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Curves of Small Genus on Certain K3 Surfaces

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Recent workers [**1**, **3**] have proved density theorems about the rational points on K3 surfaces of the form

$$V:\ X_0^4 + cX_1^4 = X_2^4 + cX_3^4$$

for certain non-zero values of c. Their arguments depend on the presence of at least two pencils of curves of genus 1 on V. Unfortunately the values of c for which the argument works are constrained by the need to exhibit explicitly a rational point on V which satisfies certain extra conditions; these in particular require it to lie outside the four obvious rational lines on V. It is therefore natural to ask whether there are other curves of genus 0 or 1 defined over $\mathbf{Q}$ on V. In the case $c = 1$ there are known to be infinitely many such curves (see [**2**]), and for general rational c the quadratic form Q on the Néron–Severi group whose value is the self-intersection number takes the values 0 and -2 infinitely often. Naively one might expect the case $c = 1$ to be typical; but this is not so. The main object of this paper is to prove the following result.

THEOREM 1. *If c is not in $\pm\mathbf{Q}^{*2}$ the only absolutely irreducible nonsingular curves of genus* 0 *on V defined over $\mathbf{Q}$ are the four obvious straight lines. The only such curves of genus* 1 *have degree* 3, 4 *or* 6.

Once this has been proved, it is straightforward to deduce the corresponding result for curves of higher genus.

THEOREM 2. *Suppose that c is not in $\pm\mathbf{Q}^{*2}$. For any fixed g there are only finitely many classes in the Néron-Severi group of V which contain absolutely irreducible nonsingular curves of genus g defined over $\mathbf{Q}$.*

The proof constructs a bound for the degree of such a class in terms of g. I believe that this finiteness property is essentially a number-theoretic phenomenon, and that nothing analogous happens over $\mathbf{C}$.

The first step in proving these theorems is to find the Néron–Severi group $\mathrm{NS}(\mathbf{Q})$ of V. It is well known that $\mathrm{NS}(\mathbf{C}) \otimes \mathbf{Q}$ for V is spanned by the classes of the 48 lines — as is most simply deduced from the fact that the corresponding quadratic form Q has rank 20. (In fact these classes span $\mathrm{NS}(\mathbf{C})$, but this is harder to prove.) It follows that $\mathrm{NS}(\mathbf{Q}) \otimes \mathbf{Q}$ is spanned by the classes of the following divisors, where $\sum$ denotes the sum of the conjugates over $\mathbf{Q}$.

Λ_0 is $\{X_0 = X_2, X_1 = X_3\}$, Λ_1 is $\{X_0 = -X_2, X_1 = X_3\}$,
Λ_2 is $\{X_0 = X_2, X_1 = -X_3\}$, Λ_3 is $\{X_0 = -X_2, X_1 = -X_3\}$;
Γ_6 is $\sum\{X_0 = X_2, X_1 = iX_3\}$, Γ_7 is $\sum\{X_0 = -X_2, X_1 = iX_3\}$,
Γ_8 is $\sum\{X_0 = iX_2, X_1 = X_3\}$, Γ_9 is $\sum\{X_0 = iX_2, X_1 = -X_3\}$,
Γ_{10} is $\sum\{X_0 = iX_2, X_1 = iX_3\}$, Γ_{11} is $\sum\{X_0 = iX_2, X_1 = -iX_3\}$;

Γ_0 is $\sum\{X_0 = rX_1, X_2 = rX_3\}$, Γ_1 is $\sum\{X_0 = rX_1, X_2 = -rX_3\}$,
Γ_4 is $\sum\{X_0 = rX_1, X_2 = irX_3\}$ where $r^4 = -c$;
Γ_2 is $\sum\{X_0 = sX_3, X_2 = sX_1\}$, Γ_3 is $\sum\{X_0 = sX_3, X_2 = -sX_1\}$,
Γ_5 is $\sum\{X_0 = sX_3, X_2 = isX_1\}$ where $s^4 = c$.

We use small greek letters for the corresponding divisor classes, and we denote by π the class of a plane section.

