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DISCRETIONARY STOPPING OF ONE-DIMENSIONAL ITÔ DIFFUSIONS WITH A STAIRCASE REWARD FUNCTION

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Abstract

We consider the problem of optimally stopping a general one-dimensional Itô diffusion X. In particular, we solve the problem that aims at maximising the performance criterion $\mathrm{E}_x[\exp(-\int_0^\tau r(X_s)\,\mathrm{d}s) f(X_\tau)]$ over all stopping times τ, where the reward function f can take only a finite number of values and has a 'staircase' form. This problem is partly motivated by applications to financial asset pricing. Our results are of an explicit analytic nature and completely characterise the optimal stopping time. Also, it turns out that the problem's value function is not C^1, which is due to the fact that the reward function f is not continuous.

Keywords: Optimal stopping; American option; principle of smooth fit; local time

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Secondary 62P05

1. Introduction

This paper is concerned with the problem of optimally stopping the one-dimensional Itô diffusion

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad X_0 = x > 0. \tag{1}$$

Here, W is a standard one-dimensional Brownian motion, and b and σ are deterministic functions such that (1) has a unique weak solution that is nonexplosive and assumes values in the interval $(0, \infty)$. The objective of the discretionary stopping problem is to maximise the performance criterion

$$\mathrm{E}_x\left[\exp\left(-\int_0^\tau r(X_s)\,\mathrm{d}s\right) f(X_\tau)\right]$$

over all stopping times τ, where $r > 0$ is a given deterministic function. The reward function f takes finite values, and is increasing and piecewise constant, so its graph looks like a staircase with a finite number of steps.

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The simplest version of this problem, which arises when $b \equiv 0$ and $\sigma \equiv 1$, i.e. when X is a standard Brownian motion, and when f can take only two values, was solved by Salminen (1985) using Martin boundary theory. The more general version of Salminen's model that arises when X is a Brownian motion with drift was recently solved by Dayanik and Karatzas (2003, Section 6.7) using a new methodology for addressing general one-dimensional discretionary stopping problems by means of a new characterisation of excessive functions that they have developed.

The investigations undertaken here have been partly motivated by the problem of pricing digital options of American type. In this context, the stochastic differential equation (1) models the underlying asset price dynamics, and the discounting rate r can be interpreted as the interest rate (i.e. the short rate). In such financial applications, r would typically be taken to be a strictly positive constant.

We have also been motivated by some general stochastic control-theoretic issues; in particular, it is of interest to observe that the problem we study provides an example in which the so-called 'principle of smooth fit', which suggests that the value function of an optimal stopping problem should be C^1, does not hold. Indeed, it turns out that the value function is not C^1 at all points that belong both to the boundary of the stopping region and to the set of points at which f is discontinuous. This phenomenon has been observed by Salminen (1985) and Dayanik and Karatzas (2003). One of the purposes of this paper is to offer a new way of addressing this issue, based on local-time techniques. At this point, we should mention that our methodology has some similarities with the analysis of Karatzas and Sudderth (1999), who solved a stochastic optimisation problem that combined discretionary stopping with control of the underlying diffusion's drift.

Incidentally, we should note that we have opted to consider the case in which f takes finite rather than infinite values only to simplify the presentation of our results. Simplicity of exposition has also been behind our assumption that f is increasing. Indeed, our construction of the solution to the problem follows a 'stepwise' approach that, at least in principle, can be adapted to account for arbitrary piecewise-constant reward functions.

2. The discretionary stopping problem

We consider a stochastic system whose state process X satisfies (1). We impose conditions (ND)$'$ and (LI)$'$ in Karatzas and Shreve (1988, Section 5.5.C); these conditions are sufficient for (1) to have a weak solution that is unique in the sense of probability law. In particular, we make the following assumption.

Assumption 1. *The deterministic functions $b, \sigma\colon (0, \infty) \to \mathbb{R}$ satisfy the following conditions:*

$$\sigma^2(x) > 0, \quad \textit{for all } x > 0,$$

$$\textit{for all } x > 0, \ \textit{there exists an } \varepsilon > 0 \textit{ such that } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|b(s)|}{\sigma^2(s)}\, \mathrm{d}s < \infty.$$

We also assume that the probability that the diffusion X hits either of the boundaries, 0 or ∞, of its state space in finite time is zero.

Assumption 2. *The diffusion X is nonexplosive.*

Feller's test for explosions provides a necessary and sufficient condition for X to be nonexplosive (see Karatzas and Shreve (1988, Theorem 5.5.29)).

We adopt a *weak* formulation of the optimal stopping problem that we study. In particular, we allow for the stopping strategy to depend, in principle, on the underlying diffusion's initial condition $x > 0$.

Definition 1. Given an initial condition $x > 0$, a *stopping strategy* is any collection $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P}_x, W, X, \tau)$, where $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P}_x, W, X)$ is a weak solution to (1) and τ is an $(\mathcal{F}_t)$-stopping time. We denote by $\mathcal{S}_x$ the family of all stopping strategies associated with a given initial condition $x > 0$.

