BULL. AUSTRAL. MATH. SOC. 46A03
VOL. 75 (2007) [247–252]

ORLICZ–PETTIS THEOREM FOR λ-MULTIPLIER CONVERGENT OPERATOR SERIES

TAO YUANHONG AND LI RONGLU

We show that the λ-multiplier convergence of operator series depends completely upon the AK property of the sequence space λ, and thus present a lot of new important theorems.

1. INTRODUCTION

Let λ be a scalar sequence space which contains c_{00}, the space of all sequences which are eventually 0, and (X,τ) a Hausdorff locally convex space. A series $\sum_i x_i$ in X is said to be λ-multiplier convergent with respect to the topology τ if the series $\sum_i t_i x_i$ converges in X for every $t = \{t_i\} \in \lambda$.

The Orlicz–Pettis Theorem for locally convex spaces, which asserts that a series $\sum_i x_i$ in the space which is subseries convergent in the weak topology is actually subseries convergent in the original topology of the space, can be interpreted as a theorem about multiplier convergent series. The literature abounds with such Orlicz–Pettis theorems and much more general λ-multiplier convergent series [**3, 7, 8, 9, 11, 12, 13**].

Let X, Y be two Hausdorff locally convex spaces and $L(X,Y)$ the space of continuous linear operators from X into Y. We say that the series $\sum_k T_k$ is λ-multiplier convergent for a locally convex topology τ on $L(X,Y)$ if the series $\sum_k t_k T_k$ is τ convergent for every $t = \{t_k\} \in \lambda$.

If $x \in X, y' \in Y'$, let $x \otimes y'$ be the linear functional on $L(X,Y)$ defined by $\langle x \otimes y', T\rangle = \langle y', Tx\rangle$ and let $X \otimes Y'$ be the linear subspace spanned by $\{x \otimes y' : x \in X, y' \in Y'\}$. The weak operator topology on $L(X,Y)$ is the weak topology from the duality between $L(X,Y)$ and $X \otimes Y'$. The strong operator topology $L_s(X,Y)$ on $L(X,Y)$ is the topology of pointwise convergence on X. Let $L_b(X,Y)$ be $L(X,Y)$ with the topology of uniform convergence on the bounded subsets of X. The topology $L_b(X,Y)$ is generated by the seminorms $p_A(T) = \sup\{p(Tx) : x \in A\}$, where p is a continuous seminorm on Y and A is a bounded subset of X.

Several Orlicz–Pettis theorems have been established for multiplier convergent operator series [**4, 8**]. In this note, we point out that the λ-multiplier convergence of operator

Received 18th October, 2006

series depends completely upon the AK property of sequence space λ and thus present many new important theorems.

We now list some definitions and terminology which will be used in the sequel. Let (X, X') be a dual pair and $\sigma(X,X')$, $\tau(X,X')$, $\beta(X,X')$ the weak topology, Mackey topology and the strong topology of X, respectively. $\mathcal{K}(X,X')$, $c(X,X')$, $v(X,X')$ denote the topologies on X of uniformly convergence on $\sigma(X',X)$ compact sets, $\sigma(X',X)$ countable compact sets and $\sigma(X',X)$ sequential compact sets, respectively. $\gamma(X,X')$ will denote the topology on X of uniform convergence on unconditionally $\sigma(X',X)$ sequential compact sets (a set $A \subseteq X'$ is unconditionally $\sigma(X',X)$ sequential compact if every sequence in A has a subsequence which is $\sigma(X',X)$ Cauchy [4]). The topology $c(X,X')$ is obviously stronger than $\mathcal{K}(X,X')$ and $v(X,X')$ and strictly stronger than the Mackey topology $\tau(X,X')$ ([**10**]). The topologies $\mathcal{K}(X,X')$ and $\gamma(X,X')$ are not comparable.

Recall that the β-dual space of λ is $\lambda^\beta = \{(u_i) : \sum_i u_i t_i$ is convergent for every $(t_i) \in \lambda\}$. It is obvious that if $c_{00} \subseteq \lambda$, then (λ, λ^β) is a dual pair with respect to the bilinear pairing $(t,u) = \sum_i u_i t_i, t = (t_i) \in \lambda, u = (u_i) \in \lambda^\beta$.

