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PERIOD POLYNOMIALS AND CONGRUENCES FOR HECKE ALGEBRAS

by WINFRIED HOHNEN

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Using the Eichler-Shimura isomorphism and the action of the Hecke operator T_2 on period polynomials, we shall give a simple and new proof of the following result (implicitly contained in the literature): let f be a normalized Hecke eigenform of weight k with respect to the full modular group with eigenvalues λ_p under the usual Hecke operators T_p (p a prime). Let K_f be the field generated over $\mathbf{Q}$ by the λ_p for all p. Let $\mathcal{P}$ be a prime of K_f lying above 5. Then

$$\lambda_2 \not\equiv 0 \pmod{\mathcal{P}}.$$

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1. Introduction

Let S_k be the space of cusp forms of weight k w.r.t. $\Gamma_1 = SL_2(\mathbf{Z})$ and let $f \in S_k$ be a normalized Hecke eigenform with eigenvalues λ_p under the usual Hecke operators T_p (p a prime). Recall that λ_p is a (totally) real algebraic integer and that the field K_f generated over $\mathbf{Q}$ by the λ_p for all p is a number field, i.e. is of finite degree over $\mathbf{Q}$.

Using the Eichler-Shimura isomorphism and the action of Hecke operators on period polynomials, we shall give a simple proof of the following result, implicitly contained in [6]:

Theorem. *Let $\mathcal{P}$ be a prime of K_f lying above 5. Then*

$$\lambda_2 \not\equiv 0 \pmod{\mathcal{P}}. \tag{1}$$

Corollary. *The Hecke operator T_2 on S_k is an isomorphism.*

Using the action of Hecke operators on periods, in [2] various congruences for λ_p modulo primes $l \leq 5$ and modulo small powers of 2 and 3 are proved. In particular, it is shown that

$$\lambda_p \equiv 1 + p \pmod{l} \qquad (l = 3, 5;\ p \equiv \pm 1 \pmod{l}), \tag{2}$$

and it is conjectured that

$$\lambda_p \equiv 1 + p \pmod 7 \qquad (p \equiv \pm 1 \pmod 7). \tag{3}$$

We would like to point out that (1) as well as (2) and (3) immediately follow from results given in [6]. In fact, let $\tilde{M}$ be the algebra of modular forms on Γ_1 modulo l. Then according to [6,§5], for $l = 3, 5, 7$ the only eigenvalue systems $\{a_p\}_{p \text{ prime}, p \neq l}$ in $\overline{\mathbf{F}}_l$ of the Hecke algebra $\langle T_p \rangle_{p \text{ prime}, p \neq l}$ acting on $\tilde{M} \otimes_{\mathbf{F}_l} \overline{\mathbf{F}}_l$ are those coming from Eisenstein series and "θ-powers" of the latter, i.e. the eigenvalues satisfy

$$a_p = p^m + p^n, \tag{4}$$

where $m, n \in \mathbf{Z}/(l-1)\mathbf{Z}$ with $m + n$ odd.

Note that (4) for $l = 5$ implies

$$\lambda_p \not\equiv 0 \pmod 5 \qquad (p \not\equiv -1 \pmod 5)). \tag{5}$$

The proof of (4) given in [6] is based on the existence of the mod l Galois representation attached to the eigenvalue system $\{a_p\}$ and its properties (cf. [1]), which is a deeper fact.

Our reason to look at the special Hecke operator T_2 from the point of view of periods, is that it is of smallest degree and partly as a consequence its action on period polynomials modulo 5 turns out to be very simple. It might be interesting to find a proof of (5) in a similar way as here for primes $p \not\equiv \pm 1 \pmod 5$ other than 2. Here eventually results given in [7] on Hecke operators and period polynomials (generalizing previous results of [5]) might be useful.

2. Proof of theorem

For basic facts on the Eichler-Shimura-Manin theory we refer to [4, 3].

