p) \subset I$, and $H([z, p])$ contains every point of the interval I, by Lemma 3.2, including (x, y). Then $H([z, p])$ contains x and y, and $H([z, p]) = H$.

The open line interval I will be called the *line interior through* (x, y).

A semiconvex set is *simple* if it admits no non-trivial congruences, and *subdirectly irreducible* if the intersection of the non-trivial congruences is not the diagonal, i.e. if there is a minimal non-trivial congruence.

Let $(-\infty, \infty]$ be the semicone $\mathbf{R} \cup \{\infty\}$, where $s + \infty = \infty$ for real s and $r \cdot \infty = \infty$ for positive r. Then as a semiconvex set, every point in $(-\infty, \infty]$ is attached to ∞.

3.10 THEOREM. *The simple semiconvex sets are the open line intervals and the two-point interval. The subdirectly irreducible semiconvex sets are the subalgebras of $(-\infty, \infty]$. Every semiconvex set may be embedded in a product of copies of $(-\infty, \infty]$.*

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PROOF. If semiconvex S is simple and has a prime ideal, it must have only two points. Otherwise, S is open by Proposition 2.6ii. For two points x, y in S, the hull $H([x, y]) = S = S^i$, and by Proposition 3.9, S is an open line interval. Any open line interval is simple, by Lemma 3.2.

If L is subdirectly irreducible but not simple, suppose x and y in L are equivalent under the smallest non-trivial congruence. Then any ideal in L has either a single point or it contains both x and y. So either x and y are internal, or the interior of L has one point, $\underline{\infty}$ say, and every point of L is attached to $\underline{\infty}$. Also (x, y) must be cancellative, since $\langle y \rangle = \langle x \rangle$.

If $z \neq \underline{\infty}$ is in L, then $\langle z \rangle$ contains x. So x is internal to $H([z, x])$, which must contain y by hypothesis, and z is in $H = H([x, y])$, by Proposition 3.9. So $L = H \cup \{\underline{\infty}\}$.

If $\varphi\colon H \to \mathbf{R}$ is the affine extension of the inverse of $\psi_{xy}\colon [0, 1] \to [x, y]$, extend φ affinely, if necessary, $\varphi\colon L \to (-\infty, \infty]$, by putting $\varphi(\underline{\infty}) = \infty$. Then φ is injective, since $\varphi(x) \neq \varphi(y)$.

Any simple semiconvex set may also be embedded in $(-\infty, \infty]$. Conversely, any subalgebra S of $(-\infty, \infty]$ is subdirectly irreducible, with minimum congruence, the congruence identifying the points of the interior of the interval $S - \{\infty\}$.

The last statement follows from Birkhoff's theorem (Grätzer (1968), p. 164).

It follows that $(-\infty, \infty]$ is a cogenerator in the variety.

3.11 DEFINITION. A *line* in semiconvex X is a maximal subalgebra which is a line interval.

If S is any line interval in X, then since $S \subseteq H([x, y])$ for any two distinct points x, y in S, $H(S) = H([x, y])$ by Proposition 3.5.

3.12 PROPOSITION. *If L is any line in semiconvex X, then either the interior of L is equal to the interior I of $H(L)$, or some endpoint of L adheres to a point in I, and $L^i = L \cap I$.*

PROOF. Assume L cancellative, otherwise I has a single point and $I = L^i$. Suppose x is in I but not in L. Choose distinct points a, b in L^i such that $a \in (x, b)$. Since $L^i \subseteq \langle a \rangle$, $L^i \subseteq I$ by Proposition 3.9. So L has at most two points not in I. For one of these points z, we must have $a \in (z, b)$, otherwise $L \cup (x, b)$ is a line interval strictly containing L (both are contained in the interval I, except for perhaps one point of L). Choosing x_1 in $H(L)$ so that $x \in (x_1, b)$, (x_1, b) and (z, b) meet in a but are not equal. By Lemma 3.6, z adheres to a point in (x_1, a).