With each stopping strategy $\mathbb{S}_x \in \mathcal{S}_x$, we associate the performance criterion

$$J(\mathbb{S}_x) = \mathrm{E}_x[\exp(-\Lambda_\tau) f(X_\tau)], \tag{2}$$

where

$$\Lambda_t = \int_0^t r(X_s)\,\mathrm{d}s. \tag{3}$$

The reward function f appearing here is assumed in the present investigation to have the form of a *finite staircase*, given by

$$f(x) = K_0\,\mathbf{1}_{(0,p_1)}(x) + \sum_{j=1}^{N-1} K_j\,\mathbf{1}_{[p_j,p_{j+1})}(x) + K_N\,\mathbf{1}_{[p_N,\infty)},$$

where $0 < p_1 < \cdots < p_N$ and $0 \le K_0 < K_1 < \cdots < K_N$ are given constants. The objective of the discretionary stopping problem is to maximise J over $\mathcal{S}_x$. Accordingly, we define the value function

$$v(x) = \sup_{\mathbb{S}_x \in \mathcal{S}_x} J(\mathbb{S}_x). \tag{4}$$

We shall also need the following assumption.

Assumption 3. *The discounting rate r is locally bounded and there exists a constant $r_0 > 0$ such that $r(x) > r_0$, for all $x > 0$.*

At this point, we should note that Assumption 3 and the fact that f is bounded imply that (2) is well defined when the event $\{\tau = \infty\}$ has positive probability. Indeed, in this case, we define

$$\exp(-\Lambda_\tau) f(X_\tau)\Big|_{\tau=\infty} := \lim_{t\to\infty} \exp(-\Lambda_t) f(X_t) = 0.$$

3. The Hamilton–Jacobi–Bellman (HJB) equation

On the basis of the standard theory of optimal stopping, we expect that the value function v should satisfy the HJB equation

$$\max\left\{\tfrac{1}{2}\sigma^2(x)v''(x) + b(x)v'(x) - r(x)v(x),\ f(x) - v(x)\right\} = 0, \quad \text{for } x > 0. \tag{5}$$

It turns out that the value function v of our discretionary stopping problem, which is defined by (4), has discontinuities in its first derivative. Therefore, it does not suffice in the present situation merely to consider classical solutions to the HJB equation (5). For this reason, we consider solutions to (5) in the sense of distributions. In particular, we consider candidates w for the value function v that are differences of convex functions; for a survey of the results needed here, see Revuz and Yor (1994, Appendix 3). If a function $w\colon (0,\infty) \to \mathbb{R}$ is the difference of two convex functions, then its left-hand derivative w'_- exists and is a function of finite variation, which implies that w'_- is locally bounded. Also, the second distributional derivative of w is a measure, which we denote by $w''(\mathrm{d}x)$. In view of this notation and these observations, we define the measure $\mathcal{L}w$ on $((0,\infty),\mathcal{B}((0,\infty)))$, where $\mathcal{B}((0,\infty))$ is the Borel σ-algebra on $(0,\infty)$, by

$$\mathcal{L}w(\mathrm{d}x) = \tfrac{1}{2}\sigma^2(x)w''(\mathrm{d}x) + b(x)w'_-(x)\,\mathrm{d}x - r(x)w(x)\,\mathrm{d}x. \tag{6}$$

Now, we consider solutions to (5) in the following sense.

Definition 2. A function $w\colon (0,\infty) \to \mathbb{R}$ satisfies the HJB equation (5) if the following conditions hold:

(i) w can be expressed as the difference of two convex functions,

(ii) $-\mathcal{L}w$ is a positive measure,

(iii) $w(x) \ge f(x)$, for all $x > 0$, and

(iv) the support of the measure $\mathcal{L}w$ is contained in the complement of the open set

$$\mathcal{C} := \{x > 0\colon w(x) > f(x)\}. \tag{7}$$

At this point, it is worth noting that the set $\mathcal{C}$ appearing in this definition is indeed open because w is continuous and f is upper semicontinuous.

Following Zervos (2003, Theorem 1), we can now establish conditions that are sufficient for optimality in our problem.

Theorem 1. *In the discretionary stopping problem formulated in Section 2, suppose that Assumptions 1–3 hold, and let $w\colon (0,\infty) \to \mathbb{R}$ be a bounded solution to the HJB equation (5) in the sense of Definition 2. Then, $v = w$ and, given any initial condition $x > 0$, a stopping strategy*

$$\mathbb{S}^*_x = (\Omega^*, \mathcal{F}^*, \mathcal{F}^*_t, \mathrm{P}^*_x, W^*, X^*, \tau^*), \tag{8}$$

such that $(\Omega^*, \mathcal{F}^*, \mathcal{F}^*_t, \mathrm{P}^*_x, W^*, X^*)$ is a weak solution to (1) and

$$\tau^* = \inf\{t \ge 0\colon X^*_t \in \mathcal{C}^{\mathrm{c}}\}, \tag{9}$$

where $\mathcal{C}$ is the open set defined by (7), is optimal.