If w is an even integer and K is a field of characteristic different from 2, we denote by $\mathbf{P}_w(K)$ the K-vector space of polynomials of degree $\leq w$ with coefficients in K. The group $GL_2(K)$ operates on $\mathbf{P}_w(K)$ by

$$\left(P|_{-w}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(X) := (cX+d)^w P\left(\frac{aX+b}{cX+d}\right) \qquad \left(P(X) \in \mathbf{P}_w(K), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)\right),$$

and this action extends to an action of the group algebra of $GL_2(K)$ over K. We usually simply write "$|$" instead of "$|_{-w}$". Note that the element -1 operates trivially.

We set $\epsilon := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then ϵ acts by $P(X) \mapsto P(-X)$ and splits up $\mathbf{P}_w(K)$ into the direct sum of the spaces $\mathbf{P}_w^+(K)$ and $\mathbf{P}_w^-(K)$ of even and odd polynomials, respectively.

Now let $w := k-2$. For $g \in S_k$ we denote by

$$r_n(g) := \int_0^{i\infty} g(z) z^n \, dz \qquad (n \in \mathbf{Z}, 0 \le n \le w)$$

the n-th period of g and put

$$r_g(X) := \sum_{n=0}^{w} (-1)^n \binom{w}{n} r_n(g) X^{w-n}.$$

We let

$$r_g^-(X) := -\sum_{0<n<w, n \text{ odd}} \binom{w}{n} r_n(g) X^{w-n}$$

and

$$r_g^+(X) := \sum_{0 \le n \le w, n \text{ even}} \binom{w}{n} r_n(g) X^{w-n}$$

be the odd and even part of $r_g(X)$, respectively.

We let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = TS \in \Gamma_1$ and for any field K of characteristic different from 2 put

$$\mathbf{V}_w(K) := \{P \in \mathbf{P}_w(K) \mid P|(1+S) = P|(1+U+U^2) = 0\}.$$

The subspace $\mathbf{V}_w(K)$ is invariant under ϵ, and we set

$$\mathbf{V}_w^{\pm}(K) := \mathbf{V}_w(K) \cap \mathbf{P}_w^{\pm}(K).$$

One has $r_g^{\pm}(X) \in V_w^{\pm}(\mathbf{C})$.

Let $S_k(\mathbf{R})$ be the $\mathbf{R}$-vector space consisting of cusp forms of weight k w.r.t. Γ_1 with real Fourier coefficients. Let $g \in S_k(\mathbf{R})$. Then $r_n(g)$ is real for n odd and is purely imaginary for n even. According to the main result of Eichler-Shimura the map

$$r^- : S_k(\mathbf{R}) \to \mathbf{V}_w^-(\mathbf{R}) = \mathbf{V}_w^-(\mathbf{Q}) \otimes \mathbf{R}, g \mapsto r_g^-(X)$$

is an isomorphism, and the map

$$r^+ : S_k(\mathbf{R}) \to \mathbf{V}_w^+(\mathbf{R}) = \mathbf{V}_w^+(\mathbf{Q}) \otimes \mathbf{R}, g \mapsto \frac{1}{i} r_g^+(X)$$

gives an isomorphism of $S_k(\mathbf{R})$ with a subspace of $\mathbf{V}_w^{+}(\mathbf{R})$ of codimension 1, defined over $\mathbf{Q}$ and not containing the element $p_0(X) := X^w - 1$.

According to Manin, the action of Hecke operators T_p on $S_k(\mathbf{R})$ can be described explicitly at the level of their periods. In particular, for $p=2$, one finds by direct calculations or by referring to [7,§ 6, examples] that

$$r^{\pm}_{g|T_2} = r^{\pm}_{g}\left|\left(\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix} + \begin{pmatrix}2 & 0\\1 & 1\end{pmatrix} + \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}\right)\right..$$

To prove the theorem it is sufficient to show the following

Proposition. *Let $K_{f,\mathcal{P}}$ be the completion of K_f at $\mathcal{P}$. Let P be a polynomial of degree $\le w$ with coefficients in the ring of $\mathcal{P}$-adic integers of $K_{f,\mathcal{P}}$. Suppose that*

$$P|\epsilon \equiv (-1)^{\frac{w}{2}-1}P \pmod{\mathcal{P}}, \tag{6}$$

$$P|S \equiv -P \pmod{\mathcal{P}} \tag{7}$$

and

$$P\left|\left(\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix} + \begin{pmatrix}2 & 0\\1 & 1\end{pmatrix} + \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}\right)\right. \equiv 0 \pmod{\mathcal{P}}. \tag{8}$$

Then $P \equiv 0 \pmod{\mathcal{P}}$.