LEMMA 1. *The classes* $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \pi, \gamma_0$ *form a base for* $\mathrm{NS}(\mathbf{Q})$.

Proof. For any class δ the self-intersection number $(\delta \cdot \delta) = 2p_a(\delta) - 2$ is even; moreover, by Riemann–Roch a class with $p_a(\delta) \geqslant 0$ and $d = \deg(\delta) > 0$ is effective. For certain divisor classes δ we shall use the notation

$$\delta = a_0\lambda_0 + a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\pi + a_5\gamma_0. \tag{1}$$

The γ_i with $i > 0$ all have this form. The corresponding values of the a_j, which can be deduced consecutively, are as follows; in each line the function in the last column can be used to derive the corresponding formula:

	a_0	a_1	a_2	a_3	a_4	a_5		
γ_6	-1	0	-1	0	1	0	using	$X_0 - X_2$
γ_7	0	-1	0	-1	1	0	using	$X_0 + X_2$
γ_8	-1	-1	0	0	1	0	using	$X_1 - X_3$
γ_9	0	0	-1	-1	1	0	using	$X_1 + X_3$
γ_{10}	-1	0	0	-1	2	-1	using	$X_0X_3 - X_1X_2$
γ_{11}	2	1	1	2	-2	1	using	$X_0^2 + X_2^2$
γ_3	1	-1	-1	1	0	1	using	$X_0X_1 + X_2X_3$
γ_1	-2	-2	-2	-2	4	-1	using	$X_0X_3 + X_1X_2$
γ_2	-3	-1	-1	-3	4	-1	using	$X_0X_1 - X_2X_3$
γ_4	2	2	2	2	0	0	using	$X_0^4 + cX_1^4$
γ_5	2	2	2	2	0	0	using	$X_0^4 - cX_3^4$

This table shows that $\mathrm{NS}(\mathbf{Q}) \otimes \mathbf{Q}$ is generated by the λ_i, π and γ_0. The intersection-number matrix of these six classes is as follows:

	λ_0	λ_1	λ_2	λ_3	π	γ_0
λ_0	-2	1	1	0	1	4
λ_1	1	-2	0	1	1	0
λ_2	1	0	-2	1	1	0
λ_3	0	1	1	-2	1	4
π	1	1	1	1	4	4
γ_0	4	0	0	4	4	-8

which has determinant -256. Hence if the theorem is false there must be a divisor class (not necessarily primitive) in $\mathrm{NS}(\mathbf{Q})$ which has the form (1) with the a_i all half-integers but not all integers. If a_4 is not an integer, consideration of the $(\delta \cdot \lambda_i)$ shows that just one of a_0 and a_3, and just one of a_1 and a_2, is an integer; and then $(\delta \cdot \delta)$ is not an integer. Again, if a_4 is an integer then for the same reason both or neither of a_0 and a_3, and both or neither of a_1 and a_2, are integers; and if one pair are integers and the other pair are not, then $(\delta \cdot \delta)$ is an odd integer. But $(\delta \cdot \delta)$ must be an even integer. So there are only three cases to consider: the a_i which are not integers must be

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(i) a_0, a_1, a_2, a_3 or
(ii) a_5 or
(iii) a_0, a_1, a_2, a_3, a_5.

If case (i) happened then $\delta = (\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)/2$ would represent a curve of degree 2 and genus 1; but no such curve can exist. Again, the only quadric which contains $\Lambda_0 + \Lambda_3 + \Gamma_{10}$ is $X_0 X_3 - X_1 X_2 = 0$, and $\gamma_0 = 2\pi - \lambda_0 - \lambda_3 - \gamma_{10}$. Hence Γ_0 is the only effective divisor in γ_0. But if case (ii) happened then $\delta = \gamma_0/2$ would be effective; so case (ii) cannot happen. A similar argument using $\gamma_3 = 2\pi - \lambda_1 - \lambda_2 - \gamma_{10}$ shows that (iii) is impossible; for the only quadric which contains $\lambda_1 + \lambda_2 + \gamma_{10}$ is $X_0 X_1 + X_2 X_3 = 0$.