Proof. Fix an initial condition $x > 0$ and a weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P}_x, W, X)$ to (1). Using the Itô–Tanaka formula (see, e.g. Revuz and Yor (1994, Theorem VI.1.5)), we obtain

$$w(X_t) = w(x) + \int_0^t b(X_s)w'_-(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)w'_-(X_s)\,\mathrm{d}W_s + \frac{1}{2}\int_0^\infty L^a_t w''(\mathrm{d}a), \tag{10}$$

where L^a is the local-time process of the diffusion X at level a. In view of the occupation times formula (Revuz and Yor (1994, Corollary VI.1.6)), we can see that

$$\int_0^\infty L^a_t \frac{b(a)w'_-(a) - r(a)w(a)}{\sigma^2(a)}\,\mathrm{d}a = \int_0^t [b(X_s)w'_-(X_s) - r(X_s)w(X_s)]\,\mathrm{d}s.$$

It follows that (10) is equivalent to

$$w(X_t) = w(x) + \int_0^t r(X_s)w(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)w'_-(X_s)\,\mathrm{d}W_s + A^{\mathcal{L}w}_t,$$

where $\mathcal{L}w$ is the measure defined by (6) and the process $A^{\mathcal{L}w}$ is defined by

$$A_t^{\mathcal{L}w} = \int_0^\infty \frac{L_t^a}{\sigma^2(a)} \mathcal{L}w(\mathrm{d}a), \quad \text{for } t \geq 0. \tag{11}$$

Also, using integration by parts, we calculate

$$\exp(-\Lambda_t)w(X_t) = w(x) + \int_0^t \exp(-\Lambda_s)\,\mathrm{d}M_s + \int_0^t \exp(-\Lambda_s)\,\mathrm{d}A_s^{\mathcal{L}w}, \tag{12}$$

where M is the stochastic integral defined by

$$M_t = \int_0^t \sigma(X_s)w'_-(X_s)\,\mathrm{d}W_s. \tag{13}$$

To proceed further, fix any admissible stopping strategy $\mathbb{S}_x \in \mathcal{S}_x$, and let (τ_m) be a sequence of $(\mathcal{F}_t)$-stopping times such that $\lim_{m\to\infty} \tau_m = \infty$ and the stopped process M^{τ_m}, where M is defined as in (13), is a uniformly integrable martingale. Rearranging terms and taking expectations in (12), we can see that

$$\begin{aligned} \mathrm{E}_x[\exp(-\Lambda_{\tau\wedge\tau_m})f(X_{\tau\wedge\tau_m})] = w(x) + \mathrm{E}_x[\exp(-\Lambda_{\tau\wedge\tau_m})[f(X_{\tau\wedge\tau_m}) - w(X_{\tau\wedge\tau_m})]] \\ + \mathrm{E}_x\biggl[\int_0^{\tau\wedge\tau_m} \exp(-\Lambda_s)\,\mathrm{d}A_s^{\mathcal{L}w}\biggr]. \end{aligned} \tag{14}$$

Now, Definition 2(ii) and the definition (11) imply that $-A^{\mathcal{L}w}$ is an increasing process because the local time L^a is an increasing process. In view of this observation and Definition 2(iii), it follows that

$$\mathrm{E}_x[\exp(-\Lambda_{\tau\wedge\tau_m})f(X_{\tau\wedge\tau_m})] \leq w(x).$$

However, by taking the limit as $m \to \infty$ in this inequality using the dominated convergence theorem, we can see that $J(\mathbb{S}_x) \leq w(x)$, which proves that $v(x) \leq w(x)$.

Now, let $\mathbb{S}_x^*$ be the stopping strategy given by (8)–(9), and let (τ_m^*) be a localising sequence for the local martingale M^*, which is defined as in (13). Since the measure $\mathrm{d}L_t^{a*}$ is supported on the set $\{t \geq 0\colon X_t^* = a\}$, the definition of τ^* implies that

$$L_t^{a*} = 0, \quad \text{for all } t \in [0, \tau^*] \text{ and } a \in \mathcal{C}^{\mathrm{c}},$$

which, in view of Definition 2(iv) and (11), implies that $A_t^{\mathcal{L}w*} = 0$, for all $t \leq \tau^*$. However, combining this observation with (14) and the fact that the set $\{x > 0\colon w(x) = f(x)\}$ is closed, which follows from the upper semicontinuity of f, we can see that

$$\mathrm{E}_x^*[\exp(-\Lambda^*_{\tau^*\wedge\tau_m^*})f(X^*_{\tau^*\wedge\tau_m^*})] = \mathrm{E}_x^*[\exp(-\Lambda^*_{\tau_m^*})[f(X^*_{\tau_m^*}) - w(X^*_{\tau_m^*})]\,\mathbf{1}_{\{\tau_m^*<\tau^*\}}] + w(x).$$

In view of the boundedness of f and w, and the uniform positivity of the discounting factor r (see Assumption 3), we can take the limit as $m \to \infty$ using the dominated convergence theorem, to conclude that $J(\mathbb{S}_x^*) = w(x)$. Combining this result with the inequality $v(x) \leq w(x)$ that we established above, we can see that $v(x) = w(x)$ and that $\mathbb{S}_x^*$ is an optimal strategy.