Proof. First observe the matrix congruences

$$\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix} \equiv 2\epsilon\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix} \pmod 5,$$

$$\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix} \equiv -ST^2S\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix} \pmod 5$$

and

$$\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix} \equiv 2\epsilon T^2\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix} \pmod 5.$$

Using them we obtain from (8)

$$P|(1+(2\epsilon)+ST^2S+(2\epsilon)T^2)|\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix} \equiv 0 \pmod{\mathcal{P}},$$

hence (since $P|ST^2S \equiv -P|T^2S \pmod{\mathcal{P}}$ by (7)) we deduce that

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$$P|(1+2^w\epsilon - T^2S + 2^w\epsilon T^2) \equiv 0 \pmod{\mathcal{P}}.$$

Since $2^w \equiv (-1)^{\frac{w}{2}} \pmod 5$, it follows by (6) that

$$P|T^2ST^{-2} \equiv -P \pmod{\mathcal{P}}. \tag{9}$$

Write

$$P(X) = \sum_{n=0}^{w} a_n X^n.$$

Since $T^2ST^{-2} \equiv \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix} \pmod 5$, we find from (9) that

$$\sum_{n=0}^{w} a_n (2X)^n (X-2)^{w-n} \equiv -\sum_{n=0}^{w} a_n X^n \pmod{\mathcal{P}}. \tag{10}$$

We will first suppose that $\frac{w}{2}$ is even. Then $P \pmod{\mathcal{P}}$ is odd, whence putting $X=2$ in (10) we deduce

$$0 \equiv -P(2) \pmod{\mathcal{P}}.$$

Now putting $X = 1$ in (10) it follows that

$$0 \equiv -P(2) \equiv \sum_{0<n<w, n \text{ odd}} a_n 2^n (-1)^{w-n} \equiv -P(1) \pmod{\mathcal{P}}.$$

Thus $P \pmod{\mathcal{P}}$ has zeros $X \equiv 0, \pm 1, \pm 2 \pmod{\mathcal{P}}$, hence we have

$$P(X) \equiv (X^5 - X) P_1(X) \pmod{\mathcal{P}} \tag{11}$$

with P_1 a polynomial of degree $\le w-6$ with coefficients in the ring of $\mathcal{P}$-adic integers of $K_{f,\mathcal{P}}$, $P_1 \pmod{\mathcal{P}}$ even.

Now observe that $X^5 - X \pmod 5$ is $|_6$ invariant under all elements in $SL_2(\mathbf{F}_5)$ (indeed, this is true for the images of S and T in $SL_2(\mathbf{F}_5)$ which are generators). From (7), (8) and (11) it therefore follows that

$$P_1|S \equiv -P_1 \pmod{\mathcal{P}} \tag{12}$$

and

$$P_1|T^2ST^{-2} \equiv -P_1 \pmod{\mathcal{P}}.$$

Put $w_1 := w-6$. Then $\frac{w_1}{2}$ is odd. Writing

$$P_1(X) = \sum_{n=0}^{w_1} b_n X^n$$

we have as above

$$\sum_{n=0}^{w_1} b_n (2X)^n (X-2)^{w_1-n} \equiv -\sum_{n=0}^{w_1} b_n X^n \pmod{\mathcal{P}}. \tag{13}$$

