V

The FBI transform and some applications

This chapter begins with a discussion of certain submanifolds in CR and hypoanalytic manifolds. We then introduce the FBI transform which is a nonlinear Fourier transform that characterizes analyticity. We also present a more general version of this transform which characterizes hypoanalyticity. We will discuss several applications of the FBI transform to the study of the regularity of solutions in locally integrable structures.

V.1 Certain submanifolds of hypoanalytic manifolds

This section discusses certain submanifolds of hypoanalytic manifolds. We begin with a discussion of CR manifolds in $\mathbb{C}^N$. CR manifolds are good models for hypoanalytic manifolds. Later in the chapter we will see that a hypoanalytic structure can be locally embedded in a CR structure. This can sometimes be useful in reducing problems about general vector fields in hypoanalytic structures to CR vector fields. We will first recall the concept of a complex linear structure on a real vector space and apply it to the real tangent bundle of real submanifolds in $\mathbb{C}^N$. Let $\mathbb{V}$ be a vector space over $\mathbb{R}$ and suppose $J:\mathbb{V} \longrightarrow \mathbb{V}$ is a linear map such that $J^2 = -\mathrm{Id}$ (where Id = the identity). Clearly J is an isomorphism and $\dim\mathbb{V}$ is even since $(\det J)^2 = \det(-\mathrm{Id}) = (-1)^{\dim\mathbb{V}}$. The map J is called a *complex structure* on $\mathbb{V}$. Indeed, with such a J, $\mathbb{V}$ becomes a complex vector space by defining $(a+ib)v = av + b(Jv)$ for $a, b \in \mathbb{R}, v \in \mathbb{V}$. Conversely, if $\mathbb{V}$ is a complex vector space, it is also a vector space over $\mathbb{R}$ and the map $Jv = iv$ is an $\mathbb{R}$-linear map with $J^2 = -\mathrm{Id}$. If $\{v_1, \ldots, v_N\}$ is a basis of $\mathbb{V}$ over $\mathbb{C}$, then $\{v_1, \ldots, v_N, Jv_1, \ldots, Jv_N\}$ is a basis of $\mathbb{V}$ over $\mathbb{R}$.

EXAMPLE V.1.1. In $\mathbb{C}^N$ let $z_j = x_j + iy_j$, $1 \le j \le N$, denote the coordinates. We will identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ by means of the map

$$(z_1, ..., z_N) \longmapsto (x_1, y_1, ..., x_N, y_N).$$

Multiplication by i in $\mathbb{C}^N$ then induces a map $J\colon \mathbb{R}^{2N} \longrightarrow \mathbb{R}^{2N}$ given by

$$J(x_1, y_1, \dots, x_N, y_N) = (-y_1, x_1, \dots, -y_N, x_N).$$

Note that $J^2 = -\mathrm{Id}$ and so J is a complex structure on $\mathbb{R}^{2N}$, called the standard complex structure on $\mathbb{R}^{2N}$.

EXAMPLE V.1.2. With notation as in the previous example, for $p \in \mathbb{C}^N$, a basis of the real tangent space $T_p\mathbb{C}^N$ is given by $\frac{\partial}{\partial x_1}|_p, \frac{\partial}{\partial y_1}|_p, \dots, \frac{\partial}{\partial x_N}|_p, \frac{\partial}{\partial y_N}|_p$. This basis can be used to identify $T_p\mathbb{C}^N$ with $\mathbb{R}^{2N}$ by choosing the usual basis

$$e_1 = (1, 0, \dots, 0), \dots, e_{2N} = (0, \dots, 0, 1) \quad \text{of} \quad \mathbb{R}^{2N}.$$

This leads to a complex structure $J\colon T_p\mathbb{C}^N \longrightarrow T_p\mathbb{C}^N$ given by

$$J\left(\frac{\partial}{\partial x_j}|_p\right) = \left.\frac{\partial}{\partial y_j}\right|_p \quad \text{and} \quad J\left(\frac{\partial}{\partial y_j}|_p\right) = -\left.\frac{\partial}{\partial x_j}\right|_p, \qquad j = 1, \dots, N.$$

This complex structure is independent of the choice of the holomorphic coordinates $(z_1, \dots, z_N)$. To see this, suppose $w = F(z)$ is a biholomorphic map defined near 0 with $F(0) = 0$ where we are assuming as we may that $p = 0$. Write $F = U + iV$ and let $w_j = u_j + iv_j$, $j = 1, \dots, N$. We need to show that $\mathrm{d}F_0 \circ J = J \circ \mathrm{d}F_0$. We have:

$$\mathrm{d}F_0\left(J\left(\frac{\partial}{\partial x_j}\right)\right) = \mathrm{d}F_0\left(\frac{\partial}{\partial y_j}\right) = \sum_l \frac{\partial U_l}{\partial y_j}\frac{\partial}{\partial u_l} + \sum_l \frac{\partial V_l}{\partial y_j}\frac{\partial}{\partial v_l}$$

and

$$\begin{aligned} J\left(\mathrm{d}F_0\left(\frac{\partial}{\partial x_j}\right)\right) &= J\left(\sum_l \frac{\partial U_l}{\partial x_j}\frac{\partial}{\partial u_l} + \sum_l \frac{\partial V_l}{\partial x_j}\frac{\partial}{\partial v_l}\right) \\ &= \sum_l \frac{\partial U_l}{\partial x_j}\frac{\partial}{\partial v_l} - \sum_l \frac{\partial V_l}{\partial x_j}\frac{\partial}{\partial u_l}, \end{aligned}$$

where everything is to be evaluated at 0. Thus an application of the Cauchy–Riemann equations to the U_j and V_j shows that $\mathrm{d}F(J(\frac{\partial}{\partial x_j})) = J(\mathrm{d}F(\frac{\partial}{\partial x_j}))$. The equality also holds in the same fashion for $\frac{\partial}{\partial y_j}$. Thus, J is independent of the holomorphic coordinates. This also means that J can be defined on the real tangent space of any complex manifold. Note that J extends to a $\mathbb{C}$-linear map from $\mathbb{C}T_p\mathbb{C}^N$ into itself and the extension still satisfies $J^2 = -\mathrm{Id}$. We will also denote this extension by J. The fact that $J^2 = -\mathrm{Id}$ implies that

$J : \mathbb{C}T_p\mathbb{C}^N \longrightarrow \mathbb{C}T_p\mathbb{C}^N$ has only two eigenvalues: i and $-i$. Define $T_p^{1,0}\mathbb{C}^N$ to be equal to the eigenspace associated with i, and $T_p^{0,1}\mathbb{C}^N$ will be the eigenspace associated with $-i$. We get corresponding vector bundles $T^{1,0}$ and $T^{0,1}$. Observe that $T^{1,0}$ is generated by $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_N}$. Hence $T^{1,0}$ is the bundle of holomorphic vector fields introduced in Chapter I (see the discussion preceding Theorem I.5.1). Likewise, $T^{0,1}$ is generated by $\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_N}$.

Definition V.1.3. *Let $\mathcal{M}$ be a real submanifold of $\mathbb{C}^N$. For $p \in \mathcal{M}$, define*

$$\mathcal{V}_p(\mathcal{M}) = \mathbb{C}T_p\mathcal{M} \cap T_p^{0,1}\mathbb{C}^N.$$

Definition V.1.4. *Let $\mathcal{M}$ be a real submanifold of $\mathbb{C}^N$ and $p \in \mathcal{M}$. The* complex tangent space *of $\mathcal{M}$ at p denoted $T_p^c\mathcal{M}$ is defined by*

$$T_p^c\mathcal{M} = T_p\mathcal{M} \cap J(T_p\mathcal{M})$$

It is easy to see that $T_p^c\mathcal{M} = \{v \in T_p\mathcal{M} : J(v) \in T_p\mathcal{M}\}$. Observe that $J : T_p^c\mathcal{M} \longrightarrow T_p^c\mathcal{M}$ and so $T_p^c\mathcal{M}$ is equipped with a complex vector space structure. It is also evident that $J : \mathbb{C}T_p^c\mathcal{M} \longrightarrow \mathbb{C}T_p^c\mathcal{M}$.

Example V.1.5. Let $\mathcal{M}$ be a hypersurface in $\mathbb{C}^N$ through the point 0. Let ρ be a defining function for $\mathcal{M}$ near 0. Since $d\rho(0) \neq 0$ and ρ is real-valued, $\partial\rho(0) \neq 0$. After a complex linear change of coordinates, we may assume that

$$\frac{\partial \rho}{\partial z}(0) = (0, \ldots, 0, 1).$$

That is, we have coordinates (z, w), $z = x + iy \in \mathbb{C}^{N-1}$, $w = s + it \in \mathbb{C}$, such that $\frac{\partial \rho}{\partial x_j}(0) = \frac{\partial \rho}{\partial y_j}(0) = 0$, $j = 1, \ldots, N-1$, $\frac{\partial \rho}{\partial s}(0) = 0$ and $\frac{\partial \rho}{\partial t}(0) \neq 0$. These conditions on the partial derivatives of ρ allow us to apply the implicit function theorem and conclude that near 0 the submanifold $\mathcal{M}$ is given by

$$\mathcal{M} = \{(z, s + i\phi(z, s))\},$$

where ϕ is real-valued, $\phi(0,0) = 0$, and $d\phi(0,0) = 0$. Hence $T_0\mathcal{M} =$ span at 0 of

$$\left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial s}, \quad j = 1, \ldots, N-1 \right\},$$

while $\mathcal{V}_0(\mathcal{M}) =$ the $\mathbb{C}$-span at 0 of

$$\left\{ \frac{\partial}{\partial \bar{z}_j} : j = 1, \ldots, N-1 \right\}$$

and $T_0^c\mathcal{M} =$ the $\mathbb{R}$-span at 0 of

$$\left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} : j = 1, \ldots, N-1 \right\}.$$

The spaces $T_p^c\mathcal{M}$ and $\mathcal{V}_p(\mathcal{M})$ are related. To see this, we recall the following result from [**BER**] where $\Re\mathcal{V}_p(\mathcal{M})$ denotes the real parts of elements of $\mathcal{V}_p(\mathcal{M})$:

PROPOSITION V.1.6. *For $p \in \mathcal{M}$,*

(a) $\Re\mathcal{V}_p(\mathcal{M}) = T_p^c\mathcal{M}$;
(b) $\mathbb{C}T_p^c\mathcal{M} = \mathcal{V}_p(\mathcal{M}) \oplus \overline{\mathcal{V}_p(\mathcal{M})}$;
(c) $\mathcal{V}_p(\mathcal{M}) = \{x + iJ(x) : x \in T_p^c\mathcal{M}\}$.

PROOF. Observe first that for any $x \in T_p\mathbb{C}^N$, $x + iJ(x) \in T_p^{0,1}\mathbb{C}^N$. Let $x \in T_p^c\mathcal{M}$. Then x and $J(x) \in T_p\mathcal{M}$ and so $x + iJ(x) \in \mathcal{V}_p(\mathcal{M})$. Thus $x \in \Re\mathcal{V}_p(\mathcal{M})$. Conversely, if $x \in \Re\mathcal{V}_p(\mathcal{M})$, then there is $y \in T_p\mathbb{C}^N$ such that $x + iy \in \mathcal{V}_p(\mathcal{M}) \subseteq \mathbb{C}T_p\mathcal{M}$ implying that $x \in T_p^c\mathcal{M}$ since $y = J(x)$ and $y \in T_p\mathcal{M}$. We have thus proved (a) and (b) follows from (a) trivially. The proof of (c) is also contained in that of (a). □

From Proposition V.1.6 we see that

$$\dim T_p^c\mathcal{M} = 2\dim_{\mathbb{C}}\mathcal{V}_p(\mathcal{M}).$$

DEFINITION V.1.7. *A submanifold $\mathcal{M}$ of $\mathbb{C}^N$ is called* CR *(for Cauchy–Riemann) if $\dim_{\mathbb{C}}\mathcal{V}_p(\mathcal{M})$ is constant as p varies in $\mathcal{M}$. In this case, $\dim_{\mathbb{C}}\mathcal{V}_p(\mathcal{M})$ is called the CR dimension of $\mathcal{M}$.*

DEFINITION V.1.8. *A CR submanifold $\mathcal{M}$ of $\mathbb{C}^N$ is called* totally real *if its CR dimension is* 0.

EXAMPLE V.1.9. The copy of $\mathbb{R}^N$ in $\mathbb{C}^N$ given by

$$\left\{x + iy \in \mathbb{C}^N : y = 0\right\}$$

is a totally real submanifold.

EXAMPLE V.1.10. Let k and N be positive integers, $1 \le k \le N$. Write the coordinates in $\mathbb{C}^N$ as (z, w), $z = x + iy \in \mathbb{C}^k$ and $w = u + iv \in \mathbb{C}^{N-k}$. Let $\phi : \mathbb{R}^k \mapsto \mathbb{R}^k$ and $\psi : \mathbb{R}^k \mapsto \mathbb{C}^{N-k}$ be smooth functions with $\phi(0) = 0$, $\mathrm{d}\phi(0) = 0$, $\psi(0) = 0$, and $\mathrm{d}\psi(0) = 0$. Then the submanifold

$$\mathcal{M} = \left\{(x + i\phi(x), \psi(x)) : x \in \mathbb{R}^k\right\}$$

is totally real near the point $0 \in \mathcal{M}$. Conversely, if $\mathcal{N}$ is any totally real submanifold of $\mathbb{C}^N$, then near each of its points, there are holomorphic coordinates in which $\mathcal{N}$ takes the form of $\mathcal{M}$ above (see proposition 1.3.8 in [**BER**]). If $\mathcal{N}$ is also real-analytic, holomorphic coordinates can be found so that $\phi \equiv 0$ and $\psi \equiv 0$.

LEMMA V.1.11. *Suppose $\mathcal{M}$ is a submanifold of $\mathbb{C}^N$ of real codimension d. Then*

$$2N-2d \leq \dim T_p^c\mathcal{M} \leq 2N-d.$$

PROOF. Since $T_p^c\mathcal{M} \subseteq T_p\mathcal{M}$,

$$\dim T_p^c\mathcal{M} \leq \dim T_p\mathcal{M} = 2N-d.$$

On the other hand, $T_p\mathcal{M}+J(T_p\mathcal{M}) \subseteq T_p\mathbb{C}^N$ and so

$$\dim T_p^c\mathcal{M} = 2\dim T_p\mathcal{M} - \dim(T_p\mathcal{M}+J(T_p\mathcal{M})) \geq 2N-2d. \qquad \square$$

EXAMPLE V.1.12. A hypersurface $\mathcal{M} \subseteq \mathbb{C}^N$ is a CR submanifold of CR dimension $N-1$. Indeed, this follows from the lemma since $T_p^c\mathcal{M}$ always has even real dimension, which when $p \in \mathcal{M}$ has to equal $2N-2$.

EXAMPLE V.1.13. Let $\mathcal{M}$ be a complex submanifold of $\mathbb{C}^N$ of complex dimension n. Then $\mathcal{M}$ is a CR submanifold of CR dimension n. This follows from the J-invariance of $T_p\mathcal{M}$. To see this, let $X \in T_p\mathcal{M}$. If $f_j = u_j + iv_j$ $(1 \leq j \leq N-n)$ are local holomorphic defining functions near $p \in \mathcal{M}$, then by the Cauchy–Riemann equations we have $J(X)u_j = J(X)v_j = 0$ for all j. Hence $J(X) \in T_p\mathcal{M}$.

Suppose $\mathcal{M} \subseteq \mathbb{C}^N$ has codimension d and is locally defined by $\rho_j = 0$, $j = 1,\dots,d$. Then a vector $v = \sum_{j=1}^N v_j \frac{\partial}{\partial \bar{z}_j} \in \mathcal{V}_p(\mathcal{M})$ if and only if $\sum_{j=1}^N v_j \frac{\partial \rho_l}{\partial \bar{z}_j} = 0$ for all l. Hence $\dim \mathcal{V}_p\mathcal{M} = N-r$ where $r =$ the dimension of the $\mathbb{C}$-span of $\{\bar{\partial}\rho_1(p),\dots,\bar{\partial}\rho_d(p)\}$.

EXAMPLE V.1.14. Let $\mathcal{M}$ be the two-dimensional submanifold of $\mathbb{C}^2$ defined by $\rho_1 = x_2 - x_1^2 = 0$ and $\rho_2 = y_2 - y_1^2 = 0$. Then by calculating $\bar{\partial}\rho_1(p)$ and $\bar{\partial}\rho_2(p)$, we easily see that $\dim \mathcal{V}_p\mathcal{M} = 0$ or 1 depending on whether $p \in \mathcal{M} \cap \{x_1 = y_1\}$. Hence $\mathcal{M}$ is not a CR manifold.

If $\mathcal{M} \subseteq \mathbb{C}^N$ has codimension d, since $2\dim_{\mathbb{C}} \mathcal{V}_p\mathcal{M} = \dim T_p^c\mathcal{M}$, Lemma V.1.11 tells us that the minimum possible value of $\dim_{\mathbb{C}} \mathcal{V}_p\mathcal{M} = N-d$. This minimum value is attained precisely when the forms $\{\bar{\partial}\rho_1(p),\dots,\bar{\partial}\rho_d(p)\}$ are linearly independent. The CR submanifolds for which $\dim \mathcal{V}_p\mathcal{M}$ has such minimal value are the generic ones introduced in Chapter I. It will be convenient to present here an equivalent definition.

DEFINITION V.1.15. *A CR submanifold $\mathcal{M} \subseteq \mathbb{C}^N$ of codimension d is called* generic *if for $p \in \mathcal{M}$, $\dim_{\mathbb{C}} \mathcal{V}_p\mathcal{M} = N-d$.*

EXAMPLE V.1.16. A hypersurface of $\mathbb{C}^N$ is a generic CR submanifold.

Example V.1.17. A complex submanifold of $\mathbb{C}^N$ that is not an open subset is a nongeneric CR submanifold.

Example V.1.18. Let (z, w) denote the coordinates of $\mathbb{C}^{n+d}$ where $z = x + iy \in \mathbb{C}^n$ and $w = s + it \in \mathbb{C}^d$. Let $\phi_1(z, s), \ldots, \phi_d(z, s)$ be smooth, real-valued functions. Then

$$\mathcal{M} = \{(z, w) : t_j - \phi_j(z, s) = 0, 1 \le j \le d\}$$

is a generic CR manifold. This is easily checked by noting that

$$\rho_j = \frac{w_j - \overline{w}_j}{2i} - \phi_j(z, s)$$

are defining functions with $\{\overline{\partial}\rho_1(p), \ldots, \overline{\partial}\rho_d(p)\}$ linearly independent at each point.

Remark V.1.19. Conversely, as we saw in Chapter I, given any generic CR submanifold $\mathcal{M}$, in appropriate holomorphic coordinates, $\mathcal{M}$ takes the form in the example.

Let $\mathcal{M}$ be a CR submanifold of $\mathbb{C}^N$ of codimension d that is not generic and assume $0 \in \mathcal{M}$. We will show that in a certain sense, near the point 0, $\mathcal{M}$ can be viewed as a generic CR submanifold of $\mathbb{C}^L$ for some $L < N$. Define

$$\mathcal{S} = T_0\mathcal{M} + J(T_0\mathcal{M}).$$

Let Y be a subspace of $T_0\mathcal{M}$ such that

$$T_0\mathcal{M} = T_0^c\mathcal{M} \oplus Y.$$

Note that $J(Y) \cap T_0\mathcal{M} = 0$. Let $\{v_1, \ldots, v_n\}$ be a $\mathbb{C}$-basis of the complex space $T_0^c\mathcal{M}$. Then: $\{v_1, \ldots, v_n, J(v_1), \ldots, J(v_n)\}$ is an $\mathbb{R}$-basis of $T_0^c\mathcal{M}$. Complete this to a basis $\{v_1, \ldots, v_n, J(v_1), \ldots, J(v_n), u_1, \ldots, u_r\}$ of $T_0\mathcal{M}$ where $2n + r + d = 2N$. Then since $J(Y) \cap T_0\mathcal{M} = 0$, it follows that

$$\mathcal{B} = \{v_1, \ldots, v_n, J(v_1), \ldots, J(v_n), u_1, \ldots, u_r, J(u_1), \ldots, J(u_r)\}$$

is a basis of $\mathcal{S}$. Extend $\mathcal{B}$ to a basis

$$\mathcal{B}' = \mathcal{B} \cup \{u_1', \ldots, u_l', J(u_1'), \ldots, J(u_l')\}$$

of $T_0\mathbb{C}^N$, $N = n + r + l$. Split the coordinates in $\mathbb{C}^N = \mathbb{C}^{n+r+l}$ as (z, w, p) where $z = x + iy \in \mathbb{C}^n$, $w = s + it \in \mathbb{C}^r$, and $p = s' + it' \in \mathbb{C}^l$. Define the map $A : T_0\mathbb{C}^N \to T_0\mathbb{C}^N$ by $A(v_i) = \frac{\partial}{\partial x_i}$, $\quad A(Jv_i) = \frac{\partial}{\partial y_i}$, $1 \le i \le n$; $A(u_k) = \frac{\partial}{\partial s_k}$, $\quad A(Ju_k) = \frac{\partial}{\partial t_k}$, $1 \le k \le r$; $A(u_j') = \frac{\partial}{\partial s_j'}$, $\quad A(Ju_j') = \frac{\partial}{\partial t_j'}$, $1 \le j \le l$. Note that the map A commutes with J and hence after a complex linear map (see Remark V.1.20 below) we are in coordinates $(z, w, p) \in \mathbb{C}^{n+r+l}$ where

$$\mathcal{S} = \text{span of} \quad \left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial s_k}, \frac{\partial}{\partial t_k} : 1 \le j \le n, 1 \le k \le r \right\}$$

and $T_0\mathcal{M} = \text{span of } \{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial s_k}\}$. It follows that near 0, $\mathcal{M}$ can be expressed as a graph of the form:

$$\mathcal{M} = \{(x+iy, s+if(x, y, s), g(x, y, s)\}$$

where f is valued in $\mathbb{R}^r$ and g is valued in $\mathbb{C}^l$. The components of the functions $s+if(x, y, s)$ and $g(x, y, s)$ are CR functions. Observe that the projection $\pi : \mathbb{C}^{n+r+l} \to \mathbb{C}^{n+r}$, $\pi(z, w, p) = (z, w)$ is a diffeomorphism of $\mathcal{M}$ onto the generic CR submanifold $\pi(\mathcal{M})$ of $\mathbb{C}^{n+r}$.