We shall also need the following result, which is a version of the classical maximum principle, for the construction of an appropriate solution to the HJB equation (5) in Section 4.

Lemma 1. *Suppose that Assumptions 1–3 hold, fix two constants $y, z \in [0, \infty]$ such that $y < z$, and suppose that $g\colon (y, z) \to \mathbb{R}$ is a bounded function such that*

(i) *g is the difference of two convex functions,*

(ii) *$\mathcal{L}g$ is a positive measure on $((y, z), \mathcal{B}((y, z)))$,*

(iii) *if $y > 0$ then $\lim_{x\downarrow y} g(x) \le 0$ and $\lim_{x\downarrow y} |g'_-(x)| < \infty$, and*

(iv) *if $z < \infty$ then $\lim_{x\uparrow z} g(x) \le 0$ and $\lim_{x\uparrow z} |g'_-(x)| < \infty$.*

Then $g(x) \le 0$, for all $x \in [y, z]$.

Proof. In view of Lemma 1(iii)–(iv), if $y > 0$ and/or $z < \infty$, then we extend g by setting $g(x) = \lim_{s\downarrow y} g(s)$ for all $x \le y$, and/or $g(x) = \lim_{s\uparrow z} g(s)$ for all $x \ge z$, and we note that the resulting function on $(0, \infty)$ is the difference of two convex functions.

Now, fix any initial condition $x \in (y, z)$ and any weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P}_x, W, X)$ to (1), and define

$$T = \inf\{t \ge 0\colon X_t \notin (y, z)\}.$$

Also, let (τ_m) be a localising sequence of $(\mathcal{F}_t)$-stopping times for the stochastic integral M that is defined in (13). Since g is the difference of two convex functions, it satisfies (12). Taking expectations in (12), and using Lemma 1(iii)–(iv), we obtain

$$\begin{aligned}\mathrm{E}_x[\exp(-\Lambda_{\tau_m}) g(X_{\tau_m})\, \mathbf{1}_{\{\tau_m < T\}}] &\ge \mathrm{E}_x[\exp(-\Lambda_{\tau_m\wedge T}) g(X_{\tau_m\wedge T})] \\ &= g(x) + \mathrm{E}_x\biggl[\int_0^{\tau_m\wedge T} \exp(-\Lambda_s)\, \mathrm{d}A_s^{\mathcal{L}g}\biggr].\end{aligned}$$

Since local times are increasing processes, we can see that Lemma 1(ii) and the definition of $A^{\mathcal{L}g}$ as in (11) imply that $A^{\mathcal{L}g}_{\cdot\wedge T}$ is an increasing process. It follows that

$$\mathrm{E}_x[\exp(-\Lambda_{\tau_m}) g(X_{\tau_m})\, \mathbf{1}_{\{\tau_m < T\}}] \ge g(x).$$

In view of Assumption 3 and the boundedness of g, we can take the limit as $m \to \infty$ using the dominated convergence theorem, to conclude that $0 \ge g(x)$.

4. The solution to the discretionary stopping problem

We solve the optimal stopping problem that we consider by constructing a solution to the HJB equation (5) that satisfies the requirements of Theorem 1. To this end, we first observe that the general solution to the homogeneous ordinary differential equation (ODE)

$$\tfrac{1}{2}\sigma^2(x) w''(x) + b(x) w'(x) - r(x) w(x) = 0, \tag{15}$$

which is associated with (5), is given by

$$w(x) = A\varphi(x) + B\psi(x),$$

where $A, B \in \mathbb{R}$ are constants. The functions ψ and φ are defined by

$$\begin{aligned}\psi(x) &= \begin{cases} \mathrm{E}_x[\exp(-\Lambda_{T_y})] & \text{for } x < y, \\ (\mathrm{E}_y[\exp(-\Lambda_{T_x})])^{-1} & \text{for } x \ge y, \end{cases} \\ \varphi(x) &= \begin{cases} (\mathrm{E}_y[\exp(-\Lambda_{T_x})])^{-1} & \text{for } x < y, \\ \mathrm{E}_x[\exp(-\Lambda_{T_y})] & \text{for } x \ge y, \end{cases}\end{aligned} \tag{16}$$

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for a given choice of $y > 0$. Here Λ is defined by (3), while T_x and T_y are, respectively, the first hitting times of $\{x\}$ and $\{y\}$. For future reference, note the following remark.

Remark 1. The functions φ and ψ are both strictly positive and C^1, their second derivative exists in the classical sense, φ is strictly decreasing, and ψ is strictly increasing.