The maximality of L ensures that no endpoint of L can be in I, so $L^i = L \cap I$.

For any pair of points x, y in semiconvex X, a line containing x and y may be constructed by adding a suitable pair of endpoints (if necessary) to a suitable subset of the line interior through (x, y).

The lines with two endpoints in the reversal cone are the sets $\{-a\} \cup (a, b) \cup \{-b\}$, for $a, b \geqslant 0$.

Uniqueness properties of the lines through a given pair of points give a criterion for cancellativeness.

3.13 COROLLARY. *If semiconvex X is semicancellative, every line through a given pair of points has the same interior. If X is not an interval, it is cancellative if and only if there is a unique line through each pair of points in X.*

PROOF. If x and y are points in semicancellative X, and L is a line through x and y, by Proposition 3.12, $L^i = H([x, y])^i$. If z_1, z_2 are two points in $H([x, y])$ not in its interior, both on the same side of $[x, y]$, then $P_r(z_1, y) = P_r(z_2, y) = x$, for some $r > 0$ (keeping in mind Lemma 3.6), and $(z_1, y) = (z_2, y)$ by 3.4. So if $H([x, y])$ is cancellative, $z_1 = z_2$, and $H([x, y])$ is a line segment, which must be the unique line through x and y.

Conversely, if $P_r(x, z) = P_r(y, z)$ for distinct x, y, z in X and r in $(0, 1)$, distinct lines through $(x, z) = (y, z)$ may be obtained by adding to the interior of $H([x, z])$, x and y respectively, together with a suitable endpoint at the other end, if necessary.

The interior of a semiconvex set coincides with the intuitive definition of interior.

3.14 PROPOSITION. *The interior of semiconvex X is the set of points in the interior of every line containing them.*

PROOF. If x is an internal point of X, and L is a line containing x, then for any $z \neq x$ in L, by Proposition 3.9, $x \in H([z, x])^i = H(L)^i$, and $x \in L^i$.

Conversely, if z is any point of X, then if x is in the interior of any (hence all) lines containing x and z, $x \in \langle z \rangle$. So if x is in the interior of all lines, it is internal.

4. The Hahn-Banach theorem

The formulation of Bourbaki (1953) Chapter II.3, may be followed with modifications.

4.1 DEFINITION. If G is a subalgebra of semiconvex X, x is an *algebraic interior point* of G if for any line L in X containing x, x is in the interior of the segment $L \cap G$.

4.2. PROPOSITION. *For G a subalgebra of X, x is an algebraic interior point of G if and only if x is an internal point of G and $H(G) = X$.*

PROOF. If x is an algebraic interior point of G and y is in X, a line L in X containing y and x has x as an interior point of $L \cap G$, so $x \in (y, g)$ for some g in G and $y \in H(G)$. If y is in G, $x \in \langle y \rangle$ (where the ideal is formed in G) and x is internal to G.

Conversely, suppose L is a line in X containing x, suppose $y \neq x \in L$; then if $H(G) = X$, there are g_1, g_2 in G such that $g_1 \in (y, g_2)$. Choose g in (g_1, g_2). If x is internal, we have g_3 such that $x \in (g, g_3)$.

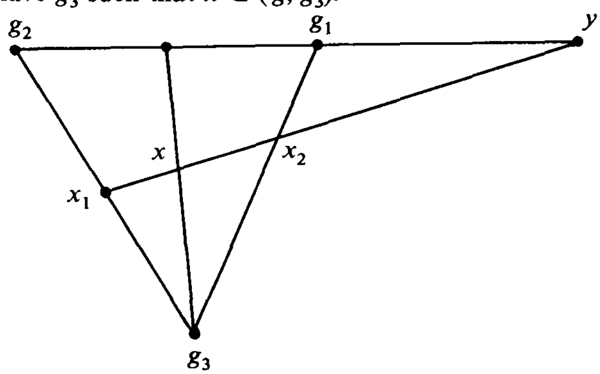

FIGURE 3

Then there is x_1 in (g_2, g_3) such that $x \in (x_1, y)$, and x_2 in (g_1, g_3) such that $x_2 \in (x_1, y)$ and $x \in (x_1, x_2)$. If (x_1, y) is not contained in L, an endpoint of L adheres to a point in (x_1, x) by Proposition 3.12, and x is in the interior of $L \cap G$.