Remark V.1.20. Recall the identification of $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ of Example V.1.1 given by $(z_1, \ldots, z_N) \mapsto (x_1, y_1, \ldots, x_N, y_N)$. With this identification, it is easy to see that a real linear map $A : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ induces a $\mathbb{C}$-linear map on $\mathbb{C}^N$ if and only if A commutes with the operator J.

Proposition V.1.21. *If $\mathcal{M}$ is a totally real submanifold of $\mathbb{C}^N$ of codimension d, then $d \ge N$ and hence $\dim \mathcal{M} \le N$. If $\mathcal{M}$ is also generic, then $d = N$. Thus, a totally real submanifold of maximal dimension has dimension $= N$.*

Proof. Let $p \in \mathcal{M}$ and $\rho_1, ..., \rho_d$ be defining functions of $\mathcal{M}$ near p. Since $\mathcal{V}_p(\mathcal{M}) = \{0\}$, we must have:

$$\text{span}_{\mathbb{C}}\{\partial\rho_1, \cdots, \partial\rho_d\} = \text{span}_{\mathbb{C}}\{dz_1, \ldots, dz_N\}$$

at the point p. Hence $d \ge N$. If $\mathcal{M}$ is also generic, then $\partial\rho_1, ..., \partial\rho_d$ are linearly independent and so $d = N$. □

The map J can be used to characterize CR, generic CR, and totally real submanifolds.

Proposition V.1.22. *Let $\mathcal{M}$ be a submanifold of $\mathbb{C}^N$. Then*

(i) *$\mathcal{M}$ is CR if and only if $\dim(T_p\mathcal{M} \cap J(T_p\mathcal{M}))$ is constant as p varies in $\mathcal{M}$.*
(ii) *$\mathcal{M}$ is totally real if and only if $T_p\mathcal{M} \cap J(T_p\mathcal{M}) = \{0\}$ for all $p \in \mathcal{M}$.*
(iii) *$\mathcal{M}$ is a generic CR submanifold if and only if*

$$T_p\mathcal{M} + J(T_p\mathcal{M}) = T_p\mathbb{C}^N \quad \textit{for all } p \in \mathcal{M}.$$

https://doi.org/10.1017/CBO9780511543067.006 Published online by Cambridge University Press

PROOF. (i) follows from the definition of $T_p^c\mathcal{M}$ and Proposition V.1.6. (ii) also follows from Proposition V.1.6. To prove (iii), if $\mathcal{M}$ is generic and $\rho_1, \ldots, \rho_d$ are local defining functions, then the linear independence of $\partial\rho_1, \ldots, \partial\rho_d$ is equivalent to:

$$\dim_{\mathbb{C}} \mathcal{V}_p(\mathcal{M}) = N - d.$$

Hence, by Proposition V.1.6, dim $(T_p\mathcal{M} \cap J(T_p\mathcal{M})) = 2(N-d)$. But then dim $(T_p\mathcal{M} + J(T_p\mathcal{M})) = 2N$, implying that $T_p\mathcal{M} + J(T_p\mathcal{M}) = T_p\mathbb{C}^N$ for all $p \in \mathcal{M}$. Conversely, if $T_p\mathcal{M} + J(T_p\mathcal{M}) = T_p\mathbb{C}^N$, then $\dim(T_p\mathcal{M} \cap J(T_p\mathcal{M})) = 2(N-d)$ and so by Proposition V.1.6, $\dim_{\mathbb{C}} \mathcal{V}_p(\mathcal{M}) = N - d$ showing that $\mathcal{M}$ is generic. $\square$

We will next describe certain submanifolds in hypoanalytic structures that play important roles in the analysis of the solutions of the sections of the associated vector bundle. Let $(\mathcal{M}, \mathcal{V})$ be a hypoanalytic structure. $\mathcal{M}$ is a smooth manifold of dimension N and $\mathcal{V}$ is an involutive sub-bundle of $\mathbb{C}T\mathcal{M}$ of fiber dimension n whose orthogonal bundle T' in $\mathbb{C}T^*\mathcal{M}$ is locally generated by the differentials of $m = N - n$ smooth functions. Recall from Chapter I that $T^0 = \bigcup_{p\in\mathcal{M}} T_p^0$ denotes the characteristic set of the structure $(\mathcal{M}, \mathcal{V})$.

DEFINITION V.1.23. *A submanifold $\mathcal{Y}$ is called* noncharacteristic *if*

$$T_p\mathcal{M} = T_p\mathcal{Y} + \Re(\mathcal{V}_p) \qquad \forall p \in \mathcal{Y}.$$

DEFINITION V.1.24. *A submanifold $\mathcal{N}$ is called* strongly noncharacteristic *if*

$$\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_p\mathcal{N} + \mathcal{V}_p \qquad \forall p \in \mathcal{N}.$$

DEFINITION V.1.25. *A submanifold $\mathcal{X}$ of $\mathcal{M}$ is called* maximally real *if*

$$\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_p\mathcal{X} \oplus \mathcal{V}_p \qquad \forall p \in \mathcal{X}.$$

Clearly, a maximally real submanifold is strongly noncharacteristic. If $\mathcal{N}$ is strongly noncharacteristic, then dim $\mathcal{N} \geq m$, while if $\mathcal{X}$ is maximally real, dim $\mathcal{X} = m$. A strongly noncharacteristic submanifold is a noncharacteristic submanifold. A noncharacteristic hypersurface in $\mathcal{M}$ is strongly noncharacteristic.

EXAMPLE V.1.26. Denote the coordinates in $\mathbb{R}^3$ by (x, y, t) and consider the structure generated by $L = \frac{\partial}{\partial t} + i\frac{\partial}{\partial y}$. The orthogonal of L is generated by dZ_1 and dZ_2 where $Z_1 = x$ and $Z_2 = t + iy$. If $S = \{(x, 0, 0)\}$, then S is a noncharacteristic submanifold that is not strongly noncharacteristic.

A CR submanifold of $\mathbb{C}^N$ is strongly noncharacteristic if and only if it is generic. It is maximally real precisely when it is totally real of maximal dimension.

The proofs of the following propositions are left to the reader.

Proposition V.1.27. *A submanifold $\mathcal{Y}$ of $\mathcal{M}$ is noncharacteristic if and only if the natural map $T^*\mathcal{M}|_{\mathcal{Y}} \longrightarrow T^*\mathcal{Y}$ maps T^0 injectively into $T^*\mathcal{Y}$.*

Proposition V.1.28. *A submanifold $\mathcal{N}$ of $\mathcal{M}$ is strongly noncharacteristic if and only if the natural map $\mathbb{C}T^*\mathcal{M}|_{\mathcal{N}} \longrightarrow \mathbb{C}T^*\mathcal{N}$ maps $T'|_{\mathcal{N}}$ injectively into $\mathbb{C}T^*\mathcal{N}$.*

Proposition V.1.29. *A submanifold $\mathcal{X}$ of $\mathcal{M}$ is maximally real if and only if the natural map $\mathbb{C}T^*\mathcal{M}|_{\mathcal{X}} \longrightarrow \mathbb{C}T^*\mathcal{X}$ induces a bijection of $T'|_{\mathcal{X}}$ onto $\mathbb{C}T^*\mathcal{X}$.*

Distribution solutions have traces on a noncharacteristic submanifold of $\mathcal{M}$ (see proposition 1.4.3 in [**T5**]). In particular, a solution can always be restricted to a maximally real manifold. The local and microlocal regularity of solutions are often studied by analyzing their restrictions to maximally real submanifolds. Instances of this will occur later in this chapter.

V.2 Microlocal analyticity and the FBI transform

The Paley–Wiener Theorem (see Theorem V.3.1 in the next section) characterizes the smoothness of a tempered distribution u in terms of the rapid decay of its Fourier transform $\hat{u}(\xi)$. This characterization is very useful in studying the local and microlocal regularity of solutions of partial differential equations with smooth coefficients. There is also a characterization of analyticity in terms of the Fourier transform ([**H8**]). However, the latter is based on estimates using a sequence of cut-off functions making it more difficult in applications. The FBI transform is a nonlinear Fourier transform which characterizes analyticity (see Theorem V.2.4 below).

Definition V.2.1. *Let $u \in \mathcal{E}'(\mathbb{R}^m)$. Define the FBI transform of u by*

$$F_u(x, \xi) = \int e^{i(x-y)\cdot\xi - |\xi||x-y|^2} u(y)\,dy \qquad \text{(V.1)}$$

for $(x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m$, where

$$(x-y)\cdot\xi = \sum_{j=1}^{m} (x_j - y_j)\xi_j.$$

https://doi.org/10.1017/CBO9780511543067.006 Published online by Cambridge University Press

The integral is to be understood in the duality sense.

THEOREM V.2.2. (Inversion with the FBI.) *Let $u \in C_c(\mathbb{R}^m)$. Then*

$$u(x) = \lim_{\epsilon\to 0^+} \frac{1}{(4\pi^3)^{\frac{m}{2}}} \iint F_u(t,\xi) e^{i(x-t)\cdot\xi - \epsilon|\xi|^2} |\xi|^{\frac{m}{2}} \, dt d\xi$$

where the convergence is uniform.

REMARK V.2.3. If $u \in \mathcal{E}'(\mathbb{R}^m)$, the theorem also holds where convergence is understood in the distribution sense.

PROOF. From the Fourier transform of the Gaussian, we have:

$$\int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi - \epsilon|\xi|^2} \, d\xi = \left(\frac{\pi}{\epsilon}\right)^{\frac{m}{2}} e^{\frac{-|x-y|^2}{4\epsilon}}.$$

Hence

$$\begin{aligned} \frac{1}{(2\pi)^m} \iint e^{i(x-y)\cdot\xi - \epsilon|\xi|^2} u(y) \, d\xi dy &= \frac{1}{2^m(\pi\epsilon)^{\frac{m}{2}}} \int e^{\frac{-|x-y|^2}{4\epsilon}} u(y) \, dy \\ &= \frac{1}{\pi^{\frac{m}{2}}} \int e^{-t^2} u(x - 2\sqrt{\epsilon} t) \, dt \\ &\to u(x) \end{aligned}$$

uniformly on $\mathbb{R}^m$ since $u \in C_c(\mathbb{R}^m)$. Thus

$$\begin{aligned} u(x) &= \lim_{\epsilon\to 0} \frac{1}{(2\pi)^m} \iint e^{i(x-y)\cdot\xi - \epsilon|\xi|^2} u(y) \, dy d\xi \\ &= \frac{1}{(4\pi^3)^{\frac{m}{2}}} \lim_{\epsilon\to 0} \iiint e^{i(x-y)\cdot\xi - |\xi||t-y|^2 - \epsilon|\xi|^2} |\xi|^{\frac{m}{2}} u(y) \, dt dy d\xi \\ &= \frac{1}{(4\pi^3)^{\frac{m}{2}}} \lim_{\epsilon\to 0} \iint F_u(t,\xi) e^{i(x-t)\cdot\xi - \epsilon|\xi|^2} |\xi|^{\frac{m}{2}} \, dt d\xi. \end{aligned}$$

□

The following characterization of analyticity by means of an exponential decay of the FBI transform may be viewed as an analogue of the Paley–Wiener Theorem.

THEOREM V.2.4. *Let $u \in \mathcal{E}'(\mathbb{R}^m)$. The following are equivalent:*

(i) *u is real-analytic at $x_0 \in \mathbb{R}^m$.*

(ii) *There exist a neighborhood V of x_0 in $\mathbb{R}^m$ and constants $c_1, c_2 > 0$ such that*

$$|F_u(x,\xi)| \le c_1 e^{-c_2|\xi|} \quad \textit{for } (x,\xi) \in V \times \mathbb{R}^m.$$

https://doi.org/10.1017/CBO9780511543067.006 Published online by Cambridge University Press

Proof. We will assume that u is continuous and leave the general case for the reader.

(i) $\Rightarrow$ (ii) Suppose u is real-analytic at x_0. Let $0 \leq \phi \leq 1$, $\phi \in C_0^\infty(\mathbb{R}^m)$, $\phi \equiv 1$ near x_0, and $\text{supp}(\phi) \subseteq \{x : u \text{ is analytic at } x\}$. The integrand in $F_u(x, \xi)$ has a holomorphic extension in a neighborhood of $y = x_0$ in $\mathbb{C}^m$. We will denote by u the holomorphic extension of u near x_0. In the integration defining $F_u(x, \xi)$, we deform the contour from $\mathbb{R}^m$ to the image of $\mathbb{R}^m$ under the map $y \mapsto \theta(y) = y - is\phi(y)\frac{\xi}{|\xi|}$ where s is chosen small enough so that u is defined on the image $\theta(\mathbb{R}^m)$. We then have

$$F_u(x, \xi) = \int_{\mathbb{R}^m} \mathrm{e}^{Q(x,y,\xi)} u(\theta(y)) \det \theta'(y)\, \mathrm{d}y \tag{V.2}$$

where $Q(x, y, \xi) = i(x - \theta(y)) \cdot \xi - |\xi|(x - \theta(y))^2$. Observe that

$$\Re Q(x, y, \xi) = -s|\xi|\phi(y)[1 - s\phi(y)] - |\xi||x - y|^2. \tag{V.3}$$

Let $\delta > 0$ such that $\phi(y) \equiv 1$ when $|y - x_0| \leq \delta$. Choose $s = \frac{\delta}{4}$. With these choices, (V.2) and (V.3), we get:

$$\begin{aligned} |F_u(x, \xi)| &\leq c \int_{|y-x_0|\leq\delta} \mathrm{e}^{\Re Q(x,y,\xi)}\, \mathrm{d}y + c \int_{y\in\text{supp}(u),\, |y-x_0|\geq\delta} \mathrm{e}^{\Re Q(x,y,\xi)}\, \mathrm{d}y \\ &= I_1(x, \xi) + I_2(x, \xi). \end{aligned}$$

Note then that

$$\begin{aligned} I_1(x, \xi) &\leq c \int_{|y-x_0|\leq\delta} \mathrm{e}^{-s|\xi|\phi(y)[1-s\phi(y)]}\, \mathrm{d}y \\ &\leq c\mathrm{e}^{\frac{-\delta}{8}|\xi|} \end{aligned}$$

for any ξ, and for $|x - x_0| \leq \frac{\delta}{2}$. Moreover,

$$\begin{aligned} I_2(x, \xi) &\leq c \int_{|y-x_0|\geq\delta} \mathrm{e}^{-|\xi||x-y|^2}\, \mathrm{d}y \\ &\leq c\mathrm{e}^{-(\frac{\delta}{2})^2|\xi|}. \end{aligned}$$

Hence, for $|x - x_0| \leq \frac{\delta}{2}$ and any $\xi \in \mathbb{R}^m$,

$$|F_u(x, \xi)| \leq c_1 \mathrm{e}^{-c_2|\xi|} \text{ for some } c_1, c_2 > 0.$$

(ii)$\Rightarrow$(i) Assume without loss of generality that $x_0 = 0$. Suppose then that for some $c_1, c_2 > 0$,

$$|F_u(x, \xi)| \leq c_1 \mathrm{e}^{-c_2|\xi|}$$

for all $\xi \in \mathbb{R}^m$, and for all x near 0. We will use the inversion given by Theorem V.2.2. Write

$$\iint F_u(t, \xi) \mathrm{e}^{i(x-t)\cdot\xi - \epsilon|\xi|^2} |\xi|^{\frac{m}{2}}\, \mathrm{d}t\mathrm{d}\xi = I_1^\epsilon(x) + I_2^\epsilon(x) + I_3^\epsilon(x) + I_4^\epsilon(x)$$

where for some A_1, A_2, B to be chosen later,

$$\begin{aligned} I_1^\epsilon(x) &= \text{ the integral over } \quad \{(t,\xi) : |t| \le A_1, \xi \in \mathbb{R}^m\}, \\ I_2^\epsilon(x) &= \text{ the integral over } \quad \{(t,\xi) : A_1 \le |t| \le A_2, |\xi| \le B\}, \\ I_3^\epsilon(x) &= \text{ the integral over } \quad \{(t,\xi) : |t| \ge A_2, \xi \in \mathbb{R}^m\}, \\ I_4^\epsilon(x) &= \text{ the integral over } \quad \{(t,\xi) : A_1 \le |t| \le A_2, |\xi| \ge B\}. \end{aligned}$$

Our goal is to show that there is a neighborhood of the origin in $\mathbb{C}^m$ to which the I_j^ϵ extend as holomorphic functions and for each j, $I_j^\epsilon(z)$ converges uniformly on this neighborhood as $\epsilon \to 0$. Consider first I_1^ϵ. Recall that $x_0 = 0$. Choose $A_1 > 0$ so that

$$|F_u(x,\xi)| \le c_1 \mathrm{e}^{-c_2|\xi|} \quad \text{for} \quad |x| \le A_1.$$

If we complexify x to $z = x + iy$ in the integrand of I_1^ϵ, we see that the integrand is bounded by a constant multiple of

$$|\xi|^{\frac{m}{2}} \mathrm{e}^{(-c_2+|y|)|\xi|}$$

which therefore has an integrable majorant for $|y| \le \frac{c_2}{2}$. Hence, as $\epsilon \to 0$, the entire functions $I_1^\epsilon(z)$ converge uniformly on a neighborhood of 0 to a holomorphic function. The functions I_2^ϵ easily extend as entire functions of z and converge uniformly on compact subsets to an entire function as $\epsilon \to 0$. Next choose A_2 so that

$$\mathrm{supp}(u) \subseteq \left\{ y : |y| \le \frac{A_2}{4} \right\}.$$

Then note that when $|t| \ge A_2$,

$$\begin{aligned} |F_u(t,\xi)| &\le c\mathrm{e}^{-|\xi|\left(|t| - \frac{A_2}{4}\right)^2} \\ &\le c\mathrm{e}^{-|\xi|\left(\frac{|t|^2}{4} + \frac{A_2^2}{16}\right)}. \end{aligned}$$

Using the latter we see that after integrating in t, the integrand in I_3^ϵ is uniformly bounded by a constant multiple of

$$\mathrm{e}^{-\frac{A_2^2}{16}|\xi|}.$$

This allows us to complexify as in I_1^ϵ to conclude that $I_3^\epsilon(z)$ converges uniformly to a holomorphic function in a neighborhood of 0. Write

$$I_4^\epsilon(x) = \iiint_R \mathrm{e}^{i(x-y)\cdot\xi - |\xi||t-y|^2 - \epsilon|\xi|^2} |\xi|^{\frac{m}{2}} u(y)\,\mathrm{d}y\mathrm{d}t\mathrm{d}\xi$$

where

$$R = \{(y,t,\xi) : |\xi| \ge B, A_1 \le |t| \le A_2, y \in \mathrm{supp}\, u\}.$$

Note that the function $\xi \mapsto |\xi|$ has a holomorphic extension $\langle\zeta\rangle$ in the region $|\Im\xi| < |\Re\xi|$, where

$$\langle\zeta\rangle = \left(\sum_{j=1}^{m} \zeta_j^2\right)^{\frac{1}{2}}$$

and an appropriate branch of the square root is taken. We change the contour in the ξ integration from $\mathbb{R}^m$ to its image under the map $\zeta(\xi) = \xi + is|\xi|(x-y)$ for s small, $s > 0$. The number s is chosen to be small enough to ensure that for $\xi \neq 0, |\Im\zeta(\xi)| < |\Re\zeta(\xi)|$. We then have, modulo entire functions that converge uniformly to an entire function,

$$I_4^{\epsilon}(x) = \iiint e^{P(x,y,t,\xi,\epsilon)} \langle\zeta(\xi)\rangle^{\frac{m}{2}} u(y)\, \mathrm{d}y\mathrm{d}t\mathrm{d}\zeta$$

where

$$P(x, y, t, \xi, \epsilon) = i(x-y)\cdot\xi - s|x-y|^2|\xi| - \langle\zeta(\xi)\rangle|t-y|^2 - \epsilon\zeta(\xi)^2.$$

Note that for s small, $\Re\zeta(\xi)^2 \geq \frac{|\xi|^2}{2}$ and $\Re\langle\zeta(\xi)\rangle \geq \frac{|\xi|}{2}$. Hence the crucial exponential term can be bounded as follows:

$$|e^{P(x,y,t,\xi,\epsilon)}| \leq e^{-s|x-y|^2|\xi| - \frac{|t-y|^2}{2}|\xi| - \frac{\epsilon}{2}|\xi|^2}.$$

In particular, when $x = 0$, since $|t| \geq A_1$, there is a constant $c > 0$ so that

$$|e^{P(0,y,t,\xi,\epsilon)}| \leq e^{-c|\xi|} \quad \text{for all } \xi.$$

This gives us enough freedom to complexify x to z and vary z near 0 to conclude that $I_4^{\epsilon}(z)$ converges uniformly to a holomorphic function near 0. □

We consider now the boundary values of holomorphic functions defined on wedges with flat edges, that is, edges that are open subsets of $\mathbb{R}^m$. Let $\Gamma \subseteq \mathbb{R}^m\backslash 0$ be an open convex cone with vertex at the origin, $V \subseteq \mathbb{R}^m$ open. For $\delta > 0$, let

$$\Gamma_\delta = \Gamma \cap \{v : |v| < \delta\}.$$

If Γ' is another cone, we write $\Gamma' \subset\subset \Gamma$ if $\overline{\Gamma'} \cap S^{m-1} \subset \Gamma \cap S^{m-1}$ where S^{m-1} denotes the unit sphere in $\mathbb{R}^m$.