To prove Theorem 1 we need to introduce further effective divisors. To define a typical one of them, note that the plane

$$X_0 - X_2 = r(X_1 - X_3) \quad \text{where} \quad r^4 = -c$$

meets V in the two lines Λ_0 and $\{X_0 = rX_1,\ X_2 = rX_3\}$ and the conic

$$2X_2^2 + 3rX_1X_2 - rX_2X_3 + 2r^2X_1^2 - r^2X_1X_3 + r^2X_3^2 = 0,$$

which is absolutely irreducible. We call the sum of this conic and its three conjugates Γ_{12}; its class is $\gamma_{12} = -4\lambda_0 + 4\pi - \gamma_0$ and its self-intersection number is -8. Similar arguments show that the classes

$$\begin{aligned}
\gamma_{13} &= -4\lambda_3 + 4\pi - \gamma_0, && \\
\gamma_{14} &= -4\lambda_1 + 4\pi - \gamma_1 &&= 2\lambda_0 - 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + \gamma_0, \\
\gamma_{15} &= -4\lambda_2 + 4\pi - \gamma_1 &&= 2\lambda_0 + 2\lambda_1 - 2\lambda_2 + 2\lambda_3 + \gamma_0, \\
\gamma_{16} &= -4\lambda_0 + 4\pi - \gamma_2 &&= -\lambda_0 + \lambda_1 + \lambda_2 + 3\lambda_3 + \gamma_0, \\
\gamma_{17} &= -4\lambda_3 + 4\pi - \gamma_2 &&= 3\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3 + \gamma_0, \\
\gamma_{18} &= -4\lambda_1 + 4\pi - \gamma_3 &&= -\lambda_0 - 3\lambda_1 + \lambda_2 - \lambda_3 + 4\pi - \gamma_0, \\
\gamma_{19} &= -4\lambda_2 + 4\pi - \gamma_3 &&= -\lambda_0 + \lambda_1 - 3\lambda_2 - \lambda_3 + 4\pi - \gamma_0
\end{aligned}$$

are all effective.

Suppose that Δ, whose class δ is given by (1), is an absolutely irreducible curve on V defined over $\mathbf{Q}$. Let $\mathbf{R}^6$ be the space whose coordinates are the a_i and let x denote distance on $\mathbf{R}^6$. Provided that Δ is not one of the lines Λ_i we must have $(\delta \cdot \theta) \geqslant 0$ where θ is any one of the λ_i or γ_j. In what follows we shall ignore the conditions coming from γ_4 and γ_5. Let $\mathcal{P} \subset \mathbf{R}^6$ be the closed set defined by the remaining conditions and let $\mathcal{P}_0 = \mathcal{P} \bigcap \{d = 1\}$. Since

$$2d = \delta \cdot (\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \gamma_6 + \gamma_7) = \delta \cdot (\lambda_0 + \lambda_3 + \gamma_0 + \gamma_{10}),$$

the only point of $\mathcal{P} \bigcap \{d \leqslant 0\}$ is the origin; hence $\mathcal{P}$ is a cone with vertex at the origin, and $\mathcal{P}_0$ is a cross-section of it. The same relations show that $\mathcal{P}_0$ is bounded, so that it is a polytope. Since

$$\left.\begin{aligned}
(\delta \cdot \delta) = {}&\tfrac{1}{4}d^2 - \tfrac{1}{4}(a_0 + a_1 + a_2 + a_3 - 4a_5)^2 - 4a_5^2 \\
&- (a_0 - a_1 - a_2 + a_3 - 2a_5)^2 - (a_0 - a_3)^2 - (a_1 - a_2)^2
\end{aligned}\right\} \tag{2}$$

is convex on $\mathcal{P}_0$, it attains its minimum only at vertices of $\mathcal{P}_0$. To find these, we can afford to proceed in a very vulgar way.

