Eigenvalues, *K*-theory and Minimal Flows

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Abstract. Let (Y, T) be a minimal suspension flow built over a dynamical system (X, S) and with (strictly positive, continuous) ceiling function $f: X \rightarrow \mathbb{R}$. We show that the eigenvalues of (Y, T) are contained in the range of a trace on the K_0-group of (X, S). Moreover, a trace gives an order isomorphism of a subgroup of $K_0(C(X) \rtimes_S \mathbb{Z})$ with the group of eigenvalues of (Y, T). Using this result, we relate the values of t for which the time-t map on the minimal suspension flow is minimal with the K-theory of the base of this suspension.

Introduction

It is known that there is a relation between the eigenvalues of a minimal continuous flow and the values of t for which the time-t map is minimal; *cf.* [8, 4.24.1]. On the other hand, it is known that the group of eigenvalues of a standard suspension built over a minimal group rotation is equal to the range of the unique normalized trace on the K_0-group of this group rotation; see [4, 12, 18]. The purpose of this paper is to use the eigenvalues of a minimal suspension flow (with ceiling function not necessarily constant) as a link to get a correspondence between the set of values of t for which the time-t map is minimal and the K-theory of the base of this suspension.

S. Schwartzman [21] made one of the first applications of algebraic topology to topological dynamics by associating to each continuous flow (Y, T) with T-invariant Borel probability measure ν, an "average asymptotic cycle" $A_\nu: H^1(Y, \mathbb{Z}) \rightarrow \mathbb{R}$. It is proved in a theorem [21, §5] that the range of A_ν contains the eigenvalues of (Y, T). Schwartzman also noted a correspondence between cross sections to (Y, T) and a certain (multiplicative) semigroup C in $H^1(Y, T)$ which we will call, following the terminology of [12], the *positive Schwartzman cone* of $H^1(Y, \mathbb{Z})$. J. Packer [13], building upon the work of Schwartzman [21], Connes [1], and Rieffel [20], showed that when (Y, T) is the standard suspension flow built over a dynamical system (X, S), the restriction of the Connes isomorphism $\Phi: K_1(C(Y)) \rightarrow K_0(C(X) \rtimes_S \mathbb{Z})$ to the positive Schwartzman cone C of $H^1(Y, \mathbb{Z})$ gives an injective semigroup homomorphism into the positive cone $K_0(C(X) \rtimes_S \mathbb{Z})^+$ of $K_0(C(X) \rtimes_S \mathbb{Z})$. Furthermore, Packer showed that if τ is a normalized trace on $C(X) \rtimes_S \mathbb{Z}$, then the images of $\tau \circ \Phi$ and A_ν coincide on $C \subset K_1(C(Y))$, and so $(\tau \circ \Phi)(C)$ contains all strictly positive eigenvalues of (Y, T). This fact made us suspect (something we prove in this work) the existence of a suitable (multiplicative) subgroup G of $H^1(Y, Z)$ such that a trace on $C(X) \rtimes_S \mathbb{Z}$ gives an isomorphism of $\Phi(G) \subset K_0(C(X) \rtimes_S \mathbb{Z})$ with the group of eigenvalues of (Y, T).

This paper is divided into three sections. Section 1 introduces notation, terminology and some results needed in the rest of the paper. The main results are stated and

proved in Section 2. In the last section these results are illustrated by examples. The idea of a connection between the values of t for which the time-t map on a minimal suspension flow is minimal, and the K-theory of the base of this suspension, was proposed in the author's Ph.D. thesis, written under the supervision of Professor N. C. Phillips.

1 Preliminaries

In this section we will provide the notation, definitions and results needed in the rest of the paper. Most of the results described here are known. The reader will be referred to their place in the literature whenever we were able to find a reference.

Consider a compact metric space Y and a continuous map $T: Y \times \mathbb{R} \to Y$. Given (y, r) in $Y \times \mathbb{R}$, we choose to write $T(y, t)$ as $T^t(y)$. If $T^s \circ T^t = T^{s+t}$ for all $s, t \in \mathbb{R}$ and $T^0 = \mathrm{Id}_Y$, then we say that (Y, T) is a *continuous flow*.