Definition V.2.5. *A holomorphic function $f \in \mathcal{O}(V + i\Gamma_\delta)$ is said to be of* tempered growth *if there is an integer k and a constant c such that*

$$|f(x+iy)| \leq \frac{c}{|y|^k}.$$

For $f \in \mathcal{O}(V + i\Gamma_\delta)$, $\varphi \in C_0^\infty(V)$, and $v \in \Gamma$, set

$$\langle f_v, \varphi \rangle = \int f(x+iv)\varphi(x)\mathrm{d}x.$$

THEOREM V.2.6. *Suppose $f \in \mathcal{O}(V + i\Gamma_\delta)$ is of tempered growth and k is as in the definition above. Then*

$$bf = \lim_{v \to 0,\, v \in \Gamma'} f_v$$

exists in $\mathcal{D}'(V)$ and is of order $k+1$.

PROOF. Assume that

$$|f(x+iy)| \le \frac{c}{|y|^k}.$$

We may assume that $\Gamma = \{y = (y_1, \ldots, y_m) : |y| < C_1 y_m\}$ for some $C_1 > 0$. Fix $y^0 \in \Gamma$. Let $\delta_0 = \frac{|y^0|}{2C_1}$. If $y \in \Gamma_{\delta_0}$, we have

(a) $y_m^0 \ge \frac{|y^0|}{C_1} \ge 2|y| \ge 2y_m$ and
(b) $|y_m^0 - y_m| \ge |y_m^0| - |y_m| \ge \frac{|y^0|}{C_1} - |y| > \frac{|y^0|}{2C_1}$.

Fix $\phi \in C_0^\infty(V)$. For $y \in \Gamma_\delta$, let

$$h(y) = \int f(x+iy)\phi(x)\,\mathrm{d}x.$$

Using the growth condition on f and the fact that f is holomorphic, we can integrate by parts and arrive at

$$|D^\alpha h(y)| \le \frac{CC_\alpha}{|y|^k} \quad \text{for all } \alpha \quad \text{where } C_\alpha = \sup|D^\alpha \phi|.$$

Let $|\beta| = k$. We will estimate $D^\beta h(y)$ on Γ_{δ_0}. Assume first that $k \ge 2$. For $y \in \Gamma_{\delta_0}$,

$$\begin{aligned}
|D^\beta h(y) - D^\beta h(y^0)| &= \left| \int_0^1 D(D^\beta h)(ty + (1-t)y^0) \cdot (y - y^0)\,\mathrm{d}t \right| \\
&\le C \left(\sum_{|\alpha|=k+1} C_\alpha \right) \int_0^1 \frac{1}{|ty + (1-t)y^0|^k}\,\mathrm{d}t \\
&\le C \left(\sum_{|\alpha|=k+1} C_\alpha \right) \int_0^1 \frac{1}{(ty_m + (1-t)y_m^0)^k}\,\mathrm{d}t \\
&\le C \left(\sum_{|\alpha|=k+1} C_\alpha \right) \frac{1}{y_m^0 - y_m} \left(\frac{1}{y_m^{k-1}} - \frac{1}{(y_m^0)^{k-1}} \right)
\end{aligned}$$

$$\leq C\left(\sum_{|\alpha|=k+1} C_\alpha\right)\frac{1}{|y|^{k-1}}.$$

We have used (a) and (b) and the fact that since Γ is convex, $ty+(1-t)y^0 \in \Gamma$. Thus there is $C>0$ such that for all β, $|\beta|=k$,

$$|D^\beta h(y)| \leq C\left(\sum_{|\alpha|=k+1} C_\alpha\right)\frac{1}{|y|^{k-1}} \quad \text{whenever } y \in \Gamma_{\delta_0}.$$

Continuing this way, we get δ_{k-2}, $C>0$ such that

$$|D^2 h(y)| \leq C\left(\sum_{|\alpha|=k+1} C_\alpha\right)\frac{1}{|y|} \quad \text{whenever } y \in \Gamma_{\delta_{k-2}}.$$

Note that this inequality also holds when $k=1$. Fix $y^{k-2} \in \Gamma_{\delta_{k-2}}$. Let $\delta_{k-1} = \frac{|y^{k-2}|}{2C_1}$. Using the preceding inequality, for $y \in \Gamma_{\delta_{k-1}}$, we can easily get:

$$|Dh(y)| \leq C\left(\sum_{|\alpha|=k+1} C_\alpha\right)|\log|y|| \quad \text{for some } C>0.$$

Let now y, $y' \in \Gamma_{\delta_{k-1}}$. We have:

$$\begin{aligned}
|h(y)-h(y')| &\leq \left|\int_0^1 Dh(ty+(1-t)y')\cdot(y-y')\,\mathrm{d}t\right| \\
&\leq C\left(\sum_{|\alpha|=k+1} C_\alpha\right)\left|\int_0^1 |\log|ty+(1-t)y'|\,\mathrm{d}t\right||y-y'| \\
&\leq C\left(\sum_{|\alpha|=k+1} C_\alpha\right)\left(\int_0^1 \frac{1}{[ty_m+(1-t)y'_m]^{\frac{1}{2}}}\,\mathrm{d}t\right)|y-y'| \\
&\leq C\left(\sum_{|\alpha|=k+1} C_\alpha\right)\left(\frac{y_m+y'_m}{\sqrt{y_m}+\sqrt{y'_m}}\right) \\
&\leq C\left(\sum_{|\alpha|=k+1} C_\alpha\right)(|y|+|y'|).
\end{aligned}$$

Hence $\lim_{\Gamma\ni y\to 0} h(y)$ exists and as $\Gamma \ni y \to 0$,

$$|h(y)| \leq C \sum_{|\alpha|\leq k+1} ||D^\alpha \phi||_{L^\infty}$$

with C independent of ϕ. □

Remark V.2.7. We note here that when $m=1$, the theorem above says that if a holomorphic function f defined on a rectangle $Q=(-a,a)\times(0,b)$ satisfies the growth condition $|f(x+iy)| \le \frac{c}{y^k}$, then the traces $f(.+iy)$ converge in $\mathcal{D}'(-a,a)$ to a distribution of order $k+1$.

EXAMPLE V.2.8. Consider $f(x,y) = \frac{1}{x+iy}$ which is holomorphic and of tempered growth in the upper half-plane $y>0$. By the theorem, f has a boundary value $bf \in \mathcal{D}'(\mathbb{R})$. It is not hard to show that in fact,

$$bf = \text{pv}\left(\frac{1}{x}\right) - i\pi\delta_0$$

where pv denotes the Cauchy principal value.

Distributions which are boundary values of holomorphic functions of tempered growth arise quite naturally. Indeed, we have:

THEOREM V.2.9. *Any $u \in \mathcal{E}'(\mathbb{R}^m)$ can be expressed as a finite sum $\sum_{j=1}^n bf_j$ where each $f_j \in \mathcal{O}(\mathbb{R}^m + i\Gamma_j')$ for some cones $\Gamma_j' \subseteq \mathbb{R}^m$, and the f_j are of tempered growth.*

PROOF. Let $u \in \mathcal{E}'(\mathbb{R}^m)$. There exist an integer N and a constant $c>0$ such that the Fourier transform $\widehat{u}(\xi)$ satisfies the estimate $|\widehat{u}(\xi)| \le c(1+|\xi|)^N$. Let $\mathcal{C}_j$, $1 \le j \le k$ be open, acute cones such that

$$\mathbb{R}^m = \bigcup_{j=1}^k \overline{\mathcal{C}_j}$$

and $\overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_l}$ has measure zero when $j \ne l$. Define the cones

$$\Gamma_j = \{v \in \mathbb{R}^m : v\cdot\xi > 0 \quad \forall \xi \in \overline{\mathcal{C}_j}\}.$$

For each $j = 1, \ldots, k$, define

$$f_j(x+iy) = \frac{1}{(2\pi)^m} \int_{\mathcal{C}_j} \mathrm{e}^{i(x+iy)\cdot\xi} \widehat{u}(\xi) \mathrm{d}\xi.$$

Note that f_j is holomorphic on $\mathbb{R}^m + i\Gamma_j$. Let Γ_j' be a cone, $\Gamma_j' \subset\subset \Gamma_j$. Then there exists $c>0$ such that $y\cdot\xi \ge c|y||\xi| \quad \forall y \in \Gamma_j', \forall \xi \in \mathcal{C}_j$. Hence for $x+iy \in \mathbb{R}^m + i\Gamma_j'$,

$$\begin{aligned} |f_j(x+iy)| &\le \int \mathrm{e}^{-c|y||\xi|} \quad |\widehat{u}(\xi)|\mathrm{d}\xi \\ &\le c\int_{\mathbb{R}^m} \mathrm{e}^{-c|y||\xi|} \quad (1+|\xi|)^N \mathrm{d}\xi \\ &\le \frac{c_j}{|y|^{m+N}}. \end{aligned}$$

https://doi.org/10.1017/CBO9780511543067.006 Published online by Cambridge University Press

Thus each f_j is of tempered growth on $\mathbb{R}^m + i\Gamma'_j$ and so by Theorem V.2.6 the f_j have boundary values $bf_j \in \mathcal{D}'(\mathbb{R}^m)$. To prove $u = \sum_{j=1}^k bf_j$, let $\varphi \in C_0^\infty(V)$. Then

$$\begin{aligned}
\langle bf_j, \varphi\rangle &= \lim_{y\to 0, y\in\Gamma'_j} \int_{\mathbb{R}^m} f_j(x+iy)\varphi(x)\mathrm{d}x \\
&= \lim_{y\to 0, y\in\Gamma'_j} \int_{\mathbb{R}^m}\int_{\mathcal{C}_j} \mathrm{e}^{i(x+iy)\cdot\xi} \quad \varphi(x)\widehat{u}(\xi)\frac{\mathrm{d}\xi}{(2\pi)^m}\mathrm{d}x \\
&= \lim_{y\to 0, y\in\Gamma'_j} \frac{1}{(2\pi)^m}\int_{\mathcal{C}_j} \mathrm{e}^{-y\cdot\xi} \quad \widehat{u}(\xi)\widehat{\varphi}(-\xi)\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^m}\int_{\mathcal{C}_j} \widehat{u}(\xi)\widehat{\varphi}(-\xi)\mathrm{d}\xi.
\end{aligned}$$

Hence

$$\langle u, \varphi\rangle = \sum_{j=1}^k \langle bf_j, \varphi\rangle. \qquad \square$$

EXAMPLE V.2.10. Let $f_1(x, y) = \frac{1}{x+iy}$ for $y > 0$ and $f_2(x, y) = -\frac{1}{x+iy}$ for $y < 0$. Then it is not hard to show that

$$-2\pi i\delta_0 = bf_1 + bf_2.$$

Granted this, since $u * \delta_0 = u$ for any $u \in \mathcal{E}'(\mathbb{R})$, we get an explicit decomposition of u as a sum of two distributions each of which is the boundary value of a tempered holomorphic function on a half-plane.

DEFINITION V.2.11. *Let $u \in \mathcal{D}'(\mathbb{R}^m)$, $x_0 \in \mathbb{R}^m$, $\xi^0 \in \mathbb{R}^m\setminus\{0\}$. We say that u is* microlocally analytic *at (x_0, ξ^0) if there exist a neighborhood V of x_0, cones $\Gamma^1, \ldots, \Gamma^N$ in $\mathbb{R}^m\setminus\{0\}$, and holomorphic functions $f_j \in \mathcal{O}(V + i\Gamma^j_\delta)$ (for some $\delta > 0$) of tempered growth such that $u = \sum_{j=1}^N bf_j$ near x_0 and $\xi^0 \cdot \Gamma^j < 0 \quad \forall j$.*

REMARK V.2.12. When $m = 1$, if we take $x_0 = 0$ and $\xi^0 = -1$, then u is microlocally analytic at $(0, -1)$ if there is a tempered holomorphic f on some rectangle $(-a, a) \times (0, b)$ such that $u = bf$ on $(-a, a)$.

DEFINITION V.2.13. *The* analytic wave front set *of a distribution u, denoted $WF_a(u)$, is defined by*

$$WF_a(u) = \{(x, \xi) : u \textit{ is not microlocally analytic at } (x, \xi)\}.$$

Observe that from Definition V.2.13 it can easily be shown that the analytic wave front set is invariant under an analytic diffeomorphism, and hence microlocal analyticity can be defined on any real-analytic manifold. The following theorem provides a very useful criterion for microlocal analyticity in terms of the FBI transform:

THEOREM V.2.14. *Let* $u \in \mathcal{D}'(\mathbb{R}^m)$, $x_0 \in \mathbb{R}^m$, $\xi^0 \in \mathbb{R}^m \setminus \{0\}$. *Then* $(x_0, \xi^0) \notin WF_a(u)$ *if and only if there is a neighborhood* V *of* x_0 *in* $\mathbb{R}^m$, *an open cone* $\Gamma \subset \mathbb{R}^m \setminus 0$, $\xi^0 \in \Gamma$ *and constants* $c_1, c_2 > 0$ *such that*

$$|F_u(x, \xi)| \le c_1 e^{-c_2|\xi|} \quad \forall (x, \xi) \in V \times \Gamma.$$

The proof uses the inversion formula of Theorem V.2.2 and ideas similar to those in the proof of Theorem V.2.4 (see also Theorem V.3.7). The reader is referred to [**Sj1**] for the proof of this theorem.

COROLLARY V.2.15. *A distribution* u *is analytic near* x_0 *if and only if for every* $\xi^0 \in \mathbb{R}^m \setminus \{0\}$, $(x_0, \xi^0) \notin WF_a(u)$.

COROLLARY V.2.16. (The edge-of-the-wedge theorem.) *Let* $V \subset \mathbb{R}^m$ *be a neighborhood of the point* p, *and* Γ^+, Γ^- *be cones such that* $\Gamma^- = -\Gamma^+$. *Suppose for some* $\delta > 0$, $f^+ \in \mathcal{O}(V + i\Gamma^+_\delta)$, $f^- \in \mathcal{O}(V + i\Gamma^-_\delta)$ *are both of tempered growth and* $bf^+ = bf^-$. *Then there exists a holomorphic function* f *defined in a neighborhood of* p *that extends both* f^+ *and* f^-. *In particular,* bf^+ *is analytic at* p.

EXAMPLE V.2.17. Let

$$u(x) = \begin{cases} |x|^{\frac{3}{2}}, & x \ge 0 \\ i|x|^{\frac{3}{2}}, & x \le 0. \end{cases}$$

Then $u(x) = bf(x)$ where $f(x, y) = (x + iy)^{\frac{3}{2}}$ for $y > 0$ and we take the principal branch of the fractional power. Since f is holomorphic for $y > 0$, it follows that $(0, -1) \notin WF_a(u)$. On the other hand, since u is not analytic (it is not even C^2), by Corollary V.2.15, $(0, 1) \in WF_a(u)$.

EXAMPLE V.2.18. Let (x, t) denote the variables in $\mathbb{R}^{m+n}$, $x \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$. Let $\phi(t) = (\phi_1(t), \dots, \phi_m(t))$ be real-analytic functions near the origin and consider the associated tube structure generated by

$$L_k = \frac{\partial}{\partial t_k} - i \sum_{j=1}^{m} \frac{\partial \phi_j}{\partial t_k} \frac{\partial}{\partial x_j}, \quad k = 1, \dots, n.$$

It was shown in [**BT5**] that this system is analytic hypoelliptic at 0, i.e., every solution u of $L_k u = 0$, $k = 1, \dots, n$ is analytic at 0 if and only if, for every $\xi \in \mathbb{R}^m$, the function

$$t \mapsto \phi(t) \cdot \xi$$

does not have a local minimum at 0. This result was proved using the FBI transform. The authors also proved a microlocal version of this result.

When a distribution u is a solution of a partial differential equation with analytic coefficients, the analyticity or microlocal analyticity of the solution can sometimes be established by using the FBI transform. Sections V.4 and V.5 contain results in this direction. The notes at the end of this chapter contain several references to such applications of the FBI transform.

V.3 Microlocal smoothness

In this section we introduce the concept of the C^∞ wave front set which is a refined way of describing the singularities of distributions. It is well known that a distribution u of compact support is C^∞ if and only if its Fourier transform $\hat{u}(\xi)$ decays rapidly as $|\xi| \to \infty$. More precisely, we recall Paley–Wiener's Theorem:

THEOREM V.3.1. (Theorem 7.3.1 in [**H2**].) *A distribution u with support in the ball $\{x \in \mathbb{R}^m : |x| \leq R\}$ is C^∞ if and only if $\hat{u}(\zeta)$ is entire on $\mathbb{C}^m$ and for each positive integer k there is C_k such that*

$$|\hat{u}(\zeta)| \leq C_k \frac{e^{R|\Im\zeta|}}{(1+|\zeta|)^k} \quad \forall \zeta \in \mathbb{C}^m.$$

DEFINITION V.3.2. *Let $u \in \mathcal{D}'(\Omega)$, $\Omega \subseteq \mathbb{R}^m$ open, $x_0 \in \Omega$, and $\xi^0 \in \mathbb{R}^m \backslash \{0\}$. We say u is microlocally smooth at (x_0, ξ_0) if there exists $\phi \in C_0^\infty(\Omega)$, $\phi \equiv 1$ near x_0 and a conic neighborhood $\Gamma \subseteq \mathbb{R}^m \backslash \{0\}$ of ξ^0 such that for all $k = 1, 2, \ldots$ and for all $\xi \in \Gamma$,*

$$|\widehat{\phi u}(\xi)| \leq \frac{C_k}{(1+|\xi|)^k} \text{ on } \Gamma.$$

DEFINITION V.3.3. *The C^∞ wave front set of a distribution u denoted $WF(u)$ is defined by*

$$WF(u) = \{(x, \xi) : u \text{ is not microlocally smooth at } (x, \xi)\}.$$

It is easy to see that a distribution u is C^∞ if and only if $WF(u) = \emptyset$. When a distribution u is a solution of a linear partial differential equation with smooth coefficients, its wave front set is constrained. We quote here a basic result along this line:

THEOREM V.3.4. (Theorem 8.3.1 in [**H2**].) *Let $P = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha$ be a smooth linear partial differential operator on an open set $\Omega \subset \mathbb{R}^m$ and suppose $u \in \mathcal{D}'(\Omega)$. Then*

$$WF(u) \subset \text{char}\, P \cup WF(Pu)$$

where the characteristic set

$$\operatorname{char} P = \left\{ (x,\xi) \in \Omega \times \mathbb{R}^m \backslash \{0\} : \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha = 0 \right\}.$$

In particular, if Pu is smooth, then $WF(u) \subset \operatorname{char} P$. If Pu is smooth, and P is elliptic, then u has to be smooth. In Section V.5 we will consider an analogous result for solutions of first-order nonlinear partial differential equations.

Definition V.3.5. *Let $f \in C^\infty(\Omega)$, $\Omega \subseteq \mathbb{R}^m$ open, and suppose $\widetilde{\Omega}$ is a neighborhood of Ω in $\mathbb{C}^m$. A function $\tilde{f}(x,y) \in C^\infty(\widetilde{\Omega})$ is called an almost analytic extension of $f(x)$ if $\tilde{f}(x,0) = f(x)\ \forall x \in \Omega$ and for each $j = 1, \ldots, m$,*

$$\frac{\partial \tilde{f}}{\partial \bar{z}_j}(x,y) = O(|y|^k) \text{ for } k = 1, 2, \ldots$$

Remark V.3.6. Lemma V.5.1 in Section V.5 shows that each smooth function of one real variable has an almost analytic extension. Such extensions also exist in higher dimensions (see [**GG**]).

The following theorem characterizes microlocal smoothness in terms of almost analytic extendability in certain wedges.