Also, the Wronskian $\mathcal{W}$ of φ and ψ, which is identified with the first derivative of the scale function of the diffusion X, is given by

$$\begin{aligned}\mathcal{W}(x) &:= \varphi(x)\psi'(x) - \varphi'(x)\psi(x)\\ &= \mathcal{W}(y)\exp\biggl(-2\int_y^x \frac{b(s)}{\sigma^2(s)}\,\mathrm{d}s\biggr), \qquad \text{for } x > 0 \text{ and for any given choice of } y > 0.\end{aligned} \tag{17}$$

These results have been known for several decades, and can be found in various forms in several references, including Feller (1952), Breiman (1968), Itô and McKean (1974), Karlin and Taylor (1981), Rogers and Williams (2000), and Borodin and Salminen (2002).

Returning now to our optimal stopping problem, we conjecture that the value function satisfies the HJB equation (5) in the *classical sense* outside the set of points at which the discontinuities of f occur, namely inside the set $(0, \infty) \setminus \{p_1, \dots, p_N\}$. This conjecture and the intuitive idea that some of the points $p_1, \dots, p_N$ (e.g. p_N) should belong to the stopping region $\mathcal{C}^{\mathrm{c}}$ of the discretionary stopping problem that we are solving motivate a 'stepwise' approach, the first objective of which is to solve the following two problems.

Problem 1. Given constants $0 < y < z$ and $0 \le K < L$, find a continuous, bounded function $\tilde{w}\colon [y, z] \to \mathbb{R}$ that is a classical solution to (5) with $f(x) = K$, for $x \in (y, z)$, and satisfies the boundary conditions

$$\tilde{w}(y) = K \quad \text{and} \quad \tilde{w}(z) = L.$$

Problem 2. Given constants $z > 0$ and $0 \le K < L$, find a continuous, bounded function $\tilde{w}\colon [0, z] \to \mathbb{R}$ that is a classical solution to (5) with $f(x) = K$, for $x \in (0, z)$, and satisfies the boundary conditions

$$\tilde{w}(0) \ge K \quad \text{and} \quad \tilde{w}(z) = L.$$

The solution to Problem 1 is associated with two qualitatively different possibilities. The first of these arises when $\tilde{w}$ satisfies the ODE (15) for all $x \in (y, z)$, in which case $\tilde{w}$ is given by

$$\tilde{w}(x) = \begin{cases} K & \text{for } x = y,\\ A\varphi(x) + B\psi(x) & \text{for } x \in (y, z),\\ L & \text{for } x = z,\end{cases} \tag{18}$$

where A and B are constants (see Figure 1(a)). The continuity of $\tilde{w}$ at the boundary of $[y, z]$ yields a linear system of two equations for the unknowns A and B, the solution of which is given by

$$A = \biggl(\frac{L}{\psi(z)} - \frac{K}{\psi(y)}\biggr)\biggl(\frac{\varphi(z)}{\psi(z)} - \frac{\varphi(y)}{\psi(y)}\biggr)^{-1}, \tag{19}$$

$$B = \biggl(\frac{L}{\varphi(z)} - \frac{K}{\varphi(y)}\biggr)\biggl(\frac{\psi(z)}{\varphi(z)} - \frac{\psi(y)}{\varphi(y)}\biggr)^{-1}. \tag{20}$$

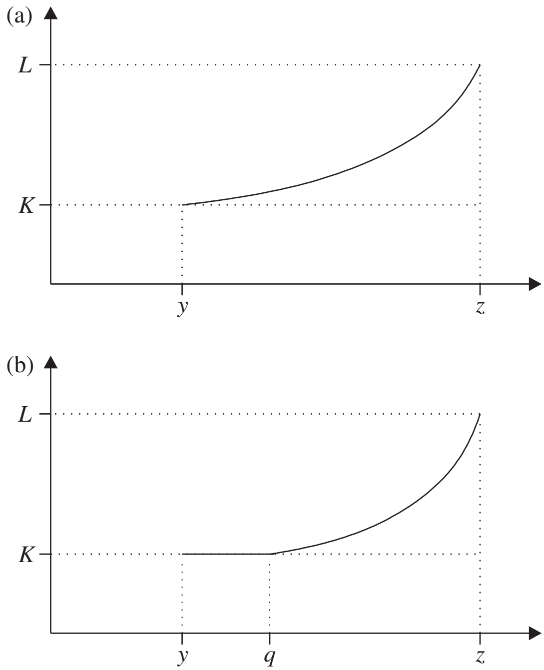

FIGURE 1: Graphs of (a) the first possible solution $\tilde{w}$ and (b) the second possible solution $\tilde{w}$ to the HJB equation (5) that satisfies the boundary conditions $\tilde{w}(y) = K$ and $\tilde{w}(z) = L > K$, when $f \equiv K$ and the independent variable x takes values in the interval (y, z), for $y < z$.

Lemma 2. *The function $\tilde{w}$ defined by (18), where A and B are given by (19) and (20), respectively, provides a solution to Problem 1 if and only if*

$$\frac{\psi'(y)}{\varphi'(y)} \leq \frac{L\psi(y) - K\psi(z)}{L\varphi(y) - K\varphi(z)}. \tag{21}$$

In Appendix A, we present the proofs of results not fully developed in the text.