A continuous function $\chi: Y \to S^1$ is said to be an eigenvector with respect to the flow (Y, T) provided there is a real number λ such that

$$\chi\left(T^t(y)\right) = e^{2\pi i \lambda t} \chi(y), \quad \forall y \in Y, \ \forall t \in \mathbb{R}.$$

We call λ the eigenvalue associated with χ or say that χ is an eigenvector for λ.

Definition 1.1 The set of eigenvalues of (Y, T) is denoted by $\Lambda(Y, T)$. We also set $\Lambda(Y, T)^+ = \{\lambda \geq 0 : \lambda \in \Lambda(Y, T)\}$.

Let (Y, T) be a continuous flow and let M be a subset of Y. We say that M is T-invariant if $T^t(M) = M$ for all $t \in \mathbb{R}$, and we say that M is *minimal* (with respect to T) if it is nonempty, closed and T-invariant, while no proper subset of M has these properties, *i.e.,* $\varnothing$ and M are the only closed T-invariant subsets of M. When Y itself is a minimal subset of Y, then (Y, T) is called a *minimal flow*. Recall that the orbit of $y \in Y$ under T is the set $\{T^t(y): t \in \mathbb{R}\}$. A continuous flow is minimal if and only if every orbit is dense, *cf.* [22, Lemma II.3.7].

Proposition 1.2 *Let (Y, T) be a minimal continuous flow.*
(i) *If χ_1 and χ_2 are two eigenvectors for the same eigenvalue λ of (Y, T), then there exists c in S^1 such that $\chi_1 = c\chi_2$.*
(ii) *The set $\Lambda(Y, T)$ is a countable subgroup of $\mathbb{R}$.*

Proof This is a standard result. A proof can be obtained by imitating the techniques in the proof of [23, Theorem 5.17]. See also [8, 2.1.5]. Here is a sketch of the proof: For part (i), just observe that χ_1/χ_2 is constant on orbits of points. For part (ii), it is clear that $\Lambda(Y, T)$ is a subgroup of $\mathbb{R}$. Let χ_1 and χ_2 be two eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2. We show that $\|\chi_1 - \chi_2\| > \frac{1}{4}$. Since $C(Y)$ is separable, this will complete the proof. As $\lambda = \lambda_1 - \lambda_2$ is a nonzero eigenvalue with eigenvector $\chi_1 \overline{\chi_2}$, we may choose t_0 in $\mathbb{R}$ and y_0 in Y such that $e^{2\pi i \lambda t_0} \chi_1 \overline{\chi_2}(y_0)$ is in

the left-hand half of the unit circle. Hence

$$\|\chi_1 - \chi_2\| = \|\chi_1\overline{\chi_2} - 1\| = \sup_{y \in Y} \|\chi_1\overline{\chi_2}(T^t(y)) - 1\|$$

$$\geq \|e^{2\pi i\lambda t_0}\chi_1\overline{\chi_2}(y_0) - 1\| > \frac{1}{4}. \qquad \blacksquare$$

The following observation can be found in [22, II.3.6(3) and II.10.3(6)].

Lemma 1.3   *If (Y, T) is a minimal flow, then Y is connected.*

Proof   Let $y \in Y$. As the orbit of y is connected, then so is the closure of such orbit which, by minimality, is all of Y. Thus Y is connected. $\qquad \blacksquare$

By a *discrete flow* or *dynamical system* we mean a pair (X, S) where X is a compact metric space and $S: X \to X$ is a homeomorphism. A discrete flow (X, S) is minimal if there is no nontrivial closed S-invariant subset of X. Given a continuous flow (Y, T), it induces, for each $t \in \mathbb{R}$, a discrete flow (Y, T^t). We sometimes refer to the map T^t as the time-t map on Y. We now prove the following.

Lemma 1.4   *Let (Y, T) be a continuous flow where Y is a connected space. Let t be a nonzero real number. The map T^t is not minimal if and only if for every rational number r the map T^{rt} is not minimal.*