Theorem V.3.7. *Let $u \in \mathcal{D}'(\mathbb{R}^m)$. Then $(x_0,\xi^0) \notin WF(u)$ if and only if there exist a neighborhood V of x_0, open acute cones $\Gamma^1, \ldots, \Gamma^N$ in $\mathbb{R}^m \backslash \{0\}$, and almost analytic functions f_j on $V + i\Gamma^j_\delta$ (for some $\delta > 0$) of tempered growth such that $u = \sum_j^N bf_j$ near x_0 and $\xi^0 \cdot \Gamma^j < 0$ for all j.*

Proof. Suppose $(x_0,\xi^0) \notin WF(u)$. Let $\phi \in C_0^\infty(\mathbb{R}^m)$, $\phi \equiv 1$ near x_0 such that $\widehat{\phi u}(\xi)$ decays rapidly in a conic neighborhood of ξ^0. By the Fourier inversion formula,

$$\phi u = \frac{1}{(2\pi)^m} \int e^{ix\cdot\xi} \widehat{\phi u}(\xi)\, d\xi$$

where the formula is understood in the duality sense, that is, for $\psi \in C_0^\infty(\mathbb{R}^m)$,

$$\langle \phi u, \psi \rangle = \frac{1}{(2\pi)^m} \int \left(\int e^{ix\cdot\xi} \psi(x)\, dx \right) \widehat{\phi u}(\xi)\, d\xi.$$

Let $\mathcal{C}_j$, $1 \le j \le N$ be open, acute cones such that

$$\mathbb{R}^m = \bigcup_{j=1}^N \overline{\mathcal{C}}_j$$

and $\overline{\mathcal{C}}_j \cap \overline{\mathcal{C}}_k$ has measure zero when $j \neq k$. We may assume that $\xi^0 \in \mathcal{C}_1$ and $\xi^0 \notin \overline{\mathcal{C}}_j$ for $j \ge 2$. This implies that we can get acute, open cones Γ^j, $2 \le j \le N$

and a constant $c > 0$ such that

$$\xi^0 \cdot \Gamma^j < 0 \quad \text{and} \quad y \cdot \xi \geq c|y||\xi| \; \forall y \in \Gamma^j, \; \forall \xi \in \mathcal{C}_j.$$

For each $j = 2, \ldots, N$, define

$$f_j(x+iy) = \frac{1}{(2\pi)^m} \int_{\mathcal{C}_j} e^{i(x+iy)\cdot\xi} \widehat{\phi u}(\xi)\, d\xi$$

and set

$$g_1(x) = \frac{1}{(2\pi)^m} \int_{\mathcal{C}_1} e^{ix\cdot\xi} \widehat{\phi u}(\xi)\, d\xi.$$

For $j \geq 2$, f_j is holomorphic on $\mathbb{R}^m + i\Gamma^j$ and as we saw in the proof of Theorem V.2.9, it is of tempered growth and hence has a boundary value $bf_j \in \mathcal{D}'(\mathbb{R}^m)$. Since $(x_0, \xi^0) \notin WF(u)$, we may assume that $\widehat{\phi u(\xi)}$ decays rapidly in the cone $\mathcal{C}_1$. It is then easy to see that g_1 is C^∞ on $\mathbb{R}^m$. By Remark V.3.6, the function g_1 has an almost analytic extension f_1 which is smooth on $\mathbb{C}^m$. It follows that $u = \sum_j^N bf_j$ near x_0 with the f_j's as asserted. For the converse, we may assume that on some neighborhood V of x_0, $u = bf$ where f is almost analytic and of tempered growth on $V + i\Gamma$, Γ is an open cone, and $\xi^0 \cdot \Gamma < 0$. Let $\phi \in C_0^\infty(V)$, $\phi \equiv 1$ near x_0. We have

$$\widehat{\phi u}(\xi) = \langle u, \phi(x)e^{-ix\cdot\xi} \rangle = \lim_{\Gamma \ni y \to 0} \int_{\mathbb{R}^m} f(x+iy)e^{-ix\cdot\xi}\phi(x)\, dx.$$

Let $\Phi(x, y)$ be an almost analytic extension of $\phi(x)$. Fix $y_0 \in \Gamma$ and let

$$D = \{x + ity_0 \in \mathbb{C}^m : x \in V, \; 0 \leq t \leq 1\}.$$

Consider the m-form

$$f(x+iy)e^{-i(x+iy)\cdot\xi}\Phi(x, y)\, dz_1 \wedge \cdots \wedge dz_m$$

where each $z_j = x_j + iy_j$, $1 \leq j \leq m$. By Stokes' theorem,

$$\widehat{\phi u}(\xi) - \int_V f(x+iy_0)e^{-i(x+iy_0)\cdot\xi}\Phi(x, y_0)\, dx = \sum_{j=1}^m \int_D \frac{\partial}{\partial \bar{z}_j}(f\Phi)e^{-i(x+iy)\cdot\xi} \, d\bar{z}_j \wedge dz_1 \wedge \cdots \wedge dz_m.$$

After contracting Γ if necessary, we may assume that for some $c > 0$, $y_0 \cdot \xi \leq -c|\xi|$ for all $\xi \in \Gamma$. This latter inequality, together with the almost analyticity of f and Φ, and the tempered growth of f, imply that on D, for any integer $k \geq 0$, we can find a constant C_k such that

$$\left| \frac{\partial}{\partial \bar{z}_j}(f\Phi)(x+ity_0) \right| \left| e^{-i(x+ity_0)\cdot\xi} \right| \leq C'_k |ty_0|^k |e^{ty_0\cdot\xi}| \leq \frac{C_k}{|\xi|^k}.$$

Observe also that the inequality $y_0 \cdot \xi \le -c|\xi|$ $(\xi \in \Gamma)$ implies that the integral

$$\int_V f(x+iy_0)\mathrm{e}^{-i(x+iy_0)\cdot\xi}\Phi(x, y_0)\,\mathrm{d}x$$

decays rapidly in Γ. It follows that $(x_0, \xi^0) \notin WF(u)$. □

Corollary V.3.8. *Let $u \in \mathcal{D}'(\mathbb{R}^m)$. If $(x_0, \xi^0) \in WF(u)$, then $(x_0, \xi^0) \in WF_a(u)$.*

V.4 Microlocal hypoanalyticity and the FBI transform

A *hypoanalytic structure* (or manifold) is an involutive structure $(\mathcal{M}, \mathcal{V})$ with charts (U_α, Z^α) where the U_α form an open covering of $\mathcal{M}$, and the $Z^\alpha = (Z_1^\alpha, \ldots, Z_m^\alpha)$ are a complete set of first integrals on U_α that are determined on the overlaps up to a local biholomorphism of $\mathbb{C}^m$. A basic example is a generic CR submanifold $\mathcal{M}$ of $\mathbb{C}^m$. A function f on a hypoanalytic manifold is said to be *hypoanalytic* if in a neighborhood of each point p it is of the form $f = h(Z_1, \ldots, Z_m)$ for some holomorphic function h defined in a neighborhood of $(Z_1(p), \ldots, Z_m(p))$ in $\mathbb{C}^m$. In the case of generic CR submanifolds of $\mathbb{C}^m$, the hypoanalytic functions are the restrictions to $\mathcal{M}$ of holomorphic functions defined in a neighborhood of $\mathcal{M}$. Hypoanalytic structures will be discussed some more in the epilogue. For more details on hypoanalytic structures, the reader is referred to [**BCT**] and [**T5**]. In this section we will briefly discuss the notion of the hypoanalytic wave front set. This notion is a generalization of the concept of microlocal analyticity we discussed in Section V.2 and the reader is referred to the work [**BCT**] for more details. We begin with the concept of a wedge in $\mathbb{C}^N$ whose edge is a generic CR manifold. Let $\mathcal{M}$ be a generic CR manifold in $\mathbb{C}^N$ of codimension d. Then $\dim \mathcal{M} = 2n+d$, $m = n+d = N$ and the bundle $T' = T'\mathcal{M}$ is generated by the differentials of the restrictions to $\mathcal{M}$ of the N complex coordinates in $\mathbb{C}^N$. Fix $p \in \mathcal{M}$ and let $h = (h_1, \ldots, h_d)$ be smooth defining functions of $\mathcal{M}$ in a neighborhood U of p in $\mathbb{C}^N$.

Definition V.4.1. *For Γ an open convex cone with vertex at $0 \in \mathbb{R}^d$, the set*

$$\mathcal{W}(U, h, \Gamma) = \{z \in U : h(z) \in \Gamma\}$$

is called a wedge with edge *$\mathcal{M}$. The wedge is said to be centered at p and to point in the direction of Γ.*

Observe that $\mathcal{W}(U, h, \Gamma)$ is an open set in $\mathbb{C}^N$ and $\mathcal{M} \cap U$ lies in its boundary. When $\mathcal{M}$ is a hypersurface, $\Gamma = (0, \infty)$ or $(-\infty, 0)$ and so a wedge with edge $\mathcal{M}$ in this case is simply a side of $\mathcal{M}$. Although the definition of a wedge involves the defining functions, the following proposition shows some independence from the defining functions.

Proposition V.4.2. (Proposition 7.1.2 in [**BER**].) *Assume that $h = (h_1, \ldots, h_d)$ and $g = (g_1, \ldots, g_d)$ are two defining functions for $\mathcal{M}$ near p. Then there is a $d \times d$ real invertible matrix B such that for every U and Γ as above, the following holds: for any open convex cone $\Gamma_1 \subseteq \mathbb{R}^d$ with $B\Gamma_1 \cap S^{d-1}$ relatively compact in $\Gamma \cap S^{d-1}$ (S^{d-1} denotes the unit sphere in $\mathbb{R}^d$), there exists a neighborhood U_1 of p in $\mathbb{C}^N$ such that*

$$\mathcal{W}(U_1, g, \Gamma_1) \subseteq \mathcal{W}(U, h, \Gamma).$$

The reader is referred to [**BER**] for the proof of this proposition. We mention that if $a(z)$ is a $d \times d$ smooth invertible matrix satisfying $g = ah$ near p, then the matrix $B = [a(p)]^{-1}$.

Definition V.4.3. *A holomorphic function f defined on a wedge $\mathcal{W} = \mathcal{W}(U, h, \Gamma)$ is said to be of* tempered growth *if there exists a constant $c > 0$ and an integer k such that*

$$|f(z)| \leq \frac{c}{|h(z)|^k} \qquad \forall z \in \mathcal{W}. \tag{V.4}$$

By using a diffeomorphism that flattens $\mathcal{M}$ near p, it is easy to see that the growth condition (V.4) is equivalent to

$$|f(z)| \leq \frac{c'}{\operatorname{dist}(z, \mathcal{M})^k} \qquad \forall z \in \mathcal{W}.$$

Recall from Chapter I that for the generic $\mathcal{M}$ we can find complex coordinates $(z_1, \ldots, z_n, w_1, \ldots, w_d)$ vanishing at $p \in \mathcal{M}$, $z = x + iy \in \mathbb{C}^n$, $w = s + it \in \mathbb{C}^d$, and smooth real-valued functions $\phi_1, \ldots, \phi_d$ defined near $(0, 0)$ in (z, s) space with $\phi_k(0) = 0$, $\mathrm{d}\phi_k(0) = 0$, $1 \leq k \leq d$ such that near 0, $\mathcal{M}$ is given by

$$\rho_k(z, w) = \phi_k(z, s) - t_k = 0, \quad 1 \leq k \leq d.$$

That is, near 0, $\mathcal{M} = \{(z, s + i\phi(z, s))\}$. By Proposition V.4.2, there exist $\epsilon > 0$ and a convex open cone $\Gamma' \subseteq \mathbb{R}^d$ such that if

$$\mathcal{W}' = \{(z, s + i\phi(z, s) + iv) : |z| < \epsilon, |s| < \epsilon, |v| < \epsilon, v \in \Gamma'\}$$

then $\mathcal{W}' \subseteq \mathcal{W}(U, h, \Gamma)$. The description of $\mathcal{W}'$ makes it clear what a wedge with edge $\mathcal{M}$ means. Observe also that a holomorphic function $f(z, w)$ on $\mathcal{W}'$ is of tempered growth if and only if it satisfies an estimate of the form

$$|f(z, s + i\phi(z, s) + iv)| \leq \frac{c}{|v|^k}$$

for $v \in \Gamma'$ small and $(z, s) \in \mathbb{C}^n \times \mathbb{R}^d$ near $(0, 0)$. Holomorphic functions of tempered growth in a wedge have distributional boundary values on the edge of the wedge. We have:

THEOREM V.4.4. (Theorem 7.2.6 in [**BER**].) *Let $f(z, w)$ be a holomorphic function of tempered growth in a wedge $\mathcal{W}'$ as above. Then there exists a CR distribution $u = bf$ defined in a neighborhood of* 0 *in $\mathcal{M}$ by*

$$\langle u, \psi \rangle = \lim_{\Gamma \ni v \longrightarrow 0} \int_{\mathbb{R}^{2n+d}} f(z, s + i\phi(z, s) + iv)\psi(x, y, s)\mathrm{d}x\mathrm{d}y\mathrm{d}s$$

for any smooth function ψ of sufficiently small compact support near the origin in $\mathbb{R}^{2n+d}$.

PROOF. The proof will use arguments similar to those used in the proof of Theorem V.2.6. For $\psi(x, y, s)$ smooth, supported near the origin, set

$$h(v) = \int_{\mathbb{R}^{2n+d}} f(z, s + i\phi(z, s) + iv)\psi(x, y, s)\mathrm{d}x\mathrm{d}y\mathrm{d}s$$

for $v \in \Gamma'$, $|v| < \epsilon$. We will estimate the derivatives of h. We have

$$\frac{\partial h}{\partial v_j}(v) = \int_{\mathbb{R}^{2n+d}} i\frac{\partial f}{\partial w_j}(z, s + i\phi(z, s) + iv)\psi(x, y, s)\mathrm{d}x\mathrm{d}y\mathrm{d}s$$

for each $j = 1, \ldots, d$. Observe that since

$$\frac{\mathrm{d}}{\mathrm{d}s_m} f(z, s + i\phi(z, s) + iv) = \sum_{k=1}^{d} \frac{\partial f}{\partial w_k}(z, s + i\phi(z, s) + iv)\left(\delta_{km} + i\frac{\partial \phi_k}{\partial s_m}\right),$$

and the matrix $I + i\phi_s$ is invertible near the origin, there are smooth functions $a_{jm}(z, s)$ such that for each $k = 1, \ldots, d$,

$$\frac{\partial f}{\partial w_k}(z, s + i\phi(z, s) + iv) = \sum_{m=1}^{d} a_{km}(z, s)\frac{\mathrm{d}}{\mathrm{d}s_m} f(z, s + i\phi(z, s) + iv).$$

It follows that

$$\frac{\partial h}{\partial v_j}(v) = \sum_{m=1}^{d} \int_{\mathbb{R}^{2n+d}} \frac{\mathrm{d}}{\mathrm{d}s_m} f(z, s + i\phi(z, s) + iv)a_{jm}(z, s)\psi(x, y, s)\mathrm{d}x\mathrm{d}y\mathrm{d}s.$$

We can thus integrate by parts and iterate the procedure to conclude that for some constant $C > 0$ and every multi-index α,

$$|D^\alpha h(v)| \le \frac{CC_\alpha}{|v|^k}$$

where $C_\alpha = \sup |D^\alpha \psi|$. It then follows, as in the proof of Theorem V.2.6, that $h(v)$ has a limit as $\Gamma' \ni v \to 0$. Set $\langle u, \psi \rangle = \lim_{v\to 0} h(v)$. Note that u

is CR since it is the distributional limit of the CR functions $\mathcal{M} \ni (z,s) \longmapsto f(z, s+i\phi(z,s)+iv)$. □

The reader is referred to [**BER**] for an invariant formulation of Theorem V.4.4 (corollary 7.2.9 in [**BER**]).

Suppose now X is a hypoanalytic structure of codimension 0. Such an X often arises as a maximally real submanifold in a hypoanalytic structure. The structure bundle of X is all of $\mathbb{C}T^*X$ and since $\mathcal{V}$ is empty, any distribution is a solution. Fix $p \in X$ and let $Z = (Z_1, \ldots, Z_m)$ be a hypoanalytic chart near p. In a neighborhood V of p in X, the map $Z: V \longrightarrow \mathbb{C}^m$ is a diffeomorphism onto $Z(V)$. $Z(V)$ is a generic submanifold of $\mathbb{C}^m$ which is totally real of maximal dimension. In what follows, we will identify V with $Z(V)$.

Definition V.4.5. *A distribution $u \in \mathcal{D}'(X)$ is* microlocally hypoanalytic *at $\sigma \in T^*_pX\backslash\{0\}$ if there exist open convex cones $\mathcal{C}_1, \ldots, \mathcal{C}_k$ in T_pX satisfying $\sigma(v) < 0 \quad \forall v \in \mathcal{C}_j$, $(1 \le j \le k)$ and wedges $\mathcal{W}_1, \ldots, \mathcal{W}_k$ in $\mathbb{C}^m$ with edge $Z(V)$ centered at p and pointing in the directions of $\Gamma_1, \ldots, \Gamma_k$ respectively such that $J\mathcal{C}_j \subseteq \Gamma_j$ and for each j, there is a holomorphic function of tempered growth u_j on $\mathcal{W}_j$ such that $u = \sum_{j=1}^k bu_j$ in V.*

Definition V.4.6. *The* hypoanalytic wave front *set of u, denoted $WF_{ha}u$ is defined by*

$$WF_{ha}u = \{\sigma \in T^*X\backslash\{0\} : u \text{ is not microlocally hypoanalytic at } \sigma\}.$$

The hypoanalytic wave front set for solutions in structures of positive codimension is defined by restriction to a maximally real submanifold as follows ([**BCT**]). Let $(\mathcal{M}, \mathcal{V})$ be a hypoanalytic structure and u a distribution solution near $p \in \mathcal{M}$. Select a maximally real submanifold $\mathcal{X}$ through p. We recall that the restriction $u|_{\mathcal{X}}$ is well-defined and by Proposition V.1.29 $\mathcal{X}$ inherits a hypoanalytic structure of codimension 0. Hence the hypoanalytic wave front set $WF_{ha}(u|_{\mathcal{X}})$ is defined and lives in $T^*\mathcal{X}\backslash\{0\}$. Since $\mathcal{X}$ is maximally real, by Propositions V.1.27 and V.1.29, the inclusion $i_{\mathcal{X}} : \mathcal{X} \to \mathcal{M}$ induces an injection $i^*_{\mathcal{X}} : T^0|_{\mathcal{X}} \to T^*\mathcal{M}$. We will say a covector $\sigma \in T^0_p\backslash\{0\}$ is in the hypoanalytic wave front set of u if $i^*_{\mathcal{X}}\sigma \in WF_{ha}(u|_{\mathcal{X}})$. This set will be denoted by $WF^{\mathcal{M}}_{ha,p}(u)$. This definition is independent of the choice of the maximally real submanifold $\mathcal{X}$ through p (see [**BCT**] for the proof) and thus for any such $\mathcal{X}$, we have a bijection $i^*_{\mathcal{X}} : WF^{\mathcal{M}}_{ha,p}(u) \to WF_{ha,p}(u|_{\mathcal{X}})$, where $WF_{ha,p}$ denotes the hypoanalytic wave front set at p.

We will next recall the FBI transform of [**BCT**] which gives a very useful Fourier transform criterion for microlocal hypoanalyticity. X is a hypoanalytic structure of codimension 0 as above. If $p \in X$, by the results in Chapter I (see

https://doi.org/10.1017/CBO9780511543067.006 Published online by Cambridge University Press

for example Corollary I.10.2), we may choose local coordinates $x_1, \dots, x_m$ for X vanishing at p so that locally, X becomes a neighborhood U of 0 in $\mathbb{R}^m$ and we may assume that a hypoanalytic chart has the form

$$Z_j = x_j + i\phi_j(x), \qquad 1 \le j \le m,$$

$\phi = (\phi_1, \dots, \phi_m)$ real-valued. For $\kappa > 0$ and $u \in \mathcal{E}'(U)$, define

$$F^\kappa(u, z, \zeta) = \int_U \mathrm{e}^{i\zeta\cdot(z-Z(y))-\kappa\langle\zeta\rangle[z-Z(y)]^2} u(y)\mathrm{d}Z(y)$$

where $z \in \mathbb{C}^m$, $[w]^2 = w_1^2 + \cdots + w_m^2$, and for any $\zeta \in \mathbb{C}^m$ with $|\Im\zeta| < |\Re\zeta|$, $\langle\zeta\rangle = (\zeta_1^2 + \cdots + \zeta_m^2)^{\frac{1}{2}}$ (the principal branch of the square root).

Definition V.4.7. *$F^\kappa(u, z, \zeta)$ is called the FBI transform of u (with parameter κ).*

In [**BCT**] the authors characterized microlocal hypoanalyticity in terms of an exponential decay of the FBI transform. In particular, when $\phi(0) = 0$ and $\mathrm{d}\phi(0) = 0$, they proved:

Theorem V.4.8. *There is a universal constant $M > 0$ such that if $\kappa > M \sup_{x\in U,\ |\alpha|=2} \partial^\alpha\phi(x)$, the following holds: for $\sigma \in \mathbb{R}^m\backslash\{0\}$, $u \in \mathcal{E}'(U)$, V a neighborhood of 0 in $\mathbb{C}^m$, $\Gamma \subseteq \mathbb{C}^m\backslash\{0\}$ a complex conic neighborhood of σ, if*

$$|F^\kappa(u, z, \zeta)| \le c_1\mathrm{e}^{-c_2|\zeta|}, \quad \forall z \in V, \quad \forall\zeta \in \Gamma$$

and for some $c_1, c_2 > 0$, then $(0, \sigma) \notin WF_{ha}u$.

Here U is a neighborhood of 0 in $\mathbb{R}^m$.

V.5 Application of the FBI transform to the C^∞ wave front set of solutions of nonlinear PDEs

In this section the FBI transform will be used to prove a result on the C^∞ wave front set of solutions of first-order nonlinear PDEs. Suppose $u = u(x, t)$ is a C^2 solution of a nonlinear pde

$$u_t = f(x, t, u, u_x)$$

where $f(x, t, \zeta_0, \zeta)$ is complex-valued, C^∞ in all the variables, and holomorphic in (ζ_0, ζ). Here x varies in an open set in $\mathbb{R}^m$, t in an interval of $\mathbb{R}$, and (ζ_0, ζ) in an open set in $\mathbb{C}^{m+1}$. We will present Asano's ([**A**]) proof of

Chemin's ([**Che**]) result that the C^∞ wave front set of any C^2 solution is contained in the characteristic set of the linearized operator

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x, t, u, u_x)\frac{\partial}{\partial x_j}.$$

We begin with some lemmas about linear vector fields:

Lemma V.5.1. *Let*

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{N} a_j(x, t, \zeta)\frac{\partial}{\partial x_j} + \sum_{k=1}^{M} b_k(x, t, \zeta)\frac{\partial}{\partial \zeta_k}$$

where the coefficients a_j and b_k are C^∞ in the variables $(x, t) \in \Omega \times J \subset \mathbb{R}^N \times \mathbb{R}$ and holomorphic in the variable $\zeta \in \mathcal{N} \subset \mathbb{C}^M$, $\mathcal{N}$ open. Let $f(x, \zeta)$ be a C^∞ function defined on $\Omega \times \mathcal{N}$, holomorphic in ζ. There exists a C^∞ function $u(x, t, \zeta)$ holomorphic in ζ which is an approximate solution of $Lu = 0$ in the sense that

$$Lu(x, t, \zeta) = O(t^k) \qquad \textit{for} \quad k = 1, 2, \ldots$$

and such that $u(x, 0, \zeta) = f(x, \zeta)$.