From (12) we obtain

$$(-1)^n b_{w_1-n} \equiv -b_n \pmod{\mathcal{P}}. \tag{14}$$

Since P_1 is even and $\frac{w_1}{2}$ is odd, congruence (14) implies

$$P_1(1) \equiv \sum_{0 \le n \le w_1, n \text{ even}} b_n \equiv 0 \pmod{\mathcal{P}}.$$

Setting $X = 1$ in (13) we get from this

$$\sum_{0 \le n \le w_1, n \text{ even}} b_n 2^n (-1)^{w_1-n} \equiv 0 \pmod{\mathcal{P}},$$

i.e.

$$P_1(2) \equiv 0 \pmod{\mathcal{P}}.$$

Finally, setting $X = 2$ in (13) we find

$$b_{w_1} 4^{w_1} \equiv -P_1(2) \equiv 0 \pmod{\mathcal{P}},$$

$$b_{w_1} \equiv 0 \pmod{\mathcal{P}}, \tag{15}$$

whence by (14)

$$b_0 \equiv 0 \pmod{\mathcal{P}}.$$

Hence again $P_1 \pmod{\mathcal{P}}$ has zeros $X \equiv 0, \pm 1, \pm 2 \pmod{\mathcal{P}}$, i.e.

$$P_1(X) \equiv (X^5 - X) P_2(X) \pmod{\mathcal{P}}$$

with P_2 of degree $\le w_1 - 6$ (by (15)) and $P_2 \pmod{\mathcal{P}}$ odd. Note that $\frac{w_1-6}{2}$ is even.

We now see that we can proceed inductively: after finitely many steps we arrive at a polynomial P_r of degree ≤ 4 whose reduction modulo $\mathcal{P}$ has zeros $0, \pm 1, \pm 2 \pmod{\mathcal{P}}$. Thus P_r must be zero modulo $\mathcal{P}$, hence P is zero modulo $\mathcal{P}$ as was to be shown.

If $\frac{w}{2}$ is even, we proceed in the same way as above, interchanging even and odd steps. This proves the proposition.

As was kindly pointed out to the author by the referee, a much simpler proof of the theorem is possible if one also makes use of the fact that $P|(1+U+U^2)=0$. Indeed, the expression on the left-hand side of (8) is congruent (mod $\mathcal{P}$) to

$$P|(1+2\cdot\epsilon)\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}+P|(SX+2\cdot\epsilon)\begin{pmatrix}2 & 2\\ 0 & 1\end{pmatrix},$$

where $X=T^2ST^{-2}=\begin{pmatrix}2 & -5\\ 1 & -2\end{pmatrix}$. Hence equations (6)–(8) imply (9), i.e. $P|X\equiv -P \pmod{\mathcal{P}}$. From this and $P|\epsilon\equiv -P \pmod{\mathcal{P}}$ it follows that $P \pmod{\mathcal{P}}$ is invariant under $(\epsilon X)^2\equiv\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix} \pmod{\mathcal{P}}$ and hence anti-invariant under $S(\epsilon X)^2\equiv U \pmod{\mathcal{P}}$, which together with $P|(1+U+U^2)=0$ immediately gives a contradiction.

3. Concluding remark

As already indicated in the Introduction, we would expect that the method introduced here can be used to study much more general congruences than (1). For example, it might be possible to give a proof of (2), (3) or (4) along the same lines as here (we have not worked out any details so far, however). Very roughly speaking, this means that if one knows an explicit representative of the action of some Hecke operator T_p given by a linear combination of matrices of determined p, then one tries to analyze in detail the quotient of the group ring of $GL_2(\mathbf{Z}/\ell\mathbf{Z})$ by the ideal generated by $\epsilon\pm 1$, $S+1$, $1+U+U^2$, and $T_p-\lambda_p$ for some hypothetical eigenvalue $\lambda_p\in\mathcal{O}_f/\mathcal{P}$ where $\mathcal{O}_f$ is the ring of integers of K_f.

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Mathematisches Institut
Universität Heidelberg
Im Neuenheimer feld 288
69120 Heidelberg
Germany