Proof   Suppose that the map T^t is not minimal and let r be a rational number. We show that the map T^{rt} is not minimal. Assume that r is nonzero. (If r is zero, then $T^{rt} = T^0$ is the identity function and so T^{rt} is not minimal.) Write $r = p/q$ with p and $q > 0$ relatively prime. We claim that $T^{t/q}$ is not minimal. This will then imply that $T^{rt} = (T^{t/q})^p$ is not minimal (because if M is a nontrivial, closed and $T^{t/q}$-invariant subset of Y, then M is also $T^{pt/q}$-invariant), which is what is wanted. To prove the claim, assume that $T^{t/q}$ is minimal. Then, since by hypothesis $T^t = (T^{t/q})^q$ is not minimal, there is a nontrivial, closed and T^t-invariant subset M of Y, cf. [23, Theorem 5.2]. Following [22, II.9.6(7)], we see that M is also open, a contradiction, since Y is assumed to be connected. This proves the claim. The converse of the lemma is obvious. $\qquad \blacksquare$

If a continuous flow (Y, T) is minimal, it is natural to ask for which t the induced discrete flow (Y, T^t) is minimal. The answer is contained in the following result which is essentially outlined in [8, 4.24.1].

Proposition 1.5   *Let (Y, T) be a minimal continuous flow. Consider the map with domain $\mathbb{Q} \otimes \Lambda(Y, T)$ and codomain $\mathbb{R}$ defined by $r \otimes \lambda \mapsto r\lambda$ for all (r, λ) in $\mathbb{Q} \times \Lambda(Y, T)$. This map is a $\mathbb{Q}$-linear monomorphism with range equal to*

$$\{0\} \cup \{t \in \mathbb{R} \setminus \{0\}: T^{\frac{1}{t}} \text{ is not minimal}\}.$$

Hence the set above is a countable $\mathbb{Q}$-linear subspace of $\mathbb{R}$ isomorphic to $\mathbb{Q} \otimes \Lambda(Y, T)$.

Proof As $\Lambda(Y, T)$ is an abelian (countable) subgroup of $\mathbb{R}$, *cf.* Lemma 1.2, one readily sees that every element in $\mathbb{Q} \otimes \Lambda(Y, T)$ can be written as a pure tensor and that the map with domain $\mathbb{Q} \otimes \Lambda(Y, T)$ and codomain $\mathbb{R}$, defined by $r \otimes \lambda \mapsto r\lambda$ for all (r, λ) in $\mathbb{Q} \times \Lambda(Y, T)$, is a $\mathbb{Q}$-linear monomorphism. We only need to check that the image of this map is equal to $\{0\} \cup \{t \in \mathbb{R} \setminus \{0\} \colon T^{\frac{1}{t}}$ is not minimal$\}$. For this purpose, we will prove that for every nonzero real number t, the time-$\frac{1}{t}$ map is not minimal if and only if there is (r, λ) in $(\mathbb{Q} \times \Lambda(Y, T)) \setminus \{(0, 0)\}$ such that $t = r\lambda$.

Let t be a nonzero real number such that $T^{1/t}$ is not minimal. Assume that t is positive. (If t is negative, then we work with $-t$ to obtain $-t = r\lambda$ and so $t = -r\lambda$.) Let M be a minimal subset for $T^{1/t}$ (*cf.* [23, Theorem 5.2]). Put

$$H = \{s \in \mathbb{R} \colon T^s(M) = M\}.$$

As H is a nonempty closed subgroup of $\mathbb{R}$ and $H \neq \mathbb{R}$ (because (Y, T) is minimal), there is $t_0 > 0$ such that $H = t_0 \mathbb{Z}$ (see [22, Lemma II.1.11]). Then

$$\bigcup_{0 \leq s < t_0} T^s(M) = T(M \times [0, t_0])$$

is a closed, nonempty and T-invariant subset of Y. As T is minimal, we obtain that

$$(1) \qquad\qquad Y = \bigcup_{0 \leq s < t_0} T^s(M).$$

Furthermore, for $0 \leq s_1, s_2 < t_0$ with $s_1 \neq s_2$ one has $T^{s_1}(M) \cap T^{s_2}(M) = \varnothing$. This together with the equality (1) lead us to conclude that $\{T^s(M) \colon 0 \leq s < t_0\}$ forms a partition of Y. Define a continuous function $\chi \colon Y \to S^1$ by the formula

$$\chi(T^s(x)) = e^{2\pi i \frac{1}{t_0} s},$$

for all $x \in M$ and for all $0 \leq s < t_0$. Then χ is an eigenvector of (Y, T) with eigenvalue $\frac{1}{t_0}$. Since $\frac{1}{t} = mt_0$ for some $m \in \mathbb{Z} \setminus \{0\}$, we get that $t = \frac{1}{m} \frac{1}{t_0}$, where $\left(\frac{1}{m}, \frac{1}{t_0}\right)$ is in $\mathbb{Q} \times \Lambda(Y, T)$.