Proof. The conditions that u has to satisfy determine the Taylor coefficients of the formal series

$$u(x, t, \zeta) = \sum_{j=0}^{\infty} u_j(x, \zeta)t^j$$

where $u_j(x, \zeta) = \frac{\partial_t^j u(x,0,\zeta)}{j!}$. Set $u_0(x, \zeta) = f(x, \zeta)$. For each j, since we want $Lu = O(t^{j+1})$, we must have $\partial_t^{j-1}(Lu)(x, 0, \zeta) = 0$. This then leads to

$$u_j(x, \zeta) = -\frac{1}{j} \sum_{p+q=j-1} \frac{1}{q!}\Bigg[\sum_{k=1}^{N} \frac{\partial u_p}{\partial x_k}(x, \zeta)\frac{\partial^q a_k}{\partial t^q}(x, 0, \zeta) + \sum_{k=1}^{M} \frac{\partial u_p}{\partial \zeta_k}(x, \zeta)\frac{\partial^q b_k}{\partial t^q}(x, 0, \zeta)\Bigg]$$

for $j \geq 1$. Note that the functions $u_j(x, \zeta)$ are C^∞ and holomorphic in ζ. Let $\chi \in C_0^\infty(\mathbb{R})$ be such that $\chi \geq 0$, $\chi \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and supp $\chi \subset [-1, 1]$. Then there exists a sequence $R_j > 1$, $R_j \nearrow +\infty$ such that the series

$$u(x, t, \zeta) = \sum_{j=1}^{\infty} \chi(R_j t)u_j(x, \zeta)t^j$$

is convergent in C^∞. It follows that u is C^∞ in all the variables and holomorphic in ζ. Moreover, from the way the functions u_j are defined, u is an approximate solution of $Lu = 0$ with the property that $u(x, 0, \zeta) = f(x, \zeta)$. $\square$

In the following lemma, WF denotes the C^∞ wave front set.

LEMMA V.5.2. *Let $X \subset \mathbb{R}^m$ be open, U an open neighborhood of $X \times \{0\}$ in $\mathbb{R}^{m+1}$, $U_+ = U \cap \mathbb{R}^{m+1}_+$. Let*

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} a_j(x, t) \frac{\partial}{\partial x_j}$$

be a C^l vector field in U for some positive integer l. Assume $f \in C^1(\overline{U_+})$ satisfies

$$|Lf(x, t)| = O(t^k), \quad k = 1, 2, \dots$$

uniformly on compact subsets of X. Suppose there exist C^l functions

$$\Psi_1(x, t), \dots, \Psi_m(x, t)$$

on U such that $Z(x, t) = x + t\Psi(x, t)$ satisfies $Z(x, 0) = x$ and

$$LZ(x, t) = O(t^k), \quad k = 1, 2, \dots$$

Let $a(x, t) = (a_1(x, t), \dots, a_m(x, t))$. Assume

$$\partial_t^j a(x, 0) = 0 \quad \forall j < l, \quad \forall x \in X$$

and that

$$\langle \partial_t^l \Im a(x_0, 0), \xi^0 \rangle > 0 \quad \textit{for some } x_0 \in X, \quad \xi^0 \in \mathbb{R}^m.$$

Then $(x_0, \xi^0) \notin WF(f(x, 0))$.

REMARK V.5.3. If L is C^∞, then Lemma V.5.1 insures that the Z_j exist and the proof below will show that in this case, we only need to assume that $f \in C([0, T], \mathcal{D}'(\mathbb{R}^m))$.

PROOF. Without loss of generality, we may assume that $x_0 = 0$. For $j = 1, \dots, m$ let $M_j = \sum_{k=1}^{m} b_{jk}(x, t) \frac{\partial}{\partial x_k}$ be vector fields satisfying

$$M_j Z_k = \delta_j^k, \qquad [M_j, M_k] = 0.$$

Note that for each j,

$$[M_j, L] = \sum_{s=1}^{m} c_{js} M_s \tag{V.5}$$

where each $c_{js} = O(t^k)$, $k = 1, 2, \ldots$ Indeed, the latter can be seen by expressing $[M_j, L]$ in terms of the basis $\{L, M_1, \ldots, M_m\}$ and applying both sides to the $m+1$ functions $\{t, Z_1, \ldots, Z_m\}$. For any C^1 function g, observe that the differential

$$\mathrm{d}g = \sum_{k=1}^{m} M_k(g)\mathrm{d}Z_k + \left(Lg - \sum_{k=1}^{m} M_k(g) LZ_k \right) \mathrm{d}t. \tag{V.6}$$

This is verified by evaluating each side at the basis vector fields

$$\{L, M_1, \ldots, M_m\}.$$

Using (V.6) we get:

$$\mathrm{d}(g\mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_m) = \left(Lg - \sum_{k=1}^{m} M_k(g) LZ_k \right) \mathrm{d}t \wedge \mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_m. \tag{V.7}$$

For $\xi \in \mathbb{R}^m$, $s \in \mathbb{R}^m$, let

$$E(s, \xi, x, t) = i\xi \cdot (s - Z(x, t)) - |\xi|[s - Z(x, t)]^2,$$

where for $w \in \mathbb{C}^m$, we write $[w]^2 = \sum_{j=1}^{m} w_j^2$. Let B denote a small ball centered at 0 in $\mathbb{R}^m$ and $\phi \in C_0^\infty(B)$, $\phi \equiv 1$ near the origin. We will apply (V.7) to the function

$$g(s, \xi, x, t) = \phi(x) f(x, t) \mathrm{e}^{E(s,\xi,x,t)}$$

where (s, ξ) are parameters. We get:

$$\mathrm{d}(g\mathrm{d}Z) = \left\{ L(\phi f) + (\phi f)LE - \sum_{k=1}^{m} (M_k(\phi f) + \phi f(M_k E))LZ_k \right\} \mathrm{e}^E \mathrm{d}t \wedge \mathrm{d}Z, \tag{V.8}$$

where $\mathrm{d}Z = \mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_m$. Next by Stokes' theorem we have, for $t_1 > 0$ small:

$$\int_B g(s, \xi, x, 0)\mathrm{d}x = \int_B g(s, \xi, x, t_1)\mathrm{d}_x Z(x, t_1) + \int_0^{t_1} \int_B \mathrm{d}(g\mathrm{d}Z). \tag{V.9}$$

We will estimate the two integrals on the right in (V.9). Write

$$Z = (Z_1, \ldots, Z_m) = x + t\Psi(x, t), \text{ and } \Psi = \Psi_1 + i\Psi_2.$$

Since the Z_j are approximate solutions of L, we have

$$\Psi + t\Psi_t + (I + t\Psi_x) \cdot a = O(t^k), \quad k = 1, 2, \ldots$$

and hence

$$\partial_t^j \Psi(x, 0) = 0, \quad j < l \text{ and } \langle \partial_t^l \Psi_2(x, 0), \xi^0 \rangle < 0 \tag{V.10}$$

for x in a neighborhood V of $\overline{B}$ (after shrinking B, if necessary). Observe that

$$\Re E(s,\xi,x,t) = t\xi\cdot\Psi_2(x,t) - |\xi|((s-x-t\Psi_1)^2 - t^2\Psi_2(x,t)^2).$$

Because of (V.10), continuity and homogeneity in ξ, we can get $c_1>0$ such that

$$\Re E(s,\xi,x,t) \le -c_1|\xi|t^{l+1}, \qquad \text{for } x\in V, \quad 0\le t\le t_1 \tag{V.11}$$

$s\in\mathbb{R}^m$ and ξ in a conic neighborhood Γ of ξ^0. Going back to the integrals in (V.9), we clearly have

$$\left|\int_B g(s,\xi,x,t_1)\mathrm{d}_x Z(x,t_1)\right| \le \mathrm{e}^{-c_2|\xi|},$$

for some $c_2>0$, for $s\in\mathbb{R}^m$ and $\xi\in\Gamma$. To estimate $\int_0^{t_1}\int_B \mathrm{d}(g\mathrm{d}Z)$, we use (V.8) and look at each term that appears there. We first consider the term $L(\phi f)\mathrm{e}^E$. For any k,

$$|\phi(Lf)\mathrm{e}^E| \le C_k t^{lk}\mathrm{e}^{-c_1t^l|\xi|} \le \frac{C'_k}{|\xi|^k}.$$

Moreover, for the x-integral

$$\int_B (L\phi)f\mathrm{e}^E\mathrm{d}Z = \langle f(.,t),(L\phi)\mathrm{e}^E\rangle$$

after decreasing t_1, we can get $\delta>0$ such that if $|s|\le\delta$ and $\xi\in\Gamma$,

$$|\langle f(.,t),(L\phi)\mathrm{e}^E\rangle| \le C\mathrm{e}^{-c|\xi|}$$

for some constants $c, C>0$. In the latter, we have used the constancy of ϕ near 0. It follows that the integral

$$\int_B\int_0^{t_1} L(\phi f)\mathrm{e}^E\mathrm{d}t\wedge\mathrm{d}Z$$

decays rapidly in ξ. The term $(\phi f)LE\mathrm{e}^E$ is estimated using the fact that for any k, $|LE|\le c_k t^k|\xi|$ for some constant c_k and that $|\mathrm{e}^E|\le \mathrm{e}^{-c_1t^l|\xi|}$. This shows that

$$\int_B\int_0^{t_1} (\phi f)LE\mathrm{e}^E\mathrm{d}t\wedge\mathrm{d}Z$$

decays rapidly in ξ. The integrals of the terms $\phi f(M_kE)LZ_k\mathrm{e}^E$ and $(M_k(\phi f))$ $LZ_k\mathrm{e}^E$ are estimated in the same fashion. Thus

$$\int_B\int_0^{t_1}\mathrm{d}(g\mathrm{d}Z)$$

has a rapid decay in ξ, and going back to (V.9), we have shown:

$$F(s,\xi)=\int_B e^{i\xi\cdot(s-x)-|\xi|[s-x]^2}\phi(x)f(x,0)\mathrm{d}x \tag{V.12}$$

decays rapidly for $|s|\le\delta$ in $\mathbb{R}^m$ and ξ in a conic neighborhood Γ of ξ^0. The function $F(s,\xi)$ is the standard FBI transform of the distribution $\phi(x)f(x,0)$. To conclude the proof, we will exploit the inversion formula for the FBI, namely,

$$\phi(x)f(x,0)=\lim_{\epsilon\to 0^+}c_m\iint e^{i(x-s)\cdot\xi-\epsilon|\xi|^2}F(s,\xi)|\xi|^{\frac{n}{2}}\mathrm{d}s\mathrm{d}\xi \tag{V.13}$$

where c_m is a dimensional constant. Assume now that $\phi(x)$ is supported in the ball centered at the origin with radius M. We will study the inversion integral in (V.13) by writing it as a sum of three pieces: $I_1(\epsilon)$, $I_2(\epsilon)$, and $I_3(\epsilon)$. The first piece consists of integration over the region $\{(\xi,s):|s|\ge 2M\}$. In the second piece we integrate over $\{(\xi,s):\delta\le|s|<2M\}$, and in the third piece over $\{(\xi,s):|s|\le\delta\}$. For the integral $I_1(\epsilon)$, after integrating in s, one gets an exponential decay in ξ independent of ϵ, and hence $\lim_{\epsilon\to 0^+}I_1(\epsilon)$ is in fact a holomorphic function near the origin in $\mathbb{C}^m$. To study the second piece, we write it as

$$I_2(\epsilon)=c_m\int_{\{(y,\xi,s):\delta\le|s|<2M\}}e^{i(x-y)\cdot\xi-|\xi|[s-y]^2-\epsilon|\xi|^2}\phi(y)f(y,0)|\xi|^{\frac{m}{2}}\,\mathrm{d}y\mathrm{d}s\mathrm{d}\xi.$$

We will use the holomorphic function $\langle\zeta\rangle=(\zeta_1^2+\cdots+\zeta_m^2)^{\frac{1}{2}}$ where we take the principal branch of the square root in the region $|\Im\zeta|<|\Re\zeta|$. Observe that this function is a holomorphic extension of $|\xi|$ away from the origin. In the ξ integration above, we can deform the contour to the image of

$$\zeta(\xi)=\xi+i\beta|\xi|(x-y)$$

where β is chosen sufficiently small. In particular, we choose β so that when x varies near the origin and y stays in the support of ϕ, then $|\Im\zeta(\xi)|<|\Re\zeta(\xi)|$, away from $\xi=0$. In the integrand of $I_2(\epsilon)$, if $|x|\le\frac{\delta}{4}$, we get an exponential decay independent of ϵ. It follows that this piece is also holomorphic near the origin in $\mathbb{C}^m$ after setting $\epsilon=0$. Finally, for the third piece, let $\Gamma_1,\ldots,\Gamma_n$ be convex cones such that with $\Gamma_0=\Gamma$,

$$\mathbb{R}^m=\bigcup_{j=0}^{n}\Gamma_j,$$

and for each $j\ge 1$ there exists a vector v_j satisfying $v_j\cdot\overline{\Gamma_j}>0$ and $v_j\cdot\xi^0<0$. We now write

$$I_3(\epsilon)=\sum_{j=0}^{n}K_j(\epsilon),$$

where K_j equals the integral over Γ_j. The decay in the FBI established in (V.12) shows us that K_0 is a smooth function even after setting $\epsilon = 0$. Each of the remaining functions K_j, after setting $\epsilon = 0$, is a boundary value of a tempered holomorphic function in a wedge whose inner product with ξ^0 is negative. Hence

$$(0, \xi^0) \notin WF_a(K_j(0+)),$$

where WF_a denotes the analytic wave front set. By Corollary V.3.8, the latter implies that

$$(0, \xi^0) \notin WF(K_j(0+)).$$

We have thus proved that

$$(0, \xi^0) \notin WF(f(x, 0)).$$

$\square$

Consider now the vector field

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} a_j(x, t) \frac{\partial}{\partial x_j}$$

where the a_j are C^1 on an open set $\Omega \subset \mathbb{R}^m_x \times \mathbb{R}_t$. To L we associate vector fields

$$L^\theta = \frac{\partial}{\partial s} - \mathrm{e}^{-i\theta} L$$

where $s \in \mathbb{R}$ is a new variable and $\theta \in [0, 2\pi)$ is a parameter. Suppose that for each $\theta \in [0, 2\pi)$ there exist C^1 functions

$$\Psi^\theta_1(x, t, s), \ldots, \Psi^\theta_m(x, t, s)$$

defined on $\Omega \times J$ ($J \subset \mathbb{R}$ is an open interval centered at the origin) such that

$$Z^\theta_j(x, t, s) = x_j + s\Psi^\theta_j\ (x, t, s), \quad j = 1, \ldots, m$$

are approximate solutions of $L^\theta Z^\theta_j = 0$ in the sense that $L^\theta Z^\theta_j$ are s-flat at $s = 0$. Define also $\Psi^\theta_{m+1}(x, t, s) = \mathrm{e}^{-i\theta}$ and $Z^\theta_{m+1}(x, t, s) = t + \mathrm{e}^{-i\theta} s$ and note that $L^\theta Z^\theta_{m+1} = 0$. If we write $\Psi^\theta = (\Psi^\theta_1, \ldots, \Psi^\theta_{m+1})$ and $Z^\theta = (Z^\theta_1, \ldots, Z^\theta_{m+1})$, then

$$Z^\theta_s(0, 0, 0) = \Psi^\theta(0, 0, 0) = -\begin{pmatrix} \mathrm{e}^{-i\theta} a(0, 0) \\ \mathrm{e}^{-i\theta} \end{pmatrix} = \mathrm{e}^{-i\theta} \begin{pmatrix} \Psi(0, 0) \\ -1 \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} \xi \\ \tau \end{pmatrix} \cdot \Im \Psi^\theta(0, 0, 0) &= \begin{pmatrix} \xi \\ \tau \end{pmatrix} \cdot \begin{pmatrix} \Im \Psi(0, 0) \cos\theta - \Re \Psi(0, 0) \sin\theta \\ \sin\theta \end{pmatrix} \\ &= \xi \cdot \Im \Psi(0, 0) \cos\theta + (\tau - \xi \cdot \Re \Psi(0, 0)) \sin\theta. \end{aligned}$$

So the condition

$$\begin{pmatrix} \xi \\ \tau \end{pmatrix} \cdot \Im \Psi^{\theta}(0,0,0) \neq 0$$

for some $\theta \in [0, 2\pi)$ is equivalent to saying that $(0, 0, \xi, \tau)$ is not in the characteristic set of L. Suppose now $h(x, t)$ is a C^1 function with the following property: there exist C^1 functions $h^{\theta}(x, t, s)$ such that $h^{\theta}(x, t, 0) = h(x, t)$ and $L^{\theta}h^{\theta}$ is s-flat at $s = 0$. If $(0, 0, \xi^0, \tau^0)$ is not in the characteristic set of L, we know that there is $\theta \in [0, 2\pi)$ such that

$$\begin{pmatrix} \xi^0 \\ \tau^0 \end{pmatrix} \cdot \Im \Psi^{\theta}(0,0,0) \neq 0.$$

By replacing θ by $\theta + \pi$ or $\theta - \pi$ if necessary, we may assume that

$$\begin{pmatrix} \xi^0 \\ \tau^0 \end{pmatrix} \cdot \Im \Psi^{\theta}(0,0,0) < 0$$

and we can apply what we saw in the proof of Lemma V.5.2 to an FBI in (x, t)-space to conclude the following: there exist a conic neighborhood Γ of (ξ^0, τ^0) in $\mathbb{R}^{m+1}\backslash\{0\}$ and a neighborhood $\mathcal{O}$ of the origin in $\mathbb{R}^{m+1}$ such that

$$\begin{aligned} &Fh^{\theta}(0; x', t', \xi, \tau) \\ &\qquad = \int_{B\times J} \mathrm{e}^{i[\xi\cdot(x'-x)+\tau(t'-t)]-|(\xi,\tau)|[\langle x'-x\rangle^2+(t'-t)^2]} h^{\theta}(x, t, 0)\mathrm{d}x\mathrm{d}t \\ &\qquad = Fh(x', t', \xi, \tau) \end{aligned}$$

is rapidly decreasing for $(\xi, \tau) \in \Gamma$ and $(x', t') \in \mathcal{O}$. We have thus proved:

LEMMA V.5.4. *For each $\theta \in [0, 2\pi)$ let $L^{\theta} = \frac{\partial}{\partial s} - \mathrm{e}^{-i\theta}L$ and suppose there exist $\Psi_1^{\theta}, \ldots, \Psi_{m+1}^{\theta} \in C^1(\Omega \times J)$ such that $Z^{\theta} = (x, t) + s\Psi^{\theta}(x, t, s)$ is an approximate solution of $L^{\theta}Z^{\theta} = 0$ in the sense that $L^{\theta}Z^{\theta}$ is s-flat at $s = 0$. Suppose moreover that there exist $h^{\theta} \in C^1(\Omega \times J)$ such that $h^{\theta}(x, t, 0) = h(x, t)$ and $L^{\theta}h^{\theta}$ is s-flat at $s = 0$. Then*

$$WF(h)\mid_0 \subset (\mathrm{char} L)\mid_0 .$$

The preceding linear results will next be applied to a nonlinear equation. Let $\Omega \subset \mathbb{R}^{m+1}$ be a neighborhood of the origin and suppose $u \in C^2(\Omega)$ is a solution of

$$u_t = f(x, t, u, u_x) \qquad \text{(V.14)}$$

where $f(x, t, \zeta_0, \zeta)$ is a C^{∞} function in the variables $(x, t) \in \Omega$ and holomorphic in the variables

$$(\zeta_0, \zeta) \in \mathcal{N} \subset \mathbb{C} \times \mathbb{C}^M, \quad (a, \omega) = (u(0,0), u_x(0,0)) \in \mathcal{N}.$$

Consider

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \left(\frac{\partial f}{\partial \zeta_j}\right)(x, t, \zeta_0, \zeta)\frac{\partial}{\partial x_j} \tag{V.15}$$

and

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \left(\frac{\partial f}{\partial \zeta_j}\right)(x, t, u, u_x)\frac{\partial}{\partial x_j}.$$

Let $v = (u, u_x)$. It is easy to check that v solves the quasi-linear system

$$L^u v = g(x, t, v) \tag{V.16}$$

where

$$g_0(x, t, \zeta_0, \zeta) = f(x, t, \zeta_0, \zeta) - \sum_{j=1}^{m} \zeta_j \frac{\partial f}{\partial \zeta_j}(x, t, \zeta_0, \zeta)$$

and

$$g_i(x, t, \zeta_0, \zeta) = f_{x_i}(x, t, \zeta_0, \zeta) - \zeta_i \frac{\partial f}{\partial \zeta_0}(x, t, \zeta_0, \zeta), \qquad i = 1, \ldots, m.$$

Consider now the principal part of the holomorphic *Hamiltonian* of (V.16)

$$H = \mathcal{L} + g_0 \frac{\partial}{\partial \zeta_0} + \sum_{j=1}^{m} g_j \frac{\partial}{\partial \zeta_j}.$$