Let (λ, τ_0) be a locally convex space, $c_{00} \subseteq \lambda$ and $t = (t_i) \in \lambda$, denote $t^{[n]} = (t_1, t_2, \cdots, t_n, 0, \cdots)$. If for every $t \in \lambda, \{t^{[n]}\}$ converges to t with respect to the topology τ_0, then (λ, τ_0) is said to be an AK-space.

2. Main Results

Henceforth, let X, Y be two Hausdorff locally convex spaces and $L(X,Y)$ the space of continuous linear operators from X into Y.

A property (P) is said to be continuous linear invariant, if the property (P) is conserved with respect to all continuous linear mappings. Hence compact sets, countable compact sets, sequential compact sets, convex compact sets, bounded sets, convergent sequences and finite sets are all continuous linear invariants.

Let $\mathcal{P} = \{D \subseteq X : D$ be finite set or D be $\sigma(X,X')$-bounded and has property $(P)\}$. Now, we denote $L_{\mathcal{P}}(X,Y)$ the topology of uniform convergent on all sets in $\mathcal{P}$. Clearly, $L_{\mathcal{P}}(X,Y)$ is an operator topology on $L(X,Y)$.

Accordingly, let $\mathcal{P}_\lambda = \{D \subseteq \lambda^\beta : D$ be finite set or D be $\sigma(\lambda^\beta, \lambda)$-bounded and has property $(P)\}$. Furthermore, we denote $\mathcal{P}_\lambda(\lambda, \lambda^\beta)$ the topology of uniform convergent on all sets in $\mathcal{P}_\lambda$. That $\mathcal{P}_\lambda(\lambda, \lambda^\beta)$ is a (λ, λ^β) polar topology is clear.

Our main results are as follows:

Theorem 1. *Let $c_{00} \subseteq \lambda$, X, Y be two Hausdorff locally convex spaces and the property (P) a continuous linear invariant. Then every λ-multiplier weak operator topology convergent operator series $\sum_i T_i$ of $L(X,Y)$ must be λ-multiplier $L_{\mathcal{P}}(X,Y)$ convergent if and only if the space $(\lambda, \mathcal{P}_\lambda(\lambda, \lambda^\beta))$ is an AK-space.*

Proof: *Sufficiency.* Let $\sum_i T_i$ be a λ-multiplier weak operator topology convergent series of $L(X,Y)$. If $\sum_i T_i$ is not λ-multiplier $L_{\mathcal{P}}(X,Y)$ convergent, then there exist $(t_i^{(0)}) \in \lambda$, $T_0 \in L(X,Y)$ and $D \in \mathcal{P}$ such that $\sum_i t_i^{(0)} T_i$ is weak operator topology convergent to T_0, but $\sum_i t_i^{(0)} T_i$ does not converge to T_0 uniformly on D. That is, there exists $\varepsilon_0 > 0$ and a continuous seminorm p of Y such that

$$\lim_{n\to\infty} \sup_{x\in D} \left\{ p\left(\sum_{i=n}^{\infty} t_i^{(0)} T_i x \right) \right\} \geqslant \varepsilon_0. \tag{1}$$

It follows from the Hahn–Banach theorem [**10**, Section 9.1] that there exists an equicontinuous subset B of Y' such that

$$\lim_{n\to\infty} \sup_{x\in D, y'\in B} \left\{ \left| \left(\sum_{i=n}^{\infty} t_i^{(0)} T_i x, y' \right) \right| \right\} \geqslant \varepsilon_0. \tag{2}$$

Note that the series $\sum_i T_i$ is λ-multiplier weak operator topology convergent, so for every $(t_i) \in \lambda$, there exists $T \in L(X,Y)$ such that for every $x \in X$ and $y' \in Y'$,

$$\left(\sum_i t_i T_i x, y' \right) = (Tx, y'). \tag{3}$$

Since (3) can be written as follows:

$$\sum_i t_i (T_i x, y') = (Tx, y'), \tag{4}$$

for every $x \in X$ and $y' \in Y'$, $\left((T_i x, y')\right)_{i=1}^{\infty} \in \lambda^\beta$. It follows from (4) and $T \in L(X,Y)$ and $D \in \mathcal{P}$ and B is an equicontinuous subset of Y' that $\left\{ \left((T_i x, y')\right)_{i=1}^{\infty} : x \in D, y' \in B \right\} \in \mathcal{P}_\lambda$. Thus, it follows from $\left(\lambda, \mathcal{P}_\lambda(\lambda, \lambda^\beta)\right)$ is an AK-space that there exists $n_0 \in \mathbf{N}$ such that whenever $n \geqslant n_0$, we have

$$\sup_{x\in D, y'\in B} \left\{ \left| \sum_{i=n}^{\infty} t_i^{(0)} (T_i x, y') \right| \right\} < \frac{\varepsilon_0}{2}.$$

This contradicts (2) and the sufficiency is proved.