To prove the converse, let (r, λ) be in $(\mathbb{Q} \times \Lambda(Y, T)) \setminus \{(0, 0)\}$. Assume that χ is a (nonconstant) eigenvector for λ. Since $\chi \circ T^t = e^{2\pi i \lambda t} \chi$ for all $t \in \mathbb{R}$, we get that $\chi \circ T^{1/\lambda} = \chi$ and so $T^{1/\lambda}$ is not minimal, *cf.* [23, Theorem 5.3]. Since Y is connected (because (Y, T) is minimal, *cf.* Lemma 1.3) we use Lemma 1.4 to conclude that $T^{\frac{1}{r\lambda}}$ is not minimal, as desired. ∎

Given a discrete flow (X, S) and a strictly positive continuous function $f \colon X \to \mathbb{R}$, consider the function $\alpha_f \colon X \times \mathbb{Z} \to \mathbb{R}$ given by the formula

$$\alpha_f(x, n) = \begin{cases} \sum_{i=0}^{n-1} f(S^i(x)) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{i=1}^{-n} f(S^{-i}(x)) & \text{if } n < 0, \end{cases}$$

for all (x, n) in $X \times \mathbb{Z}$. Then $\mathbb{R}$ and $\mathbb{Z}$ both act on the product space $X \times \mathbb{R}$ by setting

$$(x, s) \cdot n = (S^n(x), s - \alpha_f(x, n)), \quad \text{and} \quad t \cdot (x, s) = (x, s + t),$$

for all (x, s) in $X \times \mathbb{R}$ and for all (n, t) in $\mathbb{Z} \times \mathbb{R}$. The two actions commute and so we let $Y_{S,f}$ be the quotient space $(X \times \mathbb{R})/\mathbb{Z}$ and we let $T_{S,f}$ be the action of $\mathbb{R}$ on $Y_{S,f}$. The resulting continuous flow $(Y_{S,f}, T_{S,f})$ is called the *suspension flow with base S and ceiling function f*. For each (x, s) in $X \times \mathbb{R}$, we will denote by $[x, s]_f$ the image of (x, s) in $Y_{S,f}$. In this way, for the action $T_{S,f}$ of $\mathbb{R}$ on $Y_{S,f}$, we write $T_{S,f}^t([x, s]_f) = [x, s+t]_f$ for all t in $\mathbb{R}$ and for all (x, s) in $X \times \mathbb{R}$. We suppress the subindexes S and f from the notation as permitted by context. When f is the constant function 1, we refer to the induced flow as the *standard suspension of S*. A suspension flow is minimal if and only if its base is minimal. See [22, Section III.5.5] for details. For future reference we now prove the following.

Lemma 1.6 *Let (X, S) be a minimal dynamical system. If t is a real number for which the time-t map on the standard suspension of S is minimal, then t is irrational.*

Proof Let (Y, T) be the standard suspension flow of (X, S). The time-0 map is the identity map and so it is not minimal. If $t = \frac{p}{q}$ is a nonzero rational, with p and $q > 0$ relatively prime, then $M = \pi\left(\bigcup_{i=0}^{q-1} X \times \left\{\frac{i}{q}\right\}\right)$ is a closed, nontrivial and T^t-invariant subset of Y, where $\pi \colon X \times \mathbb{R} \to Y$ is the canonical quotient map. Hence the time-t map on the standard suspension of S is not minimal, as was to be proved. $\blacksquare$

Given a dynamical system (X, S), we say that a continuous function $\xi \colon X \to S^1$ is an eigenvector for S if there is $\lambda \in \mathbb{R}$ such that $\chi(S(x)) = e^{2\pi i \lambda} \chi(x)$ for all $x \in X$. We then call $e^{2\pi i \lambda}$ the eigenvalue associated with χ or say that χ is an eigenvector with eigenvalue $e^{2\pi i \lambda}$. In analogy to the case of continuous flows, Definition 1.1, we denote by $\Lambda(X, S)$ the set of all eigenvalues for (X, S). When (X, S) is minimal, the set $\Lambda(X, S)$ is actually a countable subgroup of $\mathbb{T}^1$, *cf.* [23, Theorem 5.17]. The following gives us a relation between the eigenvalues of a minimal dynamical system and the eigenvalues of its standard suspension flow.