For $\Psi(x, t, \zeta_0, \zeta)$ a C^∞ function in (x, t, ζ, ζ_0) and holomorphic in the variables $(\zeta_0, \zeta) \in \mathcal{N}$, set $\Psi^v(x, t) = \Psi(x, t, v(x, t))$ and let $\mathcal{L}^p$ denote the vector field in Ω obtained by plugging $p(x, t)$ for (ζ, ζ_0) in the coefficients of $\mathcal{L}$. Note that $\mathcal{L}^v = L^u$. Equation (V.16) implies that

$$\mathcal{L}^v \Psi^v = (H\Psi)^v$$

where $\Psi(x, t, \zeta_0, \zeta)$ is any C^∞ function in $(x, t) \in \Omega$ and holomorphic in $(\zeta_0, \zeta) \in \mathcal{N}$. Let $Z_j(x, t, \zeta_0, \zeta)$, $j = 1, \ldots, m$, and $\Xi_k(x, t, \zeta_0, \zeta)$, $k = 0, \ldots, m$ be t-flat solutions of $H\Psi = 0$ such that $Z_j(x, 0, \zeta_0, \zeta) = x_j$, $j = 1, \ldots, m$, and $\Xi_k(x, 0, \zeta_0, \zeta) = \zeta_k$, $k = 0, \ldots, m$. Let $\tilde{Z}(z, t, \zeta_0, \zeta)$ and $\tilde{\Xi}(z, t, \zeta_0, \zeta)$, $z = x + iy \in \mathbb{R}^m \oplus i\mathbb{R}^m$ be almost analytic extensions of $Z(x, t, \zeta_0, \zeta)$ and $\Xi(x, t, \zeta_0, \zeta)$ respectively, i.e., $\tilde{Z}(x, t, \zeta_0, \zeta) = Z(x, t, \zeta_0, \zeta)$, $\tilde{\Xi}(x, t, \zeta_0, \zeta) = \Xi(x, t, \zeta_0, \zeta)$ and for all $k \in \mathbb{N}$ there exists $C_k > 0$ such that for $j = 1, \ldots, m$ we have

$$\left|\frac{\partial}{\partial \bar{z}_j}\tilde{Z}(z, t, \zeta_0, \zeta)\right| \leq C_k \left|\Im z\right|^k$$

and

$$\left|\frac{\partial}{\partial \bar{z}_j}\tilde{\Xi}(z,t,\zeta_0,\zeta)\right| \le C_k\,|\Im z|^k.$$

Since the Jacobian

$$\frac{\partial(\Re\tilde{Z},\Im\tilde{Z},\Re\tilde{\Xi},\Im\tilde{\Xi})}{\partial(\Re z,\Im z,\Re\zeta_0,\Im\zeta_0,\Re\zeta,\Im\zeta)}$$

is nonsingular near $t=0$, we may solve

$$\begin{cases}\tilde{Z}(z,t,\zeta_0,\zeta) & =\tilde{Z},\\ \tilde{\Xi}(z,t,\zeta_0,\zeta) & =\tilde{\Xi}\end{cases}$$

with respect to (z,ζ_0,ζ) in a neighborhood of $(0,a,\omega)$ by the implicit function theorem and get

$$\begin{cases}z & =P(\tilde{Z},t,\tilde{\Xi}),\\ (\zeta_0,\zeta) & =Q(\tilde{Z},t,\tilde{\Xi})\end{cases}$$

with $P(0,0,a,\omega)=0$ and $Q(0,0,a,\omega)=(a,\omega)$. We get

$$\begin{cases}\tilde{Z}(P(\tilde{Z},t,\tilde{\Xi}),t,Q(\tilde{Z},t,\tilde{\Xi})) & =\tilde{Z},\\ \tilde{\Xi}(P(\tilde{Z},t,\tilde{\Xi}),t,Q(\tilde{Z},t,\tilde{\Xi})) & =\tilde{\Xi}\end{cases}$$

and differentiating with respect to $\bar{\tilde{Z}}$ we obtain

$$\frac{\partial(\tilde{Z},\tilde{\Xi})}{\partial(z,\zeta_0,\zeta)}(P(\tilde{Z},t,\tilde{\Xi}),t,Q(\tilde{Z},t,\tilde{\Xi}))\frac{\partial(P,Q)}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{\Xi})$$

$$+\frac{\partial(\tilde{Z},\tilde{\Xi})}{\partial(\bar{z},\overline{\zeta_0},\bar{\zeta})}(P(\tilde{Z},t,\tilde{\Xi}),t,Q(\tilde{Z},t,\tilde{\Xi}))\frac{\partial(\overline{P},\overline{Q})}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{\Xi})=0.$$

If $A(z,t,\zeta_0,\zeta)$ denotes a generic entry of the matrix

$$\frac{\partial(\tilde{Z},\tilde{\Xi})}{\partial(\bar{z},\overline{\zeta_0},\bar{\zeta})}(z,t,\zeta_0,\zeta),$$

then $|A(z,t,\zeta_0,\zeta)|\le C_k\,|\Im z|^k$ for all k. It follows that for each k

$$\left|\frac{\partial Q_0}{\partial\bar{\tilde{Z}}_j}(\tilde{Z},t,\tilde{\Xi})\right|\le C_k'\left|\Im P(\tilde{Z},t,\tilde{\Xi})\right|^k \qquad \forall j=1,\ldots,m$$

and Q_0 is holomorphic in (ζ_0,ζ). Now consider

$$\Psi(z,t,\zeta_0,\zeta)=Q_0(\tilde{Z}(z,t,\zeta_0,\zeta),0,\tilde{\Xi}(z,t,\zeta_0,\zeta))$$

and observe that

$$\begin{aligned}\Psi^v(x,0) &= \Psi(x,0,u(x,0),u_x(x,0))\\ &= Q_0(\tilde{Z}(x,0,u(x,0),u_x(x,0)),0,\tilde{\Xi}(x,0,u(x,0),u_x(x,0)))\\ &= u(x,0).\end{aligned}$$

Observe that $H\tilde{Z}(x,t,\zeta_0,\zeta)$ and $H\tilde{\Xi}(x,t,\zeta_0,\zeta)$ are t-flat at $t=0$. We will next show that

$$H\Psi=\sum_{j=1}^m\left(\frac{\partial Q_0}{\partial \tilde{Z}_j}H\tilde{Z}_j+\frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}H\overline{\tilde{Z}_j}\right)+\sum_{k=0}^m\left(\frac{\partial Q_0}{\partial \tilde{\Xi}_k}H\tilde{\Xi}_k+\frac{\partial Q_0}{\partial \overline{\tilde{\Xi}_k}}H\overline{\tilde{\Xi}_k}\right)$$

is t-flat. Note that

$$\begin{aligned}P(x,0,\zeta_0,\zeta) &= P(Z(x,0,\zeta_0,\zeta),0,\Xi(x,0,\zeta_0,\zeta))\\ &= P(\tilde{Z}(x,0,\zeta_0,\zeta),0,\tilde{\Xi}(x,0,\zeta_0,\zeta))\\ &= x.\end{aligned}$$

This implies that for some $C>0$,

$$\left|\Im P(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{\Xi}(x,t,\zeta_0,\zeta))\right|\le C|t|.$$

Hence $\frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{\Xi}(x,t,\zeta_0,\zeta))$ is t-flat at $t=0$, which in turn implies that for all $k\in\mathbb{N}$, there exists $C''_k>0$ such that

$$|(H\Psi)(x,t,\zeta_0,\zeta)|\le C''_k\,|t|^k.$$

Hence $L^u\Psi^v=\mathcal{L}^v\Psi^v=(H\Psi)^v$ is t-flat at $t=0$, and so we have found $h(x,t)=\Psi^v(x,t)$ such that L^uh is t-flat at $t=0$ and $h(x,0)=u(x,0)$. Now $u(x,t)$ is also a solution of the equation

$$u_s=\mathrm{e}^{-i\theta}(u_t-f(x,t,u,u_x))$$

which is of the same kind as (V.14), and the associated vector field $\mathcal{L}^\theta$ as in (V.15) is given by

$$\mathcal{L}^\theta=\frac{\partial}{\partial s}-\mathrm{e}^{-i\theta}\mathcal{L}$$

with $\mathcal{L}$ as before. Note that

$$\left(\mathcal{L}^\theta\right)^v=\frac{\partial}{\partial s}-\mathrm{e}^{-i\theta}\mathcal{L}^v=\frac{\partial}{\partial s}-\mathrm{e}^{-i\theta}L^u=(\,L^u)^\theta.$$

It follows that there exists a C^1 function $h^\theta(x,t,s)$ such that

$$(L^u)^\theta\,h^\theta=\left(\frac{\partial}{\partial s}-\mathrm{e}^{-i\theta}L^u\right)h^\theta$$

https://doi.org/10.1017/CBO9780511543067.006 Published online by Cambridge University Press

is s-flat at $s=0$ and $h^{\theta}(x,t,0)=u(x,t)$. We apply Lemma V.5.4 and conclude that $WF(u)\,|_0\subset$ char $L^u\,|_0$. By translation we may apply the same argument to all points of Ω and state

THEOREM V.5.5. *Let $u\in C^2(\Omega)$ be a solution of (V.14). Then the C^∞ wave front set of u is contained in the characteristic set of the linearized operator L^u.*

V.6 Applications to edge-of-the-wedge theory

Consider now a hypoanalytic structure $(\mathcal{M},\mathcal{V})$, $\dim\mathcal{M}=N$, fiber dimension of $\mathcal{V}=n$ and $m=N-n$. If $\mathcal{N}$ is a strongly noncharacteristic submanifold of $\mathcal{M}$, then Proposition V.1.28 shows that $\mathcal{V}$ induces a hypoanalytic structure on $\mathcal{N}$ by taking as the structure bundle in $\mathcal{N}$ the image of T' under the natural map

$$\mathbb{C}T^*\mathcal{M}|_{\mathcal{N}}\to\mathbb{C}T^*\mathcal{N}.$$

The associated bundle of vector fields will be denoted by $\mathcal{V}\mathcal{N}$ and we have $\mathcal{V}\mathcal{N}=\mathcal{V}\cap\mathbb{C}T\mathcal{N}$. Note that for any $p\in\mathcal{N}$, $\dim_{\mathbb{C}}\mathcal{V}_p\mathcal{N}=\dim\mathcal{N}-m$. For $p\in\mathcal{N}$ define

$$\mathcal{V}_p^{\mathcal{N}}=\left\{L\in\mathcal{V}_p:\Re L\in T_p\mathcal{N}\right\}.$$

LEMMA V.6.1. *$\mathcal{V}^{\mathcal{N}}$ is a real sub-bundle of $\mathcal{V}|_{\mathcal{N}}$ of rank $n+$ dim $\mathcal{N}-m$. The map $\Im$ which takes the imaginary part induces an isomorphism between $\mathcal{V}^{\mathcal{N}}/\mathcal{V}\mathcal{N}$ and $T\mathcal{M}|_{\mathcal{N}}/T\mathcal{N}$.*

PROOF. Let $p\in\mathcal{N}$. The map $\Im:\mathcal{V}_p^{\mathcal{N}}\to T_p\mathcal{M}$ induces a map $\Im:\mathcal{V}_p^{\mathcal{N}}\to T_p\mathcal{M}/T_p\mathcal{N}$. This latter map is surjective. Indeed, given $v\in T_p\mathcal{M}$, since $\mathcal{N}$ is strongly noncharacteristic, we can find $L\in\mathcal{V}_p$ and $w\in\mathbb{C}T_p\mathcal{N}$ such that $iv=L+w$. Taking real and imaginary parts, we see that $L\in\mathcal{V}_p^{\mathcal{N}}$ and $v=\Im L+\Im w$ as desired. Since the kernel of the map $\Im:\mathcal{V}_p^{\mathcal{N}}\to T_p\mathcal{M}/T_p\mathcal{N}$ is $\mathcal{V}_p\mathcal{N}$, we get an isomorphism as asserted in the lemma. Hence, dim $\mathcal{V}_p^{\mathcal{N}}=\dim T_p\mathcal{M}-\dim T_p\mathcal{N}+\dim_{\mathbb{R}}\mathcal{V}_p\mathcal{N}=n+$ dim $\mathcal{N}-m$ for any $p\in\mathcal{N}$. $\square$

DEFINITION V.6.2. *Let E be a submanifold of $\mathcal{M}$, $\dim\mathcal{M}=r+s$, $\dim E=r$. We say a subset $\mathcal{W}$ is a wedge in $\mathcal{M}$ at $p\in E$ with edge E if the following holds: there exists a diffeomorphism φ of a neighborhood V of 0 in $\mathbb{R}^{r+s}$ onto a neighborhood U of p in $\mathcal{M}$ with $\varphi(0)=p$ and a set $B\times\Gamma\subseteq V$ with B a ball centered at $0\in\mathbb{R}^r$ and Γ a truncated open convex cone in $\mathbb{R}^s$ with vertex at 0 such that $\varphi(B\times\Gamma)=\mathcal{W}$ and $\varphi(B\times\{0\})=E\cap U$.*

If $E, \mathcal{M}, \mathcal{W}$ and $p \in E$ are as in the previous definition, the direction wedge $\Gamma_p(\mathcal{W}) \subseteq T_p(\mathcal{M})$ is defined as the interior of

$$\{c'(0) \mid c : [0,1] \to \mathcal{M} \text{ smooth}, c(0) = p, \quad c(t) \in \mathcal{W} \quad \forall t > 0\}.$$

If φ is as in Definition V.6.2, $\Gamma_p(\mathcal{W}) = \{\mathrm{d}\varphi(\mathbb{R}^r \times \{\lambda v \mid v \in \Gamma\ ,\ \ \lambda > 0\})\}$. Note that $\Gamma_p(\mathcal{W})$ is a linear wedge in $T_p\mathcal{M}$ with edge equal to T_pE. Set

$$\Gamma(\mathcal{W}) = \bigcup_{p \in E \cap U} \Gamma_p(\mathcal{W}).$$

Suppose now $\mathcal{N}$ is a strongly noncharacteristic submanifold of $\mathcal{M}$ and $\mathcal{W}$ is a wedge in $\mathcal{M}$ with edge $\mathcal{N}$. Let $u \in \mathcal{D}'(\mathcal{W})$ be a solution of $\mathcal{V}$ and let $f \in \mathcal{D}'(\mathcal{N})$. In a neighborhood of $p \in \mathcal{N}$ we may choose coordinates (x, y) vanishing at p such that $y = 0$ defines $\mathcal{N}$ locally and $\mathcal{W}$ has the form $B \times \Gamma$ with B a ball centered at 0 in x-space and Γ a truncated cone in y-space with vertex at 0. Since u is a solution and $\mathcal{N}$ is noncharacteristic, by proposition 1.4.3 in [**T5**], $u(x, y)$ is a smooth function of $y \in \Gamma$ valued in $\mathcal{D}'(B)$. We say f is the boundary value of u and write $bu = f$ if $\Gamma \ni y \mapsto u(., y)$ extends continuously to $\Gamma \cup \{0\}$ with $u(., 0) = f$, and that this is true for any $p \in \mathcal{N}$. In this case, since $\mathcal{V}\mathcal{N} = \mathcal{V} \cap \mathbb{C}T\mathcal{N}$, it is readily seen that f is a solution of $\mathcal{V}\mathcal{N}$, i.e., of the induced structure on $\mathcal{N}$. If the codimension of $\mathcal{N}$ is 1, then a wedge $\mathcal{W}$ with edge $\mathcal{N}$ is simply a side of $\mathcal{N}$ and distribution solutions in $\mathcal{W}$ in this case with boundary values in $\mathcal{N}$ were studied in [**T5**]. We continue to assume that $\mathcal{W}$ is a wedge in $\mathcal{M}$ with edge $\mathcal{N}$ which is strongly noncharacteristic. For $p \in \mathcal{N}$, define

$$\Gamma_p^{\mathcal{V}}(\mathcal{W}) = \left\{\ L \in \mathcal{V}_p^{\mathcal{N}} : \Im L \in \Gamma_p(\mathcal{W})\right\},$$

and

$$\Gamma_p^{T}(\mathcal{W}) = \left\{\Re L : L \in \Gamma_p^{\mathcal{V}}(\mathcal{W})\right\}.$$

$\Gamma_p^T(\mathcal{W})$ is an open cone in $\Re\mathcal{V}_p\mathcal{M} \cap T_p\mathcal{N}$. To see this, fix $p \in \mathcal{N}$ and let $\{L_1, \ldots, L_l\}$ be an $\mathbb{R}$-basis for $\mathcal{V}_p\mathcal{N}$ and complete this to an $\mathbb{R}$-basis

$$\{L_1, \ldots, L_l, V_1, \ldots, V_k\}$$

of $\mathcal{V}_p^{\mathcal{N}}$. Observe that $\Re\mathcal{V}_p\mathcal{M} \cap T_p\mathcal{N}$ is spanned by

$$\Re L_1, \ldots, \Re L_l, \Re V_1, \ldots, \Re V_k.$$

Note also that $\Gamma_p(\mathcal{W})$ is a linear wedge in $T_p\mathcal{M}$ and hence is translation invariant by elements of $T_p\mathcal{N}$. Therefore

$$\Gamma_p^T(\mathcal{W}) = \left\{\sum_1^l a_i \Re L_i + \sum_1^k b_j \Re V_j : a_i \in \mathbb{R}, b_j \in \mathbb{R}, \sum_1^k b_j \Im V_j \in \Gamma_p(\mathcal{W})\right\}.$$

This description shows that $\Gamma_p^T(\mathcal{W})$ is an open cone in $\Re\mathcal{V}_p\mathcal{M}\cap T_p\mathcal{N}$.

LEMMA V.6.3. *Let $(\mathcal{M},\mathcal{V})$ be a CR structure, $p\in\mathcal{M}$ and $v\in T_p\mathcal{M}$. Then there is a maximally real submanifold $\mathcal{X}\subseteq\mathcal{M}$ with $p\in\mathcal{X}$ and $v\in T_p\mathcal{X}$.*

PROOF. Recall from Chapter I that there are local coordinates

$$(x_1,\dots,x_n,y_1,\dots,y_n,s_1,\dots,s_d)$$

vanishing at p and smooth, real-valued ϕ_1, ..., ϕ_d defined near the origin such that the differentials of

$$z_j=x_j+iy_j,\qquad j=1,\dots,n;$$

$$w_k=s_k+i\phi_k(x,y,s),\qquad k=1,\dots,d$$

span T' in a neighborhood of the origin, $\phi(0)=0$ and $\mathrm{d}\phi(0)=0$. Let

$$v=\sum_{k=1}^n a_k\frac{\partial}{\partial x_k}+\sum_{k=1}^n b_k\frac{\partial}{\partial y_k}+\sum_{k=1}^d c_k\frac{\partial}{\partial s_k}$$

be a real tangent vector at the origin, $v\neq 0$. If $a_j=0=b_j$ for all j, we can take $\mathcal{X}=\{(x,y,s):y=0\}$. Otherwise, assume without loss of generality that $a_1+ib_1\neq 0$. Consider the subspace S of the tangent space at the origin generated by the $n+d$ linearly independent vectors v, $\frac{\partial}{\partial s_1}$, ..., $\frac{\partial}{\partial s_d}$, $\frac{\partial}{\partial x_2}$, ..., $\frac{\partial}{\partial x_n}$. Let $\mathcal{X}$ be a submanifold of dimension $m=n+d$ through the origin so that $T_0\mathcal{X}=S$ (can take $\mathcal{X}$ to be a linear space). We claim that $\mathcal{X}$ is maximally real near the origin. To see this, suppose a one-form $\theta=\sum_{j=1}^n A_j\mathrm{d}z_j(0)+\sum_{k=1}^d B_k\mathrm{d}s_k$ is orthogonal to $T_0\mathcal{X}$. Then

$$\left\langle\theta,\frac{\partial}{\partial s_j}\right\rangle=0\qquad\forall j$$

and so $B_j=0\quad\forall j$. Moreover, since $\left\langle\theta,\frac{\partial}{\partial x_l}\right\rangle=0\quad\forall l\geq 2$, we get $A_j=0$ for $j\geq 2$. Finally, note that $0=\langle\theta,v\rangle=A_1(a_1+ib_1)$ and so since $a_1+ib_1\neq 0$, $A_1=0$ showing that $\theta=0$. Hence $\mathcal{X}$ is maximally real near 0. $\square$

We observe that Lemma V.6.3 is not valid for a general hypoanalytic structure $(\mathcal{M},\mathcal{V})$ which has a section L in $\mathcal{V}$ such that at a point $p\in\mathcal{V}$, L_p is a real vector field.