The second possibility for the solution to Problem 1 arises when there is a point $q \in (y, z)$ such that $\tilde{w}(x) = K$ for $x \in [y, q]$, and $\tilde{w}$ satisfies the ODE (15) for $x \in (q, z)$, i.e.

$$\tilde{w}(x) = \begin{cases} K & \text{for } x \in [y, q], \\ A\varphi(x) + B\psi(x) & \text{for } x \in (q, z), \\ L & \text{for } x = z, \end{cases} \tag{22}$$

where A and B are constants (see Figure 1(b)). To determine A, B, and the free boundary point q, we appeal both to the requirement that $\tilde{w}$ should satisfy (5) in the classical sense in (y, z), which implies that $\tilde{w}$ should be C^1 at q, and to the boundary condition $\tilde{w}(z) = L$. It is straightforward to see that the resulting system of equations is equivalent to the expressions

$$A = \left(\frac{L}{\psi(z)} - \frac{K}{\psi(q)}\right)\left(\frac{\varphi(z)}{\psi(z)} - \frac{\varphi(q)}{\psi(q)}\right)^{-1}, \tag{23}$$

$$B = \left(\frac{L}{\varphi(z)} - \frac{K}{\varphi(q)}\right)\left(\frac{\psi(z)}{\varphi(z)} - \frac{\psi(q)}{\varphi(q)}\right)^{-1}, \tag{24}$$

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and the algebraic equation

$$F(q) = 0, \tag{25}$$

where the function F is defined by

$$F(x) = -[L\psi(x) - K\psi(z)] + [L\varphi(x) - K\varphi(z)]\frac{\psi'(x)}{\varphi'(x)}, \quad \text{for } x \in [y, z).$$

Lemma 3. *Given any $y > 0$, (25) has a solution $q \in (y, z)$ if and only if*

$$\frac{\psi'(y)}{\varphi'(y)} > \frac{L\psi(y) - K\psi(z)}{L\varphi(y) - K\varphi(z)}. \tag{26}$$

If this condition is satisfied, then the solution q to (25) is unique and the function $\tilde{w}$ defined by (22), where A and B are given by (23) and (24), respectively, solves Problem 1.

Now, let us consider Problem 2, which is again associated with two qualitatively different solutions. Since $\lim_{x\downarrow 0}\varphi(x) = \infty$, which follows from the definition, (16), of φ and the assumption that X is nonexplosive, we can see that

$$\tilde{w}(x) = \frac{L}{\psi(z)}\psi(x), \quad \text{for } x \in [0, z], \tag{27}$$

is the appropriate choice for $\tilde{w}$ that is analogous to the solution of Problem 1 developed in Lemma 2, because it is the only bounded solution to the ODE (15) that satisfies the boundary condition $\tilde{w}(z) = L$. Taking note of the fact that ψ is strictly increasing and positive, it is straightforward to see that this choice indeed provides the solution to Problem 2 if $L\psi(0) \geq K\psi(z)$, where $\psi(0) := \lim_{x\downarrow 0}\psi(x)$. When the data in Problem 2 are such that $L\psi(0) < K\psi(z)$, which can be true only if $K > 0$, we are faced with the possibility that the solution to Problem 2 is as in Lemma 3.

Lemma 4. *Equation (25) has a unique solution $q \in (0, z)$ if and only if $L\psi(0) < K\psi(z)$. Moreover, the following two statements are true.*

(a) *If $L\psi(0) \geq K\psi(z)$ then (27) provides a solution to Problem 2.*

(b) *If $L\psi(0) < K\psi(z)$ then the function $\tilde{w}$ defined by (22)–(24), where q is the unique solution to (25) with $y = 0$, solves Problem 2.*

We can now construct a solution to the HJB equation (5) in the sense of Definition 2 that can be identified with the value function of our discretionary stopping problem, using the following algorithm.

Step 1. Set $l = 0$ and define the N-dimensional vectors

$$\boldsymbol{i}^{(l)} = (1, 2, \dots, N-1, N) \quad \text{and} \quad \boldsymbol{\rho}^{(l)} = (p_1, p_2, \dots, p_{N-1}, p_N).$$

Step 2. Define the function $w^{(l)}\colon (0, \infty) \to \mathbb{R}$ by

$$w^{(l)}(x) = w_0^{(l)}(x)\,\mathbf{1}_{(0,\rho_1^{(l)})}(x) + \sum_{j=1}^{\dim \boldsymbol{i}^{(l)}-1} w_j^{(l)}(x)\,\mathbf{1}_{[\rho_j^{(l)},\rho_{j+1}^{(l)})}(x) + K_N\,\mathbf{1}_{[p_N,\infty)},$$

where $w_0^{(l)}$ is the solution to Problem 2 with $z = \rho_1^{(l)}$, $K = K_0$, and $L = K_{i_1^{(l)}}$ given by Lemma 4, while, for $j = 1, \dots, \dim \boldsymbol{i}^{(l)} - 1$, $w_j^{(l)}$ is the solution to Problem 1 with $y = \rho_j^{(l)}$, $z = \rho_{j+1}^{(l)}$, $K = K_{i_j^{(l)}}$, and $L = K_{i_{j+1}^{(l)}}$ given by Lemmas 2 and 3.