Lemma 1.7 *Let (X, S) be a minimal dynamical system and let (Y, T) be the standard suspension of S. Then $\Lambda(X, S) = \varphi(\Lambda(Y, T))$, where $\varphi \colon \mathbb{R} \to \mathbb{T}^1$ is the canonical map $\mathbb{R} \ni x \mapsto e^{2\pi i x} \in \mathbb{T}^1$.*

Proof Suppose that λ is a real number such that $e^{2\pi i \lambda}$ is an eigenvalue of (X, S) with eigenvector χ. Then λ is an eigenvalue of (Y, T) with eigenvector $\widetilde{\chi} \colon Y \to S^1$ defined by $\widetilde{\chi}([x, s]) = e^{2\pi i s \lambda} \chi(x)$ for all $(x, s) \in X \times \mathbb{R}$. Conversely, if $\lambda \in \mathbb{R}$ is an eigenvalue of (Y, T) with eigenvector χ, then $e^{2\pi i \lambda}$ is an eigenvalue of (X, S) with eigenvector $\underline{\chi} \colon X \to S^1$ defined by $\underline{\chi}(x) = \chi([x, 0])$. $\blacksquare$

As mentioned in the introduction, Schwartzman [21] introduced, for a given continuous flow (Y, T) with T-invariant probability measure ν, an asymptotic cycle $A_\nu \colon H^1(Y, T) \to \mathbb{R}$, where $H^1(Y, \mathbb{Z})$ is the first Čech cohomology group with integer coefficients. We regard $H^1(Y, \mathbb{Z})$ as the quotient group formed by the multiplicative group of continuous functions from Y to S^1 modulo the ones that can be expressed as $e^{2\pi i h(y)}$ for some continuous function $h \colon Y \to \mathbb{R}$. The subset C consisting of all elements $[f]$ in $H^1(Y, \mathbb{Z})$ such that $A_\nu([f]) > 0$, for all T-invariant probability measure ν on Y, is a (multiplicative) semigroup. Using the terminology of J. Packer [12], we will call this semigroup C the *positive Schwartzman cone* of $H^1(Y, \mathbb{Z})$.

Definition 1.8   We define the following subset of $H^1(Y, \mathbb{Z})$, which is an abelian (multiplicative) semigroup with neutral element $H^1(Y, \mathbb{Z})^+ = C \cup \{[1]\}$. Here C is the positive Schwartzman cone of $H^1(Y, \mathbb{Z})$ and 1 is the constant function $Y \ni y \mapsto 1 \in S^1$.

Let (X, S) be a dynamical system and let (Y, T) be the standard suspension of S. We will denote by $\Phi \colon K_1(C(Y)) \to K_0(C(X) \rtimes_S \mathbb{Z})$ the Connes isomorphism [1, Corollary V.6].

A dynamical system (X, S) induces a C^*-algebra $C(X) \rtimes_\alpha \mathbb{Z}$, called the crossed product associated to (X, S), which can be described as the universal C^*-algebra generated by $C(X)$ and a unitary u satisfying $u f u^{-1} = f \circ S^{-1}$, for all $f \in C(X)$. There is a one-to-one correspondence between S-invariant Borel probability measures μ on X and normalized traces τ_μ on $C(X) \rtimes_S \mathbb{Z}$, given by $\tau_\mu(\sum_k f_k u^k) = \int_X f_0 \, d\mu(x)$. A trace τ on a C^*-algebra A induces a natural homomorphism, which we denote again by τ, from the group $K_0(A)$ to $\mathbb{R}$.

2   The Results

Here we prove our main results. Throughout this section (X, S) will denote a minimal dynamical system and $f \colon X \to \mathbb{R}$ will be a strictly positive, continuous function.