Recall next Marson's technique of locally embedding a hypoanalytic structure into a generic CR manifold ([**Ma**]). Suppose $(\mathcal{M},\mathcal{V})$ is a hypoanalytic structure with the integers m and n having their usual meaning. Let $d=\dim T_p^0$ for some $p\in\mathcal{M}$. Choose a coordinate system $(x_1,\dots,x_m,y_1,\dots,y_n)$

vanishing at p and smooth, real-valued functions $\phi_1, \ldots, \phi_d$ defined in a neighborhood U of the origin and satisfying

$$\phi_k(0) = 0, \qquad d\phi_k(0) = 0 \qquad \forall k = 1, \ldots, d$$

such that T' over U is spanned by the differentials of

$$z_j = x_j + iy_j, \qquad j = 1, \ldots, \nu;$$

$$z_{\nu+k} = x_{\nu+k} + i\phi_k(x, y, s), \qquad k = 1, \ldots, d.$$

Let $U' = U \times \mathbb{R}^{n-\nu}$ and suppose $(x_{m+1}, \ldots, x_{m+n-\nu})$ are the coordinates for $\mathbb{R}^{n-\nu}$. Define

$$z_{m+k} = x_{m+k} + iy_{\nu+k}, \qquad \text{for } k = 1, \ldots, n - \nu.$$

Let $\mathcal{V}'$ be the sub-bundle of $\mathbb{C}TU'$ that is orthogonal to the bundle generated by $dz_1, \ldots, dz_{m+n-\nu}$. It is easy to see that $(U', \mathcal{V}')$ is a CR structure and for any $L \in \mathcal{V}_p$,

$$L' = L - i\sum_{l=1}^{n-\nu}(Ly_{\nu+l})\frac{\partial}{\partial x_{m+l}} \in \mathcal{V}'_{p'}.$$

Here for $p \in U$, we write $p' \in U'$ to be any point of the form $p' = (p, x)$. Moreover, the preceding association $L \to L'$ is an isomorphism of $\mathcal{V}_p$ onto $\mathcal{V}_{p'}$. In particular, any solution of $\mathcal{V}$ is also a solution of $\mathcal{V}'$ depending on fewer variables. Characteristic covectors $\sigma \in T^0_p U$ embed into characteristic covectors $(\sigma, 0) \in T^0_{p'} U'$ for any $p' = (p, x)$. If $\mathcal{N}$ is a strongly noncharacteristic submanifold of U, then $\mathcal{N}' = \mathcal{N} \times \mathbb{R}^{n-\nu}$ is a strongly noncharacteristic submanifold of U' and if $p \in \mathcal{N}$ and $p' = (p, x) \in \mathcal{N}'$, we have:

$$\mathcal{V}^{\mathcal{N}'}_{p'} = \{L' : L \in \mathcal{V}^{\mathcal{N}}_p\}$$

where L' is determined by L as above. If $\mathcal{W}$ is a wedge with edge $\mathcal{N}$ in U, then $\mathcal{W}' = \mathcal{W} \times \mathbb{R}^{n-\nu}$ is a wedge in U' with edge $\mathcal{N}'$ and

$$\Gamma^{\mathcal{V}'}_{p'}(\mathcal{W}') = \{L' : L \in \Gamma^{\mathcal{V}}_p(\mathcal{W})\}.$$

Finally, if $u \in \mathcal{D}'(\mathcal{N})$, it may be viewed as a distribution in $\mathcal{N}'$ and it is easy to see that

$$WF^{\mathcal{N}}_{\text{ha},\, p}(u) \times \{0\} \subseteq WF^{\mathcal{N}'}_{\text{ha},\, p'}(u).$$

We are now ready to present an application of the FBI transform to the hypoanalytic wave front set of a distribution u on a strongly noncharacteristic $\mathcal{N}$ which extends to a solution in a wedge. The result is due to Eastwood and Graham ([**EG1**]).

THEOREM V.6.4. ([**EG1**]) *Let $(\mathcal{M}, \mathcal{V})$ be a hypoanalytic structure, $\mathcal{N}$ a strongly noncharacteristic submanifold, and let $\mathcal{W}$ be a wedge in $\mathcal{M}$ with edge . Suppose $f \in \mathcal{D}'(\mathcal{N})$ is the boundary value of a solution of $\mathcal{V}$ on $\mathcal{W}$. Then $WF_{\mathrm{ha}}(f) \subseteq \Gamma^T(\mathcal{W})^0 =$ the polar of $\Gamma^T(\mathcal{W})$ in the duality between $T\mathcal{N}$ and $T^*\mathcal{N}$.*

PROOF. Let $p \in \mathcal{N}$ and $\sigma \in T^*_p\mathcal{N}/\{0\}$ satisfy $\sigma \notin \Gamma^T_p(\mathcal{W})^0$. If we embed $\mathcal{M}$ near p into a CR structure as in the preceding discussion, then $\sigma' = (\sigma, 0) \notin (\Gamma^T_{p'}(\mathcal{W}'))^0$, and so because of the relation between $WF^{\mathcal{N}}_{\mathrm{ha},\,p}(f)$ and $WF^{\mathcal{N}'}_{\mathrm{ha},\,p'}(f)$, it suffices to prove the theorem under the assumption that $(\mathcal{M}, \mathcal{V})$ is CR. Since $\sigma \notin \Gamma^T_p(\mathcal{W})^0$, there is $L \in \Gamma^{\mathcal{V}}_p(\mathcal{W})$ such that $\langle \sigma, \Re L\rangle < 0$. By Lemma V.6.3, there is a maximally real submanifold $\mathcal{X} \subseteq \mathcal{N}$ with $p \in \mathcal{X}$ and $\Re L \in T_p\mathcal{X}$ (note that the induced structure on $\mathcal{N}$ is CR). Since $\mathcal{X}$ is maximally real and $L \neq 0$, $\Im L \notin T_p\mathcal{X}$. Choose a submanifold $\mathcal{Y}$ of $\mathcal{M}$ such that $\mathcal{X} \subseteq \mathcal{Y}$, and $T_p\mathcal{Y}$ is spanned by $T_p\mathcal{X}$ and $\Im L$. Thus $\mathcal{X}$ is a hypersurface in $\mathcal{Y}$. Since $\mathcal{X}$ is maximally real, $\mathcal{Y}$ inherits a hypoanalytic structure of codimension 1 from $(\mathcal{M}, \mathcal{V})$. This induced structure on $\mathcal{Y}$ is CR near p, is generated by L at p, and $\mathcal{X}$ is a maximally real submanifold of $\mathcal{Y}$. We may assume that near p, $\mathcal{X}$ divides $\mathcal{Y}$ into two components $\mathcal{Y}^+$, $\mathcal{Y}^-$ where $\mathcal{Y}^+$ is the side toward which $\Im L$ points. Since $\Im L \in \Gamma_p(\mathcal{W})$, $\mathcal{Y}^+ \subseteq \mathcal{W}$ near p. $\mathcal{Y}^+$ may be regarded as a wedge in $\mathcal{Y}$ with edge $\mathcal{X}$. If F is the solution in $\mathcal{W}$ with $bF = f$ on $\mathcal{N}$, then F restricts to $\mathcal{Y}^+$ (since $\mathcal{Y}^+$ is noncharacteristic) and this restriction is a solution for the structure on $\mathcal{Y}$. Moreover, this restriction has a boundary value equal to $f\,|_{\mathcal{X}}$. To prove the theorem, we have to show that $i^*_{\mathcal{X}}\sigma \notin WF_{\mathrm{ha},\,p}(f\,|_{\mathcal{X}})$. Note that we also have $\langle i^*_{\mathcal{X}}\sigma, \Re L\rangle < 0$. Choose local coordinates $x_1, \ldots, x_m, t$ on $\mathcal{Y}$ vanishing at p so that in these coordinates $\mathcal{X}$ is given by $t = 0$ and $L = A + i\frac{\partial}{\partial t}$ where $A = \sum_1^m A_j \frac{\partial}{\partial x_j}$ is a real vector field. We therefore need to show that if $\sigma \in T^*_0\mathbb{R}^m$ and $\langle A, \sigma\rangle < 0$, then $\sigma \notin WF_{\mathrm{ha}} bf$. This will follow from Theorem V.6.9. $\square$

COROLLARY V.6.5. *Suppose $\mathcal{X} \subset \mathcal{M}$ is a maximally real submanifold, $p \in \mathcal{X}$, and let $\mathcal{W}^+$ and $\mathcal{W}^-$ be wedges in $\mathcal{M}$ with edge $\mathcal{X}$ such that $\Gamma_p(\mathcal{W}^+) = -\Gamma_p(\mathcal{W}^-)$. If $f \in \mathcal{D}'(\mathcal{X})$ is the boundary value of a solution of $\mathcal{V}$ on $\mathcal{W}^+$ and also the boundary value of a solution of $\mathcal{V}$ on $\mathcal{W}^-$, then $WF_{\mathrm{ha},p}(f) \subset i^*_{\mathcal{X}} T^0_p\mathcal{M}$.*

PROOF. By Theorem V.6.4,

$$WF_{\mathrm{ha},p}(f) \subseteq \Gamma^T_p(\mathcal{W}^+)^0 \cap \Gamma^T_p(\mathcal{W}^-)^0.$$

Note that since $\Gamma_p(\mathcal{W}^+) = -\Gamma_p(\mathcal{W}^-)$, $\Gamma^T_p(\mathcal{W}^+) = -\Gamma^T_p(\mathcal{W}^-)$. Hence if $\sigma \in \Gamma^T_p(\mathcal{W}^+)^0 \cap \Gamma^T_p(\mathcal{W}^-)^0$, then $\langle \sigma, v\rangle = 0$ for every $v \in \Gamma^T_p(\mathcal{W}^+)$. Since $\Gamma^T_p(\mathcal{W}^+)$

is an open cone in $\Re\mathcal{V}_p \cap T_p\mathcal{X}$, it follows that $\sigma \in \left(\Re\mathcal{V}_p \cap T_p\mathcal{X}\right)^{\perp}$. Therefore the corollary follows from the fact that

$$i_{\mathcal{X}}^* T_p^0 = \left(\Re\mathcal{V}_p \cap T_p\mathcal{X}\right)^{\perp}.$$

□

COROLLARY V.6.6. (Theorem V.3.1 in [**BCT**].) *If f is defined in a full neighborhood of p and $p \in \mathcal{N}$ is strongly noncharacteristic, then*

$$WF_{\mathrm{ha},p}^{\mathcal{N}} f \subset i_{\mathcal{N}}^* T_p^0 \mathcal{M}.$$

COROLLARY V.6.7. (The edge-of-the-wedge theorem.) *If the structure $\mathcal{V}$ on $\mathcal{M}$ is an elliptic structure and f is the boundary value of solutions in two wedges $\mathcal{W}^+, \mathcal{W}^-$ with edge a maximally real $\mathcal{X}$ as in Corollary V.6.5, then f extends to a hypoanalytic function in a full neighborhood of p in $\mathcal{M}$.*

Corollary V.6.7 is a generalization of the classical edge-of-the-wedge theorem of several complex variables. The example of the structure in the plane generated by $\frac{\partial}{\partial y}$ for which the x-axis is maximally real shows that the corollary may not be valid when the structure is not elliptic.

REMARK V.6.8. Notice that in general $i_{\mathcal{N}}^* T_p^0 \mathcal{M} \subseteq \Gamma_p^T(\mathcal{W})^0$.

We will next present a result on the hypoanalytic wave front set of the trace of a solution when the vector field in question is locally integrable.

We consider a smooth vector field $L = X + iY$ where X and Y are real vector fields defined in a neighborhood U of the origin. Let Σ be an embedded hypersurface through the origin in U dividing the set U into two regions, U^+ and U^-, where U^+ denotes the region toward which X is pointing. We assume that L is noncharacteristic on Σ, which means (after multiplying L by i if necessary) that X is noncharacteristic. Our considerations will be local and so after an appropriate choice of local coordinates (x, t) and multiplication of L by a nonvanishing factor, the vector field is given by

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} a_j(x, t) \frac{\partial}{\partial x_j} \tag{V.17}$$

and Σ and U^+ are given by $t = 0$ and $t > 0$ respectively. We will need to consider the integral curve $(-\epsilon, \epsilon) \ni s \mapsto \gamma(s)$ of X that passes through the origin, i.e., $\gamma'(s) = X \circ \gamma(s)$, $\gamma(0) = 0$. It is clear that for small $\epsilon > 0$ and $|s| < \epsilon$, $\gamma(s) \in U^+$ if and only if $s > 0$, so $\gamma(-\epsilon, \epsilon) \cap U^+ = \gamma(0, \epsilon)$. To simplify the notation we will simply write γ^+ to denote $\gamma(0, \epsilon)$.

THEOREM V.6.9. *Let $L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} a_j(x, t) \frac{\partial}{\partial x_j}$ be locally integrable. Suppose $f \in \mathcal{D}'(U_+)$ has a boundary value at $t = 0$ and*

$$Lf(x, t) = 0, \qquad (x, t) \in U^+.$$

Assume that there is a sequence $p_k \in \gamma^+$, $p_k \to 0$ such that for each $k = 1, 2, \dots$, $X(p_k)$ and $Y(p_k)$ are linearly independent. Then there exists a unit vector v such that

$$\xi^0 \in \mathbb{R}^n, \quad v \cdot \xi^0 > 0 \Longrightarrow (0, \xi^0) \notin WF_{\mathrm{ha}}(bf).$$

In particular, the hypoanalytic wave front set of bf at the origin is contained in a closed half-space.

PROOF. Let $Z_1, \dots, Z_m$ be a complete set of smooth first integrals of L near the origin in U and choose new local coordinates (x, t) in which the Z_j's may be written as

$$Z_j(x, t) = x_j + i\Phi_j(x, t), \quad k = 1, \dots, m,$$

with $\Phi(0, 0) = 0$, $\Phi_x(0, 0) = 0$, and $\Phi_{xx}(0, 0) = 0$. For $j = 1, \dots, m$ let $M_j = \sum_{k=1}^{m} b_{jk}(x, t) \frac{\partial}{\partial x_k}$ be vector fields satisfying

$$M_j Z_k = \delta_j^k, \qquad [M_j, M_k] = 0.$$

It is readily checked that for each $j = 1, \dots, m$,

$$[M_j, L] = 0. \tag{V.18}$$

For any C^1 function g, the differential may be expressed as

$$\mathrm{d}g = Lg\,\mathrm{d}t + \sum_{k=1}^{m} M_k g\,\mathrm{d}Z_k. \tag{V.19}$$

Using (V.19) we get:

$$\mathrm{d}(g\mathrm{d}Z_1 \wedge \dots \wedge \mathrm{d}Z_m) = Lg\,\mathrm{d}t \wedge \mathrm{d}Z_1 \wedge \dots \wedge \mathrm{d}Z_m. \tag{V.20}$$

For $\zeta \in \mathbb{C}^m$, $z \in \mathbb{C}^m$, let

$$E(z, \zeta, x, t) = i\zeta \cdot (z - Z(x, t)) - \kappa\langle\zeta\rangle[z - Z(x, t)]^2.$$

Let B denote a small ball centered at 0 of radius r in $\mathbb{R}^m$ and $\phi \in C_0^\infty(B)$, $\phi \equiv 1$ for $|x| \le r/2$, the precise value of r as well as the value of the positive parameter κ in the definition of E will be determined later. We will apply (V.20) to the function

$$g(z, \zeta, x, t) = \phi(x) f(x, t) \mathrm{e}^{E(z, \zeta, x, t)}$$

where (z, ζ) are parameters. We get:

$$\mathrm{d}(g\mathrm{d}Z) = fL\phi\, \mathrm{e}^{E}\mathrm{d}t \wedge \mathrm{d}Z, \tag{V.21}$$

where $\mathrm{d}Z = \mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_m$. Next by Stokes' theorem we have, for $t_1 > 0$ small:

$$\int_B g(z, \zeta, x, 0)\, \mathrm{d}_x Z(x, 0) = \int_B g(z, \zeta, x, t_1)\, \mathrm{d}_x Z(x, t_1) + \int_0^{t_1} \int_B \mathrm{d}(g\mathrm{d}Z). \tag{V.22}$$

We will estimate the two integrals on the right in (V.22) and our aim is to show that for z close to the origin in complex space, both decay exponentially as $\zeta \to \infty$ in a conic neighborhood of ξ^0. Write

$$Z = (Z_1, \ldots, Z_m) = x + i\Phi(x, t), \quad \Phi = (\Phi_1, \ldots, \Phi_m).$$

Observe that, assuming without loss of generality that $|\xi^0| = 1$,

$$\Re E(0, \xi^0, x, t) = \Phi(x, t) \cdot \xi^0 - \kappa(|x|^2 - |\Phi(x, t)|^2).$$

Our main task will be to determine convenient values of t_1, κ and r such that for some $\gamma > 0$

(i) $\Re E(0, \xi^0, x, t_1) \leq -\gamma$ for $|x| \leq r$;
(ii) $\Re E(0, \xi^0, x, t) \leq -\gamma$ for $0 \leq t \leq t_1$ and $r/2 \leq |x| \leq r$.

In order to find the vector v mentioned in the statement of the theorem we will need

Lemma V.6.10. *There exists a sequence $t_k \searrow 0$ such that*

(1) $\Phi(0, t_k) \neq 0$;
(2) $|\Phi(0, t)| \leq |\Phi(0, t_k)|$ *for* $0 \leq t \leq t_k$;
(3) $\lim_{t_k \to 0} \Phi(0, t_k)/|\Phi(0, t_k)| = -v$.

We will postpone the proof of Lemma V.6.10 and continue our reasoning with v given by (3) in Lemma V.6.10. The assumptions on Φ allow us to write

$$\Phi(x, t) = \Phi(0, t) + e(x, t), \qquad |e(x, t)| \leq A|xt| + B|x|^2 \tag{V.23}$$

for some positive constants A and B. Suppose first $\Phi_t(0, 0) \neq 0$, which is the case that is needed for Theorem V.6.4. Then there is $\lambda < 0$ such that $\Phi_t(0, 0) = \lambda v$. Since $\Phi(0, 0) = 0$ and $\Phi_x(0, 0) = 0$, we can write

$$\begin{aligned} \Phi(x, t) \cdot \xi^0 &= \Phi_t(0, 0) \cdot \xi^0 + O(|x|^2 + t^2) \\ &= \lambda v \cdot \xi^0 + O(|x|^2 + t^2). \end{aligned}$$

https://doi.org/10.1017/CBO9780511543067.006 Published online by Cambridge University Press

Hence given $\kappa > 0$, we can find t_1, r and $\gamma > 0$ such that (i) and (ii) above hold. We may therefore assume that $\Phi_t(0,0) = 0$ and so the quotient $|\Phi(0,t)|/t^2 \leq C$ for $(0,t) \in U^+$. We have $\Phi(0,t_k) + |\Phi(0,t_k)|\, v = o(|\Phi(0,t_k)|)$. We recall that by hypothesis $\xi^0 \cdot v > 0$. Hence,

$$\begin{aligned}\Phi(0,t_k)\cdot\xi^0 &= -|\Phi(0,t_k)|\, v\cdot\xi^0 + o(|\Phi(0,t_k)|)\\ &< -|\Phi(0,t_k)|\, v\cdot\xi^0/2 = -c|\Phi(0,t_k)|,\end{aligned}$$

for t_k small and $0 < c < 1$. We now take $r = \alpha|\Phi(0,t_k)|/t_k$, with α and t_k small to be chosen later. Hence, for $|x| \leq r$ and $0 \leq t \leq t_k$, we can choose α small enough (depending on A, B and C but not on t_k) so that

$$\begin{aligned}|e(x,t)| &\leq A\alpha\,|\Phi(0,t_k)|\,\frac{t}{t_k} + B\alpha^2\,\frac{|\Phi(0,t_k)|}{t_k^2}\,|\Phi(0,t_k)|\\ &\leq c\,\frac{|\Phi(0,t_k)|}{2}.\end{aligned} \tag{V.24}$$

This implies that on the support of $\phi(x)$ we have

$$-(1+c)|\Phi(0,t_k)| \leq \Phi(x,t_k)\cdot\xi^0 \leq -\frac{c}{2}|\Phi(0,t_k)|.$$

Let $\kappa = \epsilon/|\Phi(0,t_k)|$. A consequence of (V.23), (V.24) and the fact that $|\Phi(0,t)| \leq |\Phi(0,t_k)|$ for $0 \leq t \leq t_k$ is

$$\begin{aligned}|\Phi(x,t)| &\leq (1+c)|\Phi(0,t_k)|,\\ |\Phi(x,t)|^2 &\leq (1+c)^2|\Phi(0,t_k)|^2,\\ \kappa|\Phi(x,t)|^2 &\leq \epsilon(1+c)^2|\Phi(0,t_k)|\end{aligned} \tag{V.25}$$

for x in the support of $\phi(x)$ and $0 \leq t \leq t_k$. Choosing $\epsilon = c/4(1+c)^2$ (thus, independent of t_k), we get, on the support of $\phi(x)$,

$$\begin{aligned}\Phi(x,t_k)\cdot\xi^0 + \kappa|\Phi(x,t_k)|^2 &\leq -\frac{c}{2}|\Phi(0,t_k)| + \epsilon(1+c)^2|\Phi(0,t_k)|\\ &\leq -\frac{c}{4}|\Phi(0,t_k)|\end{aligned}$$

which leads to an exponential decay in the first integral on the right of (V.22) for z complex near 0 and ζ in a complex conic neighborhood of ξ^0, as soon as we replace t_1 by t_k. For the second integral, note that for $0 \leq t \leq t_k$ and x in the support of ϕ, we may invoke again (V.25) to estimate the size of $|\Phi(x,t)|$ and $\kappa|\Phi(x,t)|^2$ which gives, in view of the previous choice of ϵ,

$$|\Phi(x,t)| + \kappa|\Phi(x,t)|^2 \leq (1+c)|\Phi(0,t_k)| + \frac{c}{4}|\Phi(0,t_k)| \leq (1+2c)|\Phi(0,t_k)|$$

while on the support of $L\phi$, $|x| \geq r/2 = \alpha|\Phi(0, t_k)|/2t_k$ so

$$\kappa|x|^2 \geq \frac{\epsilon\alpha^2|\Phi(0, t_k)|}{4t_k^2}$$

and

$$\Phi(x, t)\cdot\xi^0 - \kappa(|x|^2 - |\Phi(x, t)|^2) \leq \left(1 + 2c - \frac{\epsilon\alpha^2}{4t_k^2}\right)|\Phi(0, t_k)|.$$

Hence, if t_k is chosen sufficiently small, we also get exponential decay for the second integral on the right-hand side of (V.22) with t_1 replaced by t_k.