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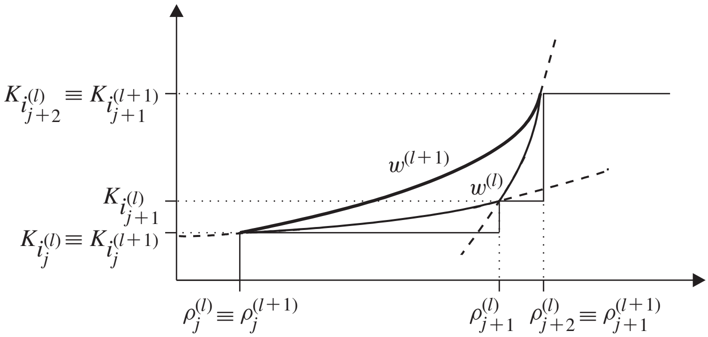

FIGURE 2: Illustration of two successive iterations of the algorithm that provides the solution to the HJB equation (5).

Step 3. Let m be index of the first element of the vector $\boldsymbol{i}^{(l)}$ such that

$$\lim_{x\uparrow\rho^{(l)}_m} \frac{\mathrm{d}}{\mathrm{d}x} w^{(l)}(x) < \lim_{x\downarrow\rho^{(l)}_m} \frac{\mathrm{d}}{\mathrm{d}x} w^{(l)}(x) \quad \Longleftrightarrow \quad \mathcal{L}w^{(l)}(\rho^{(l)}_m) > 0.$$

If no such index exists, then set $w = w^{(l)}$ and STOP. Otherwise, let $\boldsymbol{i}^{(l+1)}$ and $\boldsymbol{\rho}^{(l+1)}$ be the vectors obtained by deleting the mth entries of the vectors $\boldsymbol{i}^{(l)}$ and $\boldsymbol{\rho}^{(l)}$, respectively, set $l = l+1$, and go back to Step 2.

Plainly, this algorithm terminates after, at most, $N-1$ steps and each of the functions $w^{(l)}$ that the algorithm produces is the difference of two convex functions. Also, any functions $w^{(l)}$ and $w^{(l+1)}$ produced by two consecutive iterations of the algorithm satisfy $w^{(l)} \le w^{(l+1)}$, thanks to Lemma 1 (see also Figure 2). Furthermore, we can easily check that the resulting function w satisfies the assumptions of Theorem 1 and, therefore, can be identified with our problem's value function. We conclude with the main result of the paper.

Theorem 2. *The value function of the discretionary stopping problem formulated in Section 2 can be identified with the function w resulting from the algorithm above, and an optimal stopping strategy is given by (8)–(9) in Theorem 1.*

Appendix A.

Proof of Lemma 2. By construction, we will show that $\tilde{w}$ satisfies the HJB equation (5) for $x \in (y, z)$ if we prove that

$$\tilde{w}(x) \ge K, \quad \text{for all } x \in (y, z). \tag{28}$$

To this end, we first note that the facts $y < z$ and $0 \le K < L$, Remark 1, and the definition of B in (20) imply that $B > 0$. In view of this observation and Remark 1, we can see that

$$\tilde{w}'(x) \equiv A\varphi'(x) + B\psi'(x) \ge 0, \quad \text{for all } x \in (y, z), \tag{29}$$

if and only if

$$-\frac{\psi'(x)}{\varphi'(x)} \ge \frac{A}{B}, \quad \text{for all } x \in (y, z). \tag{30}$$

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Now, using the fact that φ and ψ satisfy the ODE (15) and the expression (17) for their Wronskian, we can see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{\psi'(x)}{\varphi'(x)}\right) &= -\frac{\psi''(x)\varphi'(x)-\psi'(x)\varphi''(x)}{[\varphi'(x)]^2}\\ &= \frac{2r(x)\mathcal{W}(x)}{[\sigma(x)\varphi'(x)]^2}\\ &> 0, \quad \text{for all } x \in (y, z). \end{aligned} \tag{31}$$

This inequality shows that (29)–(30) are both true if and only if

$$-\frac{\psi'(y)}{\varphi'(y)} \geq \frac{A}{B}. \tag{32}$$

Moreover, if (32) is not true then $\tilde{w}'(x) < 0$ for all x sufficiently close to y. Combining this observation with the fact that $\tilde{w}(y) = K$, we can see that (28) fails to be true. We conclude that (28) is true if and only if (32) holds, which, in view of the definitions of A and B in (19) and (20), respectively, is equivalent to (21) holding; thus, the proof is complete.