Necessity. If $\left(\lambda, \mathcal{P}_\lambda(\lambda, \lambda^\beta)\right)$ is not AK-space, there exist $(t_i^{(1)}) \in \lambda$ and $D \in \mathcal{P}_\lambda$ such that

$$\lim_n \sup \left\{ \left| \sum_{i=n}^{\infty} t_i^{(1)} u_i \right| : (u_i) \in D \right\} > 0. \tag{5}$$

Let $X = \left(\lambda^\beta, \sigma(\lambda^\beta, \lambda)\right)$ and Y be the complex number field $\mathbb{C}$, define $T_i : \lambda^\beta \longrightarrow \mathbb{C}$ by $T_i u = u_i$ for each $u = (u_i) \in \lambda^\beta$. Clearly, for each $i \in \mathbf{N}$, $T_i \in L(\lambda^\beta, \mathbb{C})$. $\sum_i T_i$ is λ-multiplier weak operator topology convergent since for each $t = (t_i) \in \lambda$ and $u = (u_i) \in \lambda^\beta$, $\lim_n \sum_{i=n}^{\infty} t_i T_i u = \lim_n \sum_{i=n}^{\infty} t_i u_i = 0$.

https://doi.org/10.1017/S0004972700039186 Published online by Cambridge University Press

On the other hand, it follows from (5) that $\sum_i T_i$ is not λ-multiplier $L_P(\lambda^\beta, \mathbb{C})$ convergent. This is a contradiction and the theorem is proved. $\square$

REMARK 1. A vector (non-operator) version of Theorem 1 was proved in both [**13**, Theorem 1] and [**4**, Theorem 3]. The following Lemma is such a result which will be used later.

LEMMA 1. ([**13**, Theorem 1]; [**4**, Theorem 3].) *Let $c_{00} \subseteq \lambda$. Then every λ-multiplier $\sigma(X, X')$ convergent series $\sum_i x_i$ in X must be λ-multiplier $\tau(X, X')$ convergent if and only if $(\lambda, \tau(\lambda, \lambda^\beta))$ is an AK-space.*

Now, we give some applications of Theorem 1.

Let $c_{00} \subseteq \lambda$. An interval in $\mathbb{N}$ is a set of the form $[m, n] = \{k \in \mathbb{N} : m \leqslant k \leqslant n\}$. If I is an interval, then χ_I will be the characteristic function of I, and if $t = \{t_k\} \in \lambda$, then $\chi_I t$ denote the coordinatewise product of χ_I and t. A sequence of interval $\{I_j\}$ in $\mathbb{N}$ is increasing if $\max I_j < \min I_{j+1}$ for all j.

The space λ has the signed weak gliding hump property ([**5**]) if $t \in \lambda$ and $\{I_k\}$ an increasing sequence of intervals implies there exist a subsequence $\{I_{n_k}\}$ of $\{I_k\}$ and a sequence of signs $s_k = \pm 1$ such that the coordinatewise sum $\sum_{k=1}^{\infty} s_k \chi_{I_{n_k}} t \in \lambda$. The space λ has the weak gliding hump property ([**1**]) if the signs above can be chosen with $s_k = 1$ for all k. For example, any monotone sequence space such as $\ell^p (0 < p \leqslant \infty), m_0, c_0$ has the weak gliding hump property ([**1**]), the space bs of bounded series has signed-weak gliding hump property but not weak gliding hump property. There are a large class of spaces having weak gliding hump property ([**1**]) or signed-weak gliding hump property ([**5**]).

