Canad. Math. Bull. Vol. **25** (2), 1982

A NEW PROOF OF THE SNAKE THEOREM

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ABSTRACT. The Snake Theorem (terminology of Krein), due to Karlin in its original form, has been periodically improved. The theorem shows under appropriate conditions the existence of a function p^* from a Tchebycheff space T, with a graph that alternately "touches" the graphs of functions f and g where $f<g$ and $f \le p^* \le g$ on a compact interval $[a, b]$. The number of "touchings" depends upon the dimension of T. In this paper the conditions assumed are not the weakest known (see Gopinath and Kurshan, J. of Approximation Theory **21** (1977), 151–173), but the apparently new proof offered is elementary and fairly short. f and g are not assumed continuous.

1. One of the most beautiful theorems of analysis is the oscillation theorem due to Karlin [3], [4], descriptively termed the "Snake Theorem" by Krein and Nudel'man [6]. A number of alternate proofs of this theorem have appeared in the literature, notably those of Krein and Nudel'man [6], Pinkus [7], and Gopinath and Kurshan [2]. In fact the last authors prove a generalized oscillation theorem wherein Karlin's original hypotheses are substantially weakened. The snake theorem is usually considered a "deep" theorem and the proofs in the literature are either long and complex [2], or rely upon other "deep" theorems such as the Brouwer Fixed Point Theorem [4]. In this note we use a different technique to prove the theorem in an elementary way. Our hypotheses are weaker than those of [4] or [7], but stronger than those of [2]. The reader is referred to [4] for the definitions of Tchebycheff space, nodal zero and related concepts. We assume the uniform topology is used on $C[a, b]$.

2. DEFINITION. Two functions u and v, defined on $[a, b]$, are said to touch at x_0 in $[a, b]$ if there is a sequence $\langle x_i \rangle$ in $[a, b]$ such that $x_i \to x_0$ and $u(x_i) - v(x_i) \to 0$.

THEOREM 1. *Let f and g be two functions defined on $[a, b]$ and let T be an n-dimensional Tchebycheff space of continuous functions on $[a, b]$. Assume there is a function w in T and $\varepsilon > 0$ such that $f(x) + \varepsilon \le w(x) \le g(x) - \varepsilon$ for all $x \varepsilon [a, b]$. Then there is an element p^* in T and points $x_1 < x_2 < \cdots < x_n$ in $[a, b]$*

Received by the editors March 25, 1980.

AMS (1980) Classification: 41A05; 41A99.
Key words: Tchebycheff Space, Snake Theorem.

such that

(a) $f(x) \le p^*(x) \le g(x)$ *for all* x in $[a, b]$
(b) f *touches* p^* at x_i *for* i *odd*, $1 \le i \le n$.
(c) g *touches* p^* at x_i *for* i *even*, $2 \le i \le n$.

Furthermore, there is a function q^* *in* T *that satisfies conditions* (a′), (b′), (c′) *obtained from* (a), (b), (c) *by replacing* p^* *by* q^* *and interchanging* f *and* g *in* (b) *and* (c). *The functions* p^* *and* q^* *are the only functions in* T *satisfying* (a), (b), (c) *and* (a′), (b′), (c′) respectively.

Proof. We show only the existence of p^*, the q^* case being similar. The uniqueness of p^* and q^* can be shown by standard zero-counting arguments; (see [4], p. 70). Without loss of generality we may assume $f < 0$, $g > 0$ with both f and g bounded away from zero. Let $M \subset T$ be the set of all functions p in T for which there exist n points in $[a, b]$, $z_1 < z_2 < \cdots < z_n$ and n sequences $\langle z_j^i \rangle_{i=1}^{\infty}$, $j = 1, 2, \ldots, n$ such that

$$z_j^i \to z_j, \quad 1 \le j \le n \tag{1}$$

$$f(z_j^i) \ge p(z_j) - \frac{1}{i}, \quad j \text{ odd}, 1 \le j \le n, 1 \le i < \infty. \tag{2}$$

$$g(z_j^i) \le p(z_j) + \frac{1}{i}, \quad j \text{ even}, 2 \le j \le n, 1 \le i < \infty. \tag{3}$$

The interpolation property of T guarantees that M is not empty. Define the functional F for each p in M by

$$F(p) = \max\{\sup\{p(x) - g(x) : x \in [a, b]\}, \sup\{f(x) - p(x) : x \in [a, b]\}\}$$