Let $(Y_{S,f}, T_{S,f})$ and $(Y_{S,1}, T_{S,1})$ denote the suspension flows with the same base S and with ceiling functions f and 1, respectively. For each λ in $\Lambda(Y_{S,f}, T_{S,f})^+$ and eigenvector χ of λ, we consider a continuous map $U_\lambda \colon X \times \mathbb{R} \to S^1$ defined by $U_\lambda(x, s) = \chi([x, sf(x)]_f)$ for all $(x, s) \in X \times \mathbb{R}$. Given x in X we have, by definition, that $[x, f(x)]_f = [S(x), 0]_f$, and so $U_\lambda(x, 1) = U_\lambda(S(x), 0)$. Hence the map U_λ induces a continuous map with domain $Y_{S,1}$ and codomain S^1, which we denote again by U_λ, given by the formula

$$
\begin{aligned}
U_\lambda([x, s]_1) &= \chi([x, sf(x)]_f) \\
&= \chi(T_{S,f}^{sf(x)}([x, 0]_f)) \\
&= e^{2\pi i \lambda s f(x)} \chi([x, 0]_f),
\end{aligned}
$$

for all $(x, s) \in X \times \mathbb{R}$. In the following lemma we show that the class $[U_\lambda]$ in $H^1(Y_{S,1}, \mathbb{Z})$ does not depend on the choice of the eigenvector χ of λ; moreover, $[U_\lambda]$ belongs to the positive Schwartzman cone of $H^1(Y_{S,1}, \mathbb{Z})$, except when λ is zero.

Lemma 2.1   *Let $(Y_{S,f}, T_{S,f})$ and $(Y_{S,1}, T_{S,1})$ denote the suspension flows with the same base S and with ceiling functions f and 1, respectively. For each λ in $\Lambda(Y_{S,f}, T_{S,f})^+$ and eigenvector χ of λ, consider the map $U_\lambda \colon Y_{S,1} \to S^1$ as defined above. Then the class $[U_\lambda]$ in $H^1(Y_{S,1}, \mathbb{Z})$ does not depend on the choice of the eigenvector χ of λ; moreover, $[U_\lambda]$ belongs to the positive Schwartzman cone of $H^1(Y_{S,1}, \mathbb{Z})$, except when λ is zero.*

Proof   Suppose that χ_1 and χ_2 are two different eigenvectors for λ. Using Proposition 1.2, there is a constant c in S^1 such that $\chi_1 = c\chi_2$. Therefore, for each (x, s) in $X \times \mathbb{R}$, it follows that $\chi_1([x, sf(x)]_f) = c\chi_2([x, sf(x)]_f)$. This proves that the class of U_λ in $H^1(Y_{S,1}, \mathbb{Z})$ does not depend on the choice of the eigenvector χ of λ, as wanted.

Now, observe that for each (x, s) in $X \times \mathbb{R}$ we can write

$$U_\lambda([x, s]_1) = \underline{\chi}(x)e^{2\pi i s h(x)},$$

where $\underline{\chi} \colon X \to S^1$ and $h \colon X \to \mathbb{R}$ are two continuous functions on X which are defined by the formulas $\underline{\chi}(x) = \chi([x, 0]_f)$ and $h(x) = \lambda f(x)$, for all $x \in X$. Furthermore, for each x in X, the functions $\underline{\chi}$ and h satisfy $\underline{\chi}(S(x)) = \underline{\chi}(x)e^{2\pi i h(x)}$. Using [13, Theorem 1.1], we have that $[U_\lambda]$ is in the positive Schwartzman cone of $H^1(Y, \mathbb{Z})$ if and only if $\int_X h(x)\,\mathrm{d}\mu(x)$ is strictly positive for every S-invariant Borel probability measure μ on X. We claim that this is the case, unless $\lambda = 0$. Since f is strictly positive, it follows that $\int_X f(x)\,\mathrm{d}\mu(s)$ is strictly positive for every S-invariant Borel probability measure μ on X. Therefore the integral $\int_X h(x)\,\mathrm{d}\mu(x) = \int_X \lambda f(x)\,\mathrm{d}\mu(x)$ is strictly positive for every S-invariant Borel probability measure μ on X, unless λ is zero. This proves the claim and completes the proof of the lemma.   ∎

Proposition 2.2   *Let the notation be as in Lemma 2.1. Define a function*

$$\Gamma_{S,f}^+ \colon \Lambda(Y_{S,f}, T_{S,f})^+ \to H^1(Y_{S,1}, \mathbb{Z})^+$$

by $\Gamma_{S,f}^+(\lambda) = [U_\lambda]$ for all $\lambda \in \Lambda(Y_{S,f}, T_{S,f})^+$. Then the map $\Gamma_{S,f}^+$ is a homomorphism of abelian semigroups with neutral elements.