Wu and Swartz both discussed the following result:

LEMMA 2. ([**13**, Lemma 2]; [**4**, Theorem 4].) *Let $c_{00} \subseteq \lambda$ and λ has signed-weak gliding hump property. Then $(\lambda, \tau(\lambda, \lambda^\beta))$ is an AK-space.*

Thus, the following theorem can follow from Theorem 1 and Lemma 2, but, for clarity, we give its proof, too.

THEOREM 2. *Let $c_{00} \subseteq \lambda$ and λ has signed-weak gliding hump property. Then every λ-multiplier weak operator topology convergent operator series $\sum_i T_i$ of $L(X, Y)$ must be λ-multiplier $L_s(X, Y)$ convergent.*

PROOF: It follows from λ has signed-weak gliding hump property and Lemma 2 that $(\lambda, \tau(\lambda, \lambda^\beta))$ is an AK-space.

Let τ_0 be the original topology of Y and $\sum_i T_i$ a λ-multiplier weak operator topology convergent operator series of $L(X, Y)$, then for each $x \in X$, $\sum_i T_i x$ is λ-multiplier weak topology $\sigma(Y, Y')$ convergent in Y.

It follows from Lemma 1 that $\sum_i T_i x$ is λ-multiplier Mackey topology $\tau(Y, Y')$ convergent in Y, hence is λ-multiplier τ_0-convergent in Y, that is, $\sum_i T_i$ is λ-multiplier

https://doi.org/10.1017/S0004972700039186 Published online by Cambridge University Press

$L_s(X,Y)$ convergent and so the theorem is proved. ∎

Lemma 3. ([13, Corollary 2]; [4, Theorem 4].) *Let $c_{00} \subseteq \lambda$ and λ has signed-weak gliding hump property. Then $\left(\lambda, \mathcal{K}(\lambda, \lambda^\beta)\right)$ is an AK-space.*

Lemma 4. ([13, Lemma 1].) $\mathcal{K}(\lambda, \lambda^\beta) = c(\lambda, \lambda^\beta) = v(\lambda, \lambda^\beta)$.

By Theorem 1, Lemma 3 and Lemma 4 we have:

Theorem 3. *Let $c_{00} \subseteq \lambda$ and λ has signed-weak gliding hump property. Then every λ-multiplier weak operator topology convergent operator series $\sum_i T_i$ of $L(X,Y)$ must be λ-multiplier $L_C(X,Y)$ convergent, where*

$$C = \left\{D \subseteq X : D \text{ is } \sigma(X,X')\text{-countable compact set}\right\}.$$

Lemma 5. ([4, Theorem 4].) *If $c_{00} \subseteq \lambda$ and λ has signed-weak gliding hump property, then $\left(\lambda, \gamma(\lambda, \lambda^\beta)\right)$ is an AK-space.*

Thus, it follows from Theorem 1 and Lemma 5 immediately:

Theorem 4. *Let $c_{00} \subseteq \lambda$ and λ has signed-weak gliding hump property. Then every λ-multiplier weak operator topology convergent operator series $\sum_i T_i$ of $L(X,Y)$ must be λ-multiplier $L_\gamma(X,Y)$ convergent, where $\gamma = \left\{D \subseteq X : D \text{ is unconditional } \sigma(X,X')\text{-sequentially compact set}\right\}$.*

Let $\mathcal{B} = \left\{B \subseteq X\text{: if } \{x_k\} \subseteq B, \text{ then } \lim_k Tx_k \text{ exist for every } T \in L(X,Y)\right\}$ and $L_{\mathcal{B}}(X,Y)$ be $L(X,Y)$ with the topology of uniform convergent on elements of $\mathcal{B}$.

Lemma 6. ([4, Theorem 9].) *Assume $c_{00} \subseteq \lambda$ and λ has signed-weak gliding hump property. If the operator series $\sum_k T_k$ of $L(X,Y)$ is λ-multiplier convergent in $L_s(X,Y)$, then $\sum_k T_k$ is λ-multiplier $L_{\mathcal{B}}(X,Y)$ convergent.*

It follows from Theorem 1 and Lemma 6 that

Theorem 5. *If $c_{00} \subseteq \lambda$ and λ has signed-weak gliding hump property, then $\left(\lambda, B_\lambda(\lambda, \lambda^\beta)\right)$ is an AK-space, where $B_\lambda = \left\{B \subseteq \lambda^\beta\text{: if } \{x_k\} \subseteq B, \text{ then } \lim_k Tx_k \text{ exists for every } T \in L(\lambda^\beta, \lambda)\right\}$.*

We now consider two of the most common topologies on $L(X,Y)$. Let $\xi = \left\{\{x_k\} : x_k \longrightarrow 0\right\}$ in X and $L_{\to 0}(X,Y)$ be $L(X,Y)$ with the topology of uniform convergent on the elements of ξ. Let $L_{PC}(X,Y)$ be $L(X,Y)$ with the topology of uniform convergent on precompact subsets of X.