The theorem will be established if we can show that $F(p^*) = 0$ for some $p^* \in M$. Let $\rho = \inf\{F(p) : p \in M\}$ and choose $\langle p_k \rangle$ in M so that $F(p_k) \to \rho$. It can be shown that if $\langle \|p_k\| \rangle$ is unbounded, $\langle p_k(x) \rangle$ is unbounded at all but at most $n-1$ points in $[a, b]$, (see [5]), so we may assume $\langle p_k \rangle$ is bounded. Since T is finite dimensional, we may assume, by taking subsequences if necessary, that $p_k \to \bar{p} \in T$. We show that $\bar{p} \in M$. For each k, let $\langle z_{jk}^i \rangle_{i=1}^{\infty}$ and z_{jk}, for $j = 1, 2, \ldots, n$, be the n sequences and sequence limits associated with p_k from the definition of M. Assume without loss of generality that $|z_{jk}^i - z_{jk}| \le 1/i$ for all k and all j. Again by taking subsequences if necessary, we may assume that $z_{jk} \to \bar{z}_j$ for $1 \le j \le n$, where $\bar{z}_1 \le \bar{z}_2 \le \cdots \le \bar{z}_n$ are in $[a, b]$. Finally, we may assume without loss of generality that $|z_{jk} - \bar{z}_j| \le 1/k$ for all k and j. For each j, we find a sequence $\langle y_j^m \rangle_{m=1}^{\infty}$ and a sequence $\langle s_j^m \rangle_{m=1}^{\infty}$ such that $s_j^m \to 0$ and $y_j^m \to \bar{z}_j$ and either $f(y_j^m) \ge \bar{p}(\bar{z}_j) - s_j^m$ if j is odd or $g(y_j^m) \le \bar{p}(\bar{z}_j) + s_j^m$ if j is even. Indeed define $\langle y_j^m \rangle_{m=1}^{\infty}$ by $y_j^m = z_{jm}^m$ for all m. Then

$$|z_{jm}^m - \bar{z}_j| \le |z_{jm}^m - z_{jm}| + |z_{jm} - \bar{z}_j| \le \frac{1}{m} + \frac{1}{m} = \frac{2}{m}$$

https://doi.org/10.4153/CMB-1982-033-8 Published online by Cambridge University Press

so $y_j^m \to \bar{z}_j$ for each j. Suppose that j is odd. Then

$$f(z_{jm}^m) \geq p_m(z_{jm}) - \frac{1}{m} = \bar{p}(\bar{z}_j) - [(\bar{p}(\bar{z}_j) - \bar{p}(z_{jm})) + (\bar{p}(z_{jm}) - p_m(z_{jm}))] - \frac{1}{m}$$

Setting $s_j^m = [(\bar{p}(\bar{z}_j) - \bar{p}(z_{jm})) + (\bar{p}(z_{jm}) - p_m(z_{jm}))] + 1/m$, we observe that $s_j^m \to 0$ for each j by the uniform convergence of $\langle p_m \rangle$ to $\bar{p}$ and the continuity of $\bar{p}$. A similar treatment is used for j even. By taking subsequences if necessary, we may find sequences that satisfy the conditions in the definition of M. It is clear from the continuity of $\bar{p}$ that we in fact have $\bar{z}_1 < \bar{z}_2 < \cdots < \bar{z}_n$. Thus $\bar{p} \in M$. If $\rho = F(\bar{p}) = 0$ we are done, so suppose for a contradiction that $F(\bar{p}) > 0$. Using standard arguments and the fact that T is a Tchebycheff space, we may find n disjoint intervals $I_j = [a_j, b_j]$, $j = 1, 2, \ldots, n$ and a $\delta > 0$ such that $\bar{z}_j \in I_j$ and $\bar{p}(x) \leq g(x) - \delta$ if $x \notin \cup\{I_j : j \text{ is even}\}$ or $\bar{p}(x) \geq f(x) + \delta$ for $x \notin \cup\{I_j : j \text{ is odd}\}$. Define the n-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ by

$$\sigma_j = \begin{cases} 1 & \text{if } j \text{ is even and } \bar{p}(x) > g(x) \text{ for some } x \in I_j \\ -1 & \text{if } j \text{ is odd and } \bar{p}(x) < f(x) \text{ for some } x \in I_j \\ 0 & \text{otherwise} \end{cases}$$

Let $d_j = \frac{1}{2}(b_j + a_{j+1})$ for $j = 1, 2, \ldots, n-1$. It is possible to find an element h of T such that h has a nodal zero at each d_j for which $|\sigma_j - \sigma_{j+1}| \neq 1$ and has no other zeros in [a, b]. ([8], theorem 6.5). By supposition, for some $j = j^*$, $\sigma_j^* \neq 0$. For sufficiently small $\varepsilon > 0$, $\hat{p} = \bar{p} - \varepsilon\sigma_j^* \operatorname{sgn}(h(\bar{z}_j^*))h$ is such that $0 < \sup\{\hat{p}(x) - g(x) : x \in I_j^*\} < \sup\{\bar{p}(x) - g(x) : x \in I_j^*\}$ if j^* is even or $0 < \sup\{f(x) - \hat{p}(x) : x \in I_j^*\} < \sup\{f(x) - \bar{p}(x) : x \in I_j^*\}$ if j^* is odd. From the definitions of h and $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, we may further deduce by working step by step from I_j^* through the intervals to the right and left of I_j^*, that for sufficiently small $\varepsilon > 0$, inequalities of the above form hold for all I_j with $\sigma_j \neq 0$, that $\hat{p} \in M$, that $\hat{p}(x) < g(x)$ if $x \notin \cup\{I_j : j \text{ even}\}$ and $\hat{p}(x) > f(x)$ if $x \notin \cup\{I_j : j \text{ is odd}\}$. These facts together imply that $F(\hat{p}) < \rho$, a contradiction and the theorem follows.

We comment that if f and g are assumed continuous, the proof is substantially simplified.

The technique used above can also be employed to prove an old result of Davis [1], though it is not clear if the many more recent generalizations of this result can be similarly achieved.

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https://doi.org/10.4153/CMB-1982-033-8 Published online by Cambridge University Press