Proof of Lemma 3. In view of Remark 1 and the fact that $0 \leq K < L$, we can see that

$$F(z) = -\psi(z)[L-K] + \varphi(z)[L-K]\frac{\psi'(z)}{\varphi'(z)} < 0.$$

Also, with reference to (31), we calculate

$$F'(x) = [L\varphi(x) - K\varphi(z)]\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi'(x)}{\varphi'(x)}\right) < 0, \quad \text{for } x \in (y, z).$$

It follows that the equation $F(q) = 0$ has a unique solution $q \in (y, z)$ if and only if $F(y) > 0$, which is equivalent to (26).

With regard to its construction, we can see that the function $\tilde{w}$ satisfies the HJB equation (5) for $x \in (y, z)$ if and only if

$$\tilde{w}(x) \geq K, \quad \text{for all } x \in [q, z). \tag{33}$$

Now, following the same reasoning as in the proof of Lemma 2, we obtain

$$\tilde{w}'(x) \geq 0, \quad \text{for all } x \in (q, z) \qquad \Longleftrightarrow \qquad -\frac{\psi'(q)}{\varphi'(q)} \geq \frac{A}{B}.$$

However, combining this observation with the fact that $\tilde{w}$ is C^1 at q, which implies that

$$\tilde{w}(q) = K \quad \text{and} \quad \tilde{w}'(q) \equiv A\varphi'(q) + B\psi'(q) = 0,$$

we can see that (33) is true, and the proof is complete.

Proof of Lemma 4. With reference to the proof of Lemma 3, we can see that (25) has a unique solution $q \in (0, z)$ if and only if

$$\lim_{x\downarrow 0} F(x) \equiv \lim_{x\downarrow 0}\left[K\psi(z) + L\frac{\mathcal{W}(x)}{\varphi'(x)} - K\varphi(z)\frac{\psi'(x)}{\varphi'(x)}\right] > 0, \tag{34}$$

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where $\mathcal{W}$ is the Wronskian of φ and ψ defined by (17). To establish conditions under which this inequality is true, we calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathcal{W}(x)}{\varphi'(x)}\right) = -\frac{2r(x)\mathcal{W}(x)\varphi(x)}{[\sigma(x)\varphi'(x)]^2} < 0.$$

This result, combined with the inequality $\mathcal{W}(x)/\varphi'(x) < 0$, which is true for all $x > 0$, implies that $\lim_{x\downarrow 0}\mathcal{W}(x)/\varphi'(x)$ exists in $(-\infty, 0]$. However, this observation, the fact that $\lim_{x\downarrow 0}\psi(x)$ exists in $[0, \infty)$ because ψ is strictly positive and increasing, and the expression

$$\frac{\varphi(x)\psi'(x)}{\varphi'(x)} = \frac{\mathcal{W}(x)}{\varphi'(x)} + \psi(x), \quad \text{for } x > 0, \tag{35}$$

which follows immediately from the definition (17) of $\mathcal{W}$, imply that

$$\lim_{x\downarrow 0}\frac{\varphi(x)\psi'(x)}{\varphi'(x)} \in (-\infty, 0].$$

Now, we use a contradiction argument to show that this limit is actually equal to 0. To this end, we suppose that

$$\lim_{x\downarrow 0}\frac{\varphi(x)\psi'(x)}{\varphi'(x)} = -2\varepsilon, \quad \text{for some } \varepsilon > 0. \tag{36}$$

This assumption implies that there exists an $x_1 > 0$ such that

$$-\frac{\varphi'(s)}{\varphi(s)} \le \frac{1}{\varepsilon}\psi'(s), \quad \text{for all } s \in (0, x_1].$$

In view of this inequality, we can see that

$$\begin{aligned}\ln\varphi(x) &= \ln\varphi(y) + \int_x^y -\frac{\varphi'(s)}{\varphi(s)}\,\mathrm{d}s \\ &\le \ln\varphi(y) + \frac{1}{\varepsilon}[\psi(y) - \psi(x)], \quad \text{for all } x, y,\ 0 < x < y \le x_1,\end{aligned}$$

which implies that

$$\varphi(x) \le \varphi(y)\exp\left(\frac{1}{\varepsilon}[\psi(y) - \psi(x)]\right), \quad \text{for all } x, y,\ 0 < x < y \le x_1. \tag{37}$$

For fixed y, the right-hand side of this inequality remains bounded as $x \downarrow 0$ because ψ is positive and increasing, which implies that (37) cannot be true because $\lim_{x\downarrow 0}\varphi(x) = \infty$. It follows that (36) is false and, therefore,

$$\lim_{x\downarrow 0}\frac{\varphi(x)\psi'(x)}{\varphi'(x)} = 0 \quad \Longrightarrow \quad \lim_{x\downarrow 0}\frac{\psi'(x)}{\varphi'(x)} = 0.$$

However, these limits and (35) imply that (34) is equivalent to the inequality $K\psi(z) - L\psi(0) > 0$, which establishes the claim regarding the solvability of (25).

Now, Lemma 4(a) is obvious, while Lemma 4(b) follows by a straightforward adaptation of the arguments used to establish the corresponding claim in Lemma 3.

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