It follows also from Theorem 1 and Theorem 5 that

Corollary 1. *Assume $c_{00} \subseteq \lambda$ and λ has signed-weak gliding hump property, then $\left(\lambda, \xi_\lambda(\lambda, \lambda^\beta)\right)$ is an AK-space, where $\xi_\lambda = \left\{\{x_k\} : x_k \longrightarrow 0\right\}$ in $\left(\lambda^\beta, \sigma(\lambda^\beta, \lambda)\right)$.*

In [2], conditions on the space X are given which guarantee that the space $L_{\to 0}(X,Y)$ and $L_{PC}(X,Y)$ coincide. Using this result and Corollary 1, we have

Corollary 2. *Let $c_{00} \subseteq \lambda$ and λ has signed-weak gliding hump property. If X is either metrisable or the hyper strict inductive limit of such spaces, then $\left(\lambda, PC_\lambda(\lambda, \lambda^\beta)\right)$ is an AK-space, where PC_λ is the family of precompact subsets of $\left(\lambda^\beta, \sigma(\lambda^\beta, \lambda)\right)$.*

REMARK 2. Corollaries 1 and 2 are similiar to [**4**, Theorem 4] with different topologies on λ.

REFERENCES

[1] H. Boos, C. Staurt and C. Swartz, 'Gliding hump properties of matrix domains', *Anal. Math.* **30** (2004), 243–257.

[2] H.G. Garnier, M. Dewilde and J. Schmets, (in French), *Analyse fonctionnelle théorie constructive des espaces linéaires à semi-normes. Tome 1: Théorie général* (Birkhauser, Basel, 1968).

[3] R. Li, C. Cui and M. Cho, 'An invariant with respect to all admissible (X, X')-polar topologies (Chinese)', *Chinese Ann. Math. Ser. A* **19** (1998), 289–294.

[4] C. Stuart and C. Swartz, 'Generalization of the Orlicz–Pettis Theorems', *Proyecciones* **24** (2005), 37–48.

[5] C. Swartz, *Infinite matrices and the gliding hump* (World Scientific Co., River Edge, N.J., 1996).

[6] C. Swartz, 'Orlicz–Pettis Theorems for multiplier convergent operator valued series', *Proyecciones* **23** (2004), 61-72.

[7] C. Swartz and C. Stuart, 'Orlicz–Pettis Theorems for multiplier convergent', *Z. Anal. Anwendungen* **17** (1998), 805–811.

[8] S. Wen, C. Cui and R. Li, 's-multiplier convergence and theorems of the Orlicz–Pettis-type', *Acta Math. Sinica (China)* **43** (2000), 275–282.

[9] S. Wen, C. Jin, C. Cui and R. Li, 's-multiplier convergence and its invariance for admissible polar topology', *J. Systems Sci. Math. Sci. (China)* **20** (2000), 474–479.

[10] A. Wilansky, *Modern methods in topological vector spaces* (McGraw-Hill, New York, 1978).

[11] J. Wu and R. Li, 'An Orlicz–Pettis Theorem with applications to $\mathcal{A}$-spaces', *Studia Sci. Math. Hungar.* **35** (1999), 353–358.

[12] J. Wu, W. Qu, and C. Cui, 'On the invariant of λ-multiplier convergent series', *Adv. in Math. (China)* **30** (2002), 279–283.

[13] J. Wu and S. Lu, 'A general Orlicz–Pettis Theorem', *Taiwanese J. Math.* **6** (2002), 443–440.

Department of Mathematics
Qingdao University of Science and Technology
Qingdao 266061
Peoples Republic of China

Department of Mathematics
Harbin Institute of Technology
Harbin 150006
Peoples Republic of China