4.3 DEFINITION. If C is a subalgebra of semiconvex X, define

$$C^a = \{x \in X : \exists y \in C \text{ such that } (x, y) \subseteq C\}.$$

The points of C^a are the *points adherent to* C.

If $(x, y) \subseteq C$ and $(x', y') \subseteq C$, the segment $(P_r(x, x'), P_r(y, y'))$ is contained in C, using identity (7), so that C^a is semiconvex.

4.4 LEMMA. $(C^a)^i = C^i$, *for any subalgebra C of semiconvex X.*

PROOF. If $y \in C^a$, then $\langle y \rangle \cap C$ is a non-empty ideal in C, which must contain C^i, so $(C^a)^i \supseteq C^i$. If $(c, y) \subseteq C$ for some c in C, and x is in C^i, choose z in C^i such that $x \in (z, c)$.

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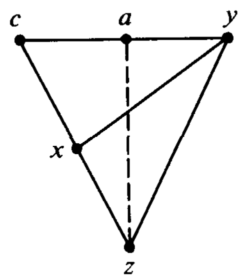

FIGURE 4

Then $(x, y) \subseteq C^i$, since each (z, a) is, for a in (c, y) and C^i is an ideal in C^a, so $C^i = (C^a)^i$.

Say G and G', subsets of semiconvex X, are *separated by* a function f if $f(G)$ does not meet $f(G')$.

4.5 THEOREM. *Let C be a subalgebra of semiconvex X which contains an algebraic interior point, and suppose C' is a subalgebra of X containing no internal point of C. Then C^i and C' are separated by a prime ideal or a real-valued affine function.*

PROOF. By Zorn's Lemma, find a subalgebra of X containing C', maximal with respect to exclusion of C^i. Then its complement will be semiconvex, by Proposition 2.6(i), and we may assume that C' is the complement of C.

If C' contains no internal point of X, there is a prime ideal in X not meeting C', by Proposition 2.6(ii), and $C^i \subseteq X^i \subseteq P$, by Propositions 4.2 and 3.14, so P separates C^i and C'.

Let $H = C^a \cap C'^a$. If $a \in C^i$ and $c' \in C'$, a is in the interior of $L \cap C$ for any line L in X containing c' and a, so a is not in H. In fact, a is not in $H(H)$, since $\langle a \rangle = (C^a)^i$ in C^a, by Lemma 4.4, and $(C^a)^i$ does not meet H. Therefore, for any x in C'^a, there is unique r such that $P_r(x, a) \in H$ (take $r = \sup\{s\colon P_s(x, a) \in C\}$).

Define a real function f on C'^a by $f(x) = 1 - r^{-1}$. So $f < 0$, and $f = 0$ on H. If for x' in C'^a, $P_s(x', a) \in H$, then for all t in $(0, 1)$

$$P_\mu(P_t(x, x'), a) = P_\nu(P_r(x, a), P_s(x, a)) \in H,$$

where $\mu^{-1} = P_t(r^{-1}, s^{-1})$, by identity (6), and $f(P_t(x, x')) = 1 - \mu^{-1} = P_t(1 - r^{-1}, 1 - s^{-1})$, so that f is affine.

If c in C' is in X^i, there is b in X so that $c \in (a, b)$. Then $b \in C'$, $b \notin H$, $f(b) < 0$, and for any y in X, (y, c) meets C'. So $H(C') = X$. Extend f to all of X by Proposition 2.7. If d is in C^i, (d, b) meets H, so $0 \in f((d, b))$ and $f(d) > 0$. Therefore f separates C^i and C'.

If X is open, this is the standard Hahn-Banach result.

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