Proof   Let λ_1 and λ_2 be two elements in $\Lambda(Y_{S,f}, T_{S,f})^+$ with eigenvectors χ_1 and χ_2, respectively. Then $\chi_1\chi_2$ is an eigenvector for $\lambda_1 + \lambda_2$. Since by Lemma 2.1 the class of $\left[U_{\lambda_1+\lambda_2}\right]$ does not depend on the choice of an eigenvector for $\lambda_1 + \lambda_2$, we get

$$\Gamma_{S,f}^+(\lambda_1 + \lambda_2) = [U_{\lambda_1+\lambda_2}] = [(\chi_1\chi_2)([x, sf(x)]_f)]$$

$$= [\chi_1([x, sf(x)]_f)]\,[\chi_2([x, sf(x)]_f)] = [U_{\lambda_1}]\,[U_{\lambda_2}]$$

$$= \Gamma_{S,f}^+(\lambda_1)\,\Gamma_{S,f}^+(\lambda_2).$$

Hence $\Gamma_{S,f}^+$ is a homomorphism, as wanted.   ∎

Observe that, given a continuous flow (Y, T), the group $\Lambda(Y, T)$ can be seen as the Grothendieck group of $\Lambda(Y, T)^+$. Furthermore, the Grothendieck group of (the multiplicative semigroup) $H^1(Y, \mathbb{Z})^+$ is a subgroup of $H^1(Y, \mathbb{Z})$. In particular, we can consider the induced group homomorphism $\Gamma_{S,f}$ of the map $\Gamma_{S,f}^+$ of Proposition 2.2. We state this observation as the following proposition.

Proposition 2.3 *Let the notation be as in Proposition 2.2. There exists a unique group homomorphism* $\Gamma_{S,f} \colon \Lambda(Y_{S,f}, T_{S,f}) \to H^1(Y_{S,1}, \mathbb{Z})$ *that makes the following diagram commute.*

$$
\begin{array}{ccc}
\Lambda\left(Y_{S,f}, T_{S,f}\right)^+ & \xrightarrow{\;\;\Gamma_{S,f}^+\;\;} & H^1\left(Y_{S,1}, \mathbb{Z}\right)^+ \\[2mm]
\iota \downarrow & & \downarrow \iota \\[2mm]
\Lambda\left(Y_{S,f}, T_{S,f}\right) & \dashrightarrow[\;\Gamma_{S,f}\;] & H^1\left(Y_{S,1}, \mathbb{Z}\right).
\end{array}
$$

Proof Let G be the Grothendieck group of $H^1\left(Y_{S,1}, \mathbb{Z}\right)^+$. Then the existence of a unique homomorphism $\Gamma_{S,f} \colon \Lambda(Y_{S,f}, T_{S,f}) \to G \subset H^1(Y_{S,1}, \mathbb{Z})$ such that the diagram above commutes is a consequence of the fact that $\Lambda\left(Y_{S,f}, T_{S,f}\right)$ is the Grothendieck group of $\Lambda(Y_{S,f}, T_{S,f})^+$, see a corollary in [24, Appendix D, p. 297]. ∎

Considering $H^1(Y_{S,1}, \mathbb{Z})$ as a direct summand of $K_1(C(Y_{S,1}))$, we can now bring K-theory into the picture by composing the map $\Gamma \colon \Lambda(Y_{S,f}, T_{S,f}) \to H^1(Y_{S,1}, \mathbb{Z})$ in Proposition 2.3 with the Connes isomorphism $\Phi \colon K_1(C(Y_{S,1})) \to K_0(C(X) \rtimes_S \mathbb{Z})$. Relying on the work of Packer [13], we will show next how the composition map $\Phi \circ \Gamma \colon \Lambda(Y_{S,f}, T_{S,f}) \to K_0(C(X) \rtimes_S \mathbb{Z})$ has a left inverse given by a map induced by a trace on $C(X) \rtimes_S \mathbb{Z}$.