The faces of $\mathcal{P}_0$ are among the 22 hyperplanes $(\delta \cdot \theta) = 0$, where θ is as above. Any vertex is the intersection of 5 or more of these hyperplanes; we therefore examine each of the 26334 sets of 5 hyperplanes. It turns out that for 1964 of these sets the hyperplane equations are

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linearly dependent; and of the remainder only 362 determine a point which lies in $\mathcal{P}_0$. There is some redundancy because some vertices lie on more than 5 faces, and we only obtain 82 distinct vertices. None of these have $(\delta \cdot \delta) < 0$; and 14 have $(\delta \cdot \delta) = 0$. After rescaling to make the a_i integers with highest common factor 1, the corresponding δ become

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3, \quad -\lambda_0 - \lambda_1 - \lambda_2 - \lambda_3 + 2\pi,$$

four like $\pi - \lambda_0$ and two like each of

$$2\lambda_0 + \gamma_0, \quad \lambda_0 - \lambda_1 + \lambda_2 + \lambda_3 + \gamma_0, \quad -\lambda_0 - \lambda_1 - \lambda_2 - 3\lambda_3 + 4\pi - \gamma_0,$$
$$-2\lambda_0 - 2\lambda_2 - 2\lambda_3 + 4\pi - \gamma_0.$$

These last eight classes are mapped to each other by the obvious symmetries of V. This completes the proof of Theorem 1.

We now turn to Theorem 2. Let A_0 be a vertex of $\mathcal{P}_0$ at which $(\delta \cdot \delta) = 0$, and let $c > 0$ be an integer such that $A = cA_0$ has integer coordinates. Let L be any line through A_0 which for small positive x measured from A_0 lies in the closed polytope $\mathcal{P}_0$; and let $\epsilon > 0$ be independent of L and such that the part of L with $0 \leqslant x \leqslant \epsilon$ lies in $\mathcal{P}_0$. On L we have

$$f_2(L, x) = d^2(\delta \cdot \delta)/dx^2 < 0$$

by (2); so $f_1(L) = d(\delta \cdot \delta)/dx > 0$ on L at A_0 because $(\delta \cdot \delta) > 0$ for small positive x. By compactness there is a constant $C_1 > 0$ such that $f_1(L) \geqslant C_1$. Similarly there is a constant C_2 such that $f_2(L, x) \geqslant -C_2$ for $0 \leqslant x \leqslant \epsilon$. Hence

$$(\delta \cdot \delta) \geqslant C_1 x - \tfrac{1}{2}C_2 x^2 \geqslant \tfrac{1}{2}C_1 x \tag{3}$$

provided $0 \leqslant x \leqslant \epsilon_1 = \min(\epsilon, C_1/C_2)$. Of course C_1, C_2, ϵ and ϵ_1 may depend on the choice of A_0.

Now let B be a point of $\mathbf{R}^6$ which corresponds to an absolutely irreducible curve of genus $g > 0$ on V defined over $\mathbf{Q}$, and let B_0 in $\mathcal{P}_0$ be the point on the line OB at which $d = 1$. Let $\mathcal{S}$ be the closed subset of $\mathcal{P}_0$ obtained by removing those points which are a distance less than ϵ_1 from A_0 for some A_0. In $\mathcal{S}$ we have $(\delta \cdot \delta) > 0$, so by compactness there is a constant $C_3 > 0$ such that $(\delta \cdot \delta) \geqslant C_3$ in $\mathcal{S}$. Hence if B_0 is in $\mathcal{S}$ then $d^2 \leqslant (2g-2)/C_3$. If B_0 is not in $\mathcal{S}$ then there is an A_0 such that B_0 is a distance x_0 from A_0 where $0 < x_0 < \epsilon_1$. The coordinates of A_0 have denominator at most d and those of B_0 have denominator at most c; so at least one of the coordinates of B_0 differs from the corresponding coordinate of A_0 by y where $(cd)^{-1} \leqslant |y| \leqslant x_0$. Hence $dx_0 \geqslant c^{-1}$; and by (3)

$$2g - 2 = (\delta \cdot \delta) \geqslant \tfrac{1}{2}C_1 x_0 d^2 \geqslant \tfrac{1}{2}C_1 d c^{-1}$$

at B. Hence in all cases d is bounded when g is fixed; and by (2) this gives bounds for all the a_i.

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