We have thus shown that the function

$$F(z, \zeta) = \int_B e^{E(z,\zeta,x,0)}\phi(x)f(x, 0)\,d_xZ(x, 0)$$

satisfies an exponential decay of the form

$$|F(z, \zeta)| \leq Ce^{-R|\zeta|}$$

for z near 0 in $\mathbb{C}^m$ and ζ in a complex conic neighborhood of ξ^0 in $\mathbb{C}^m$. In particular, since $Z(0, 0) = 0$ and $d_x\Phi(0, 0) = 0$, by Theorem V.4.8, $(0, \xi^0) \notin WF_{ha}(bf)$.

We now return to the proof of Lemma V.6.10; it is here that we use the fact that X and Y are linearly independent on a sequence $p_k \in \gamma^+$ that approaches the origin. We will show that $\Phi(0, t)$ cannot vanish identically on any interval $(0, \epsilon')$. Let us write $L = \partial_t + a\cdot\partial_x$, $Z = x + i\Phi$, $Z_x = I + i\,{}^t\Phi_x$ and recall that ${}^t\Phi_x$ has small norm for (x, t) close to 0. Now $LZ = 0$ leads to $a = -i(I + i\,{}^t\Phi_x)^{-1}\,\Phi_t$. If $\Phi(0, t)$ vanishes identically on $[0, \epsilon']$ we will have, for those values of t, that $\Phi_t(0, t) = 0$, $a(0, t) = 0$, and $Y(0, t) = \Im a(0, t) = 0$. Furthermore, $X(0, t) = \partial_t$ for $0 < t < \epsilon'$, showing that $\gamma(s) = (0, \ldots, 0, s)$ for $0 < s < \epsilon'$. Thus, $X(\gamma(s))$ and $Y(\gamma(s))$ are linearly dependent for $0 < s < \epsilon'$, a contradiction. Therefore, there exists a sequence $s_k \searrow 0$ such that $|\Phi(0, s_k)| > 0$ and since $\Phi(0, 0) = 0$ there is another sequence $t_k \searrow 0$ satisfying (1) and (2), which in turn possesses a subsequence that satisfies (1), (2), and (3). $\square$

V.7 Application to the F. and M. Riesz theorem

The classical F. and M. Riesz theorem states that a complex measure μ defined on the boundary $\mathbb{T}$ of the unit disk Δ all of whose negative Fourier coefficients vanish, i.e.,

$$\widehat{\mu}(k) = \int_0^{2\pi} \exp(-ik\theta)\,d\mu(\theta) = 0, \qquad k = -1, -2, \ldots, \tag{V.26}$$

is absolutely continuous with respect to Lebesgue measure $d\theta$.

Observe that condition (V.26) is equivalent to the existence of a holomorphic function $f(z)$ defined on Δ whose weak boundary value is μ. In other words, the theorem asserts that if a holomorphic function f on Δ has a weak boundary value bf that is a measure, then in fact $bf \in L^1(\mathbb{T})$.

The F. and M. Riesz theorem has inspired an extensive generalization in two different directions: (i) generalized analytic function algebras, which has as a starting point the fact that (V.26) means that μ is orthogonal to the algebra of continuous functions f on $\mathbb{T}$ that extend holomorphically to F on Δ with $F(0) = 0$; (ii) ordered groups, which emphasizes instead the role of the group structure of $\mathbb{T}$ in the classical result. We will next briefly describe these two directions.

Let A denote the algebra of continuous functions f on $\mathbb{T}$ which have a holomorphic extension F into Δ. The map $f \longmapsto F(0)$ is a continuous homomorphism ϕ of A and so there is a set M_ϕ of measures on $\mathbb{T}$ each of which represents ϕ. In this case, it is clear that the normalized Lebesgue measure $d\theta$ is the unique element of M_ϕ. The kernel of ϕ is the closure of the linear span A_0 of $\exp(in\theta)$, $n > 0$. Hence the condition $\widehat{\mu}(n) = 0$ for all $n < 0$ is equivalent to $\mu \in A_0^\perp$. Such a μ decomposes as $\mu = \mu_a + \mu_s$, where μ_a (resp. μ_s) is absolutely continuous (resp. singular) with respect to $d\theta$, that is, with respect to every measure in M_ϕ. The classical F. and M. Riesz theorem consists of two parts: $\mu \in A_0^\perp \Rightarrow \mu_s \in A_0^\perp$ and $\mu_s \in A_0^\perp \Rightarrow \mu_s = 0$.

For function algebras A on compact Hausdorff spaces X other than $\mathbb{T}$, one looks at continuous homomorphisms ϕ of A and their sets of representing measures M_ϕ. It is known that any measure μ on X can be decomposed as $\mu = \mu_a + \mu_s$, with μ_a (resp. μ_s) absolutely continuous (resp. singular) with respect to every measure in M_ϕ. Under a variety of hypotheses on A or M_ϕ, the implication $\mu \in A_0^\perp \Rightarrow \mu_s \in A_0^\perp$ has been proved and this kind of result turns out to be a crucial ingredient in the theory of generalized analyticity in the algebra A. For more details on this, we mention the book [**BK**] by Klaus Barbey and Heinz Konig.

In the second direction of generalization, one starts with a locally compact abelian group G. Its dual group $\widehat{G}$, written additively, is assumed to contain an order, that is, a semigroup P which satisfies $P \cup -P = \widehat{G}$. Denote by $M(E)$ the convolution algebra of complex Borel measures on G whose Fourier transforms vanish on the subset E of $\widehat{G}$. Each measure μ decomposes as $\mu_a + \mu_s$ with respect to Haar measure on G. In this set-up, the implication $\mu \in M(P) \Rightarrow \mu_s \in M(P)$ has been proved. Under some conditions on G and P, the implication $\mu \in M(P) \Rightarrow \mu_s = 0$ has also been proved. There are also results for compact groups (see [**K1**] and [**K2**]).

Thus, although absolute continuity with respect to Lebesgue measure is a local property, the generalizations mentioned above involve global objects: function algebras and groups.

In the paper [**B**], Brummelhuis used microlocal analysis to prove generalizations of a local version of the theorem of F. and M. Riesz. Among other things, in [**B**] it is shown that if a CR measure on a hypersurface of $\mathbb{C}^n$ is the boundary value of a holomorphic function defined on a side, then it is absolutely continuous with respect to Lebesgue measure. It is easy to use his methods to get a similar result for CR measures on CR submanifolds of any codimension whenever the measure is the boundary value of a holomorphic function defined in a wedge. Another proof of this result was given by Rosay in [**Ro**]. There are also results when the edge of the wedge has lower regularity ([**CR2**] and [**BH8**]). Another way of stating the F. and M. Riesz theorem is to say that if a holomorphic function $f(z)$ defined on a smoothly bounded domain D of the complex plane has tempered growth at the boundary and its weak boundary value is a measure, then the measure is absolutely continuous with respect to Lebesgue measure.

If we regard holomorphic functions as solutions of the homogeneous equation $\overline{\partial} f = 0$, it is natural to ask for which complex vector fields L it is possible to draw the same conclusion for solutions of the equation $Lf = 0$. We will present here an extension of the F. and M. Riesz theorem to all locally integrable, smooth complex vector fields in the plane for smooth domains at the noncharacteristic part of the boundary. We recall that a nowhere vanishing smooth vector field

$$L = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$$

is said to be locally integrable in an open set Ω if each $p \in \Omega$ is contained in a neighborhood which admits a smooth function Z with the properties that $LZ = 0$ and the differential $\mathrm{d}Z \neq 0$.

THEOREM V.7.1. *Suppose* $L = \frac{\partial}{\partial t} + a(x, t)\frac{\partial}{\partial x}$ *is smooth in a neighborhood* U *of the origin in the plane. Let* $U_+ = U \cap \mathbb{R}^2_+$, *and suppose* $f \in C(U_+)$ *satisfies* $Lf = 0$ *in* U_+ *and for some integer* N,

$$|f(x, t)| = O(t^{-N}) \qquad \text{as } t \to 0^+.$$

Assume that L *is locally integrable in* U. *If the trace* $bf = f(x, 0)$ *is a measure, then it is absolutely continuous with respect to Lebesgue measure.*

The existence of the trace $bf = f(x, 0)$ under the assumptions on f follows from theorem 1.1 in [**BH1**]. In his work [**B**], the author gives a microlocal criterion for the absolute continuity of a measure analogous to (V.26) based on Uchiyama's deep characterization of BMO ($\mathbb{R}^n$) [**U**]. Similarly, one of the main steps in the generalization of the F. and M. Riesz theorem is Theorem V.6.9, which involves the location of the hypoanalytic wave front set of the trace of a solution of a locally integrable vector field in $\mathbb{R}^n$. On the other hand, while in the classical case and the generalizations in [**B**] the location of the wave front set of the measure under consideration always satisfies a restrictive hypothesis which leads to absolute continuity, this restriction is not fulfilled in general by the trace of a solution of an arbitrary locally integrable vector field even if the solution is smooth (an example concerning a vector field with real-analytic coefficients is shown in example 4.3 of [**BH1**]). Thus, we need to deal as well with points where the wave front set of the measure may contain all directions; at those points, the vector field L exhibits a behavior close to that of a real vector field (in a sense made precise in Lemma V.7.2 below) and absolute continuity may be proved directly.

LEMMA V.7.2. *Let*

$$L = \frac{\partial}{\partial t} + i\sum_{j=1}^{n} b_j(x, t)\frac{\partial}{\partial x_j}$$

be smooth on a neighborhood $U = B(0, a) \times (-T, T)$ of the origin in $\mathbb{R}^{n+1}$ with $B(0, a) = \{x \in \mathbb{R}^n : \ |x| < a\}$. We will assume that the coefficients $b_j(x, t)$, $j = 1, \dots, n$ are real and that all of them vanish on $F \times [0, T)$, where $F \subset B(0, a)$ is a closed set. Assume that $f \in C(U^+)$ satisfies $Lf = 0$ on $U^+ = B(0, a) \times (0, T)$, has tempered growth as $t \searrow 0$ and its boundary value $bf(x) = f(x, 0)$ is a Radon measure μ. Then the restriction μ_F of μ to F defined on Borel sets $X \subset B(0, a)$ by $\mu_F(X) = \mu(X \cap F)$ is absolutely continuous with respect to Lebesgue measure.

PROOF. If $\tilde{x}$ is an arbitrary point in F we may write

$$b_j(x, t) = \sum_{k=1}^{n}(x_k - \tilde{x}_k)\beta_{jk}(x, \tilde{x}, t)$$

with $\beta_{jk}(x, \tilde{x}, t)$ real and smooth. The proof of theorem 1.1 in [**BH1**] shows that for any $\phi \in \mathbb{C}^\infty(-a, a)$ we have

$$\langle \mu, \phi \rangle = \int f(x, T)\Phi^k(x, T)\,\mathrm{d}s + \int_0^T \int_{B(0,a)} f(x, t)\, L^t\Phi^k(x, t)\,\mathrm{d}x\mathrm{d}t, \quad \text{(V.27)}$$

$$\text{where} \quad \Phi^k(x, t) = \sum_{j=0}^{k} \phi_j(x, t)\frac{t^j}{j!}, \ \ \phi_0(x, t) = \phi(x), \quad \text{(V.28)}$$

$$\text{and}\quad \phi_j(x,t) = -\frac{\partial}{\partial t}\phi^{\epsilon}_{j-1}(x,t) - \sum_{s,\ell=1}^{n}\frac{\partial}{\partial x_s}(x_\ell - \tilde{x}_\ell)\beta_{j\ell}(x,\tilde{x},t)\phi_{j-1}(x,t)$$

for $j = 1, \ldots, k$, with k a convenient and fixed positive integer. We can write

$$\Phi^k(x,t) = A(x,t,D_x)\phi(x) \tag{V.29}$$

where $A(x,t,D_x) = \sum_{|\alpha|\leq k} a_\alpha(x,t)D_x^\alpha$ is a linear differential operator of order k in the x variables with coefficients depending smoothly on t. The coefficients a_α are obtained from the coefficients $b_j(x,t)$ of L by means of algebraic operations and differentiations with respect to x and t. Observe that given any point $\tilde{x} \in F$, $A(x,t,D_x)$ may be written as

$$A(x,t,D_x) = \sum_{|\alpha|\leq k}\sum_{\ell=1}^{n} A_{\alpha\ell}(x,\tilde{x},t)\,((x_\ell - \tilde{x}_\ell)D_x)^\alpha. \tag{V.30}$$

Notice that $|A_{\alpha\ell}(x,\tilde{x},t)| \leq C$, for $x \in B(0,a)$, $\tilde{x} \in F$, $t \in [0,T)$, $|\alpha| \leq k$, and $\ell = 1, \ldots, n$ because the coefficients of L have uniformly bounded derivatives on $B(0,a)$. Hence, we obtain from (V.29) and (V.30) the estimate

$$\left|\int f(x,T)\Phi^k(x,T)\,\mathrm{d}x\right| \leq C\sum_{|\alpha|\leq k+1}\int_{B(0,a)} d(x,F)^{|\alpha|}\,|D_x^\alpha\phi(x)|\,\mathrm{d}x, \tag{V.31}$$

where $d(x,F) = \inf_{\tilde{x}\in F}|x-\tilde{x}|$. We next consider the second integral on the right in (V.27). We will first show that for any j,

$$L^t(\Phi^j) = \frac{\phi_{j+1}}{j!}t^j. \tag{V.32}$$

To see this, note first that (V.32) holds for $j = 0$ from the definition of ϕ_1. To proceed by induction, assume (V.32) for $j \leq m$. Then

$$\begin{aligned} L^t(\Phi^{m+1}) &= L^t(\Phi^m) + L^t\left(\frac{\phi_{m+1}}{(m+1)!}t^{m+1}\right) \\ &= \frac{\phi_{m+1}}{m!}t^m + L^t\left(\frac{\phi_{m+1}}{(m+1)!}t^{m+1}\right) \\ &= \frac{L^t(\phi_{m+1})}{(m+1)!}t^{m+1} \\ &= \frac{\phi_{m+2}}{(m+1)!}t^{m+1}. \end{aligned}$$

This proves (V.32). Next we observe that since the coefficients $b_j(x,t)$ vanish on $F \times [0,T]$, each ϕ_j has the form

$$\phi_j(x,t) = \sum_{|\alpha|\leq j} c_\alpha(x,t)D_x^\alpha\phi(x) \tag{V.33}$$

where the c_α are smooth and satisfy the estimate

$$|c_\alpha| \le C d(x, F)^{|\alpha|}.$$

The form (V.33) is clearly valid for $\phi_0 = \phi$. Assume it is valid for ϕ_j. Then it will also be valid for ϕ_{j+1} since by definition, $\phi_{j+1} = L^t \phi_j$. If we now choose $k = N+1$, (V.32) and (V.33) imply that

$$\begin{aligned} \left| \int_0^T \int_{B(0,a)} f(x,t)\, L^t \Phi^k(x,t)\, \mathrm{d}x\mathrm{d}t \right| &\le \int_0^T \int_{B(0,a)} |f(x,t)| \frac{\phi_{k+1}(x,t)}{k!} t^k\, \mathrm{d}x\mathrm{d}t \\ &\le C \int_0^T \int_{B(0,a)} |\phi_{k+1}(x,t)|\, \mathrm{d}x\mathrm{d}t \qquad \text{(V.34)} \\ &\le C \sum_{|\alpha|\le k+1} \int_{B(0,a)} d(x,F)^{|\alpha|} |D_x^\alpha \phi(x)|\, \mathrm{d}x. \end{aligned}$$

Thus the second integral on the right-hand side of (V.27) also satisfies an estimate of the kind in (V.31). Consider now a compact subset $K \subset F$ with Lebesgue measure $|K| = 0$ and choose a sequence

$$0 \le \phi_\epsilon(x) \le 1 \in C^\infty(B(0,a)) \quad \epsilon \to 0,$$

such that (i) $\phi_\epsilon(x) = 1$ for all $x \in K$; (ii) $\phi_\epsilon(x) = 0$ if $d(x,K) > \epsilon$; (iii) $|D_x^\alpha \phi_\epsilon(x)| \le C_\alpha \epsilon^{-|\alpha|}$. Note that $\phi_\epsilon(x)$ converges pointwise to the characteristic function of K as $\epsilon \to 0$ while $D^\alpha \phi_\epsilon(x) \to 0$ pointwise if $|\alpha| > 0$. Let $\psi \in C^\infty(B(0,a))$ and use (V.31) and (V.34) with $\phi = \phi_\epsilon \psi$ keeping in mind the trivial estimate $d(x,F) \le d(x,K)$. By the dominated convergence theorem,

$$\langle \mu, \phi_\epsilon \psi \rangle \to \int_K \psi\, \mathrm{d}\mu$$

while

$$|d(x,K)^{|\alpha|} D_x^\alpha \phi_\epsilon(x)\|_{L^1} \le \|\epsilon^{|\alpha|} D_x^\alpha \phi_\epsilon(x)\|_{L^1} \to 0$$

as $\epsilon \to 0$ (when $\alpha = 0$ one uses the fact that $|K| = 0$). Thus, (V.31) and (V.34) show that

$$\int_K \psi\, \mathrm{d}\mu = 0, \quad \psi \in C^\infty(B(0,a)),$$

which implies that the same conclusion holds for any continuous function ψ on K (first extend ψ to a compactly supported function on $B(0,a)$ and then approximate the extension by test functions). Thus the total variation $|\mu|(K)$ of μ on K is zero and by the regularity of μ it follows that $|\mu|(F') = 0$ whenever $F' \subset F$ is a Borel set with $|F'| = 0$. This proves that μ_F is absolutely continuous with respect to Lebesgue measure. □

We now consider the set

$$F_0 = \{x \in B(0,a) : \quad \exists \epsilon > 0 : b_j(x,t) = 0,\ \forall t \in [0,\epsilon],\ j = 0,\dots,n\}$$

which is a countable union of the closed sets

$$F_k = \{x \in B(0,a) : \quad b_j(x,t) = 0,\ \forall\, 0 \le t \le \frac{1}{k},\ j = 0,\dots,n\}$$

to which we can apply Lemma V.7.2 and conclude that μ_{F_k} is absolutely continuous with respect to Lebesgue measure. Thus, μ_{F_0} is also absolutely continuous with respect to Lebesgue measure and the Radon–Nikodym theorem implies that there exists $g \in L^1_{\mathrm{loc}}(B(0,a))$ such that

$$\mu_{F_0}(X) = \int_X g(x)\,\mathrm{d}x, \qquad X \subset B(0,a) \text{ a Borel set.}$$

Theorem V.6.9 and Lemma V.7.2 imply Theorem V.7.1:

End of the proof of Theorem V.7.1. We may assume that the vector field has the form

$$L = \frac{\partial}{\partial t} + ib(x,t)\frac{\partial}{\partial x}$$

where $b(x,t)$ is real and smooth on a neighborhood of $U = B(-a,a) \times (-T,T)$ of the origin in $\mathbb{R}^2$. Since the trace bf is a measure, by the Radon–Nikodym theorem, we may write

$$bf = g + \mu$$

where g is a locally integrable function and μ is a measure supported on a set E of Lebesgue measure zero. Suppose x_0 is a point for which we can find a sequence t_j converging to 0 with $b(x_0,t_j) \neq 0$. Let $Z(x,t)$ be a first integral satisfying $Z(x_0,0) = 0$, and $Z_x(x_0,0) = 1$. If $\Im Z_t(x_0,0) \neq 0$, then L will be elliptic in a neighborhood of $(x_0,0)$ and so by the classical F. and M. Riesz theorem, we can conclude that bf is absolutely continuous near $(x_0,0)$. Otherwise, the proof of Theorem V.6.9 shows that the FBI transform with this Z as a first integral and arbitrarily large κ decays exponentially in a complex conic neighborhood of (x_0,ξ_0), for some nonzero covector. By theorem 2.2 in [**BCT**], it follows that near the point x_0, modulo a smooth nonvanishing multiple, the trace bf is the weak boundary value of a holomorphic function F defined on a side of the curve $x \longmapsto Z(x,0)$. But then, again by the classical F. and M. Riesz theorem, bf is locally integrable near x_0, that is, $x_0 \notin E$. Hence the set E is contained in the set

$$F_0 = \{x \in B(0,a) : \quad \exists \epsilon > 0 : b_j(x,t) = 0, \forall t \in [0,\epsilon], j = 0,\dots,n\}.$$

But we already observed that the restriction of bf to F_0 is absolutely continuous with respect to Lebesgue measure which implies that μ is zero.

Notes

For a more detailed account of CR manifolds the reader is referred to the books [**Bog**] and [**BER**]. The book [**T5**] contains a detailed discussion of hypoanalytic manifolds. The characterization of microlocal analyticity (Theorem V.2.14) was proved by Bony. Microlocal analyticity was generalized to microlocal hypoanalyticity in the work [**BCT**]. Several mathematicians have used the FBI transform to study the regularity of solutions in involutive structures and higher-order partial differential equations. Some of these applications can be found in the works [**BCT**], [**BT3**], [**BRT**], [**Hi**] and [**HaT**], [**Sj1**], and [**EG1**]. Theorem V.5.5 was proved by Chemin [**Che**] by using para-differential calculus. The main ideas for the proof presented here are due to Hanges and Treves ([**HaT**]), who proved the analytic version of Chemin's result. Subsequently, Asano [**A**] used the techniques in [**HaT**] to give a new proof of Chemin's result. Most of the material in Section V.6 is taken from a paper of Eastwood and Graham ([**EG1**]). Section V.7 is taken from [**BH1**]. For a generalization of the F. and M. Riesz theorem to systems of vector fields, we refer the reader to [**BH7**].