Proposition 2.4 *Let the notation be as in Proposition 2.3. If μ is an S-invariant Borel probability measure on X, then the map $\tau_{\mu,f} \colon K_0(C(X) \rtimes_S \mathbb{Z}) \to \mathbb{R}$, defined by $\tau_{\mu,f}(x) = \frac{\tau_\mu(x)}{\tau_\mu(f)}$ for all x in $K_0(C(X) \rtimes_S \mathbb{Z})$, is a left inverse for the map $\Phi \circ \Gamma_{S,f} \colon \Lambda(Y_{S,f}, T_{S,f}) \to K_0(C(X) \rtimes_S \mathbb{Z})$.*

Proof We must show that $(\tau_{\mu,f} \circ \Phi \circ \Gamma_{S,f})(\lambda) = \lambda$ for all λ in $\Lambda(Y_{S,f}, T_{S,f})$. Since the maps involved are group homomorphisms, it will suffice to do it for strictly positive λ. Let $\lambda \in \Lambda(Y_{S,f}, T_{S,f})^+ \setminus \{0\}$ and say χ is an eigenvector for λ. By Lemma 2.1 and Proposition 2.2, we have that $\Gamma_{S,f}(\lambda) = [U_\lambda] = [\chi([x,0]_f)e^{2\pi i \lambda s f(x)}]$ is in the positive Schwartzman cone of $H^1(Y_{S,1}, \mathbb{Z})$. Then, by [13, Theorem 3.4], there is a projection p in $M_n(C(X) \rtimes_S \mathbb{Z})$ such that $\Phi([U_\lambda]) = [p]$. Furthermore, [13, Theorem 2.3] gives us that

$$
\tau_\mu(p) = \int_X \lambda f(x)\, \mathrm{d}\mu(x) = \lambda \tau_\mu(f).
$$

Thus

$$
\left(\tau_{\mu,f} \circ \Phi \circ \Gamma_{S,f}\right)(\lambda) = \tau_{\mu,f}\left(\Phi([U_\lambda])\right) = \tau_{\mu,f}\left(p\right) = \frac{\tau_\mu\left(p\right)}{\tau_\mu\left(f\right)} = \lambda. \qquad \blacksquare
$$

We are ready to prove our two main results.

Theorem 2.5 *Let (Y, T) be a minimal suspension flow with base (X, S) and with (strictly positive, continuous) ceiling function $f : X \to \mathbb{R}$. Let (Y_S, T_S) denote the standard suspension of S. Consider the homomorphism $\Gamma : \Lambda(Y, T) \to H^1(Y_S, \mathbb{Z})$ of Proposition 2.3 and put $\Lambda K(X, S, f) = (\Phi \circ \Gamma)(\Lambda(Y, T))$. If μ is an S-invariant Borel probability measure on X, then the map $\tau_{\mu, f} : K_0(C(X) \rtimes_S \mathbb{Z}) \to \mathbb{R}$ defined by the formula $\tau_{\mu, f}(x) = \frac{\tau_\mu(x)}{\tau_\mu(f)}$, for all $x \in K_0(C(X) \rtimes_S \mathbb{Z})$, satisfies $\tau_{\mu, f}(\Lambda K(X, S, f)) = \Lambda(Y, T)$. Moreover, the map $\tau_{\mu, f} : \Lambda K(X, S, f) \to \Lambda(Y, T)$ is an order isomorphism of ordered abelian groups, where $\Lambda(Y, T)$ inherits the order from $\mathbb{R}$.*

Proof   Using Proposition 2.4, we get that

$$(2) \qquad \Phi \circ \Gamma : \Lambda(Y, T) \to \Lambda K(X, S, f) \subset K_0(C(X) \rtimes_S \mathbb{Z})$$

is an isomorphism with inverse $\tau_{\mu, f} : \Lambda K(X, S, f) \to \Lambda(Y, T)$. In fact, the isomorphism in (2) can be seen to be an order isomorphism since, by [13, Theorem 3.4] (see also [13, Remark 3.6]), we have that $(\Phi \circ \Gamma)(\Lambda(Y, T)^+)$ lies in the positive cone $K_0(C(X) \rtimes_S \mathbb{Z})^+$ of $K_0(C(X) \rtimes_S \mathbb{Z})$. Thus $\tau_{\mu, f} : \Lambda K(X, S, f) \to \Lambda(Y, T)$ is also an order isomorphism, as was to be proved.   ∎

Theorem 2.6 *Let (Y, T) be a minimal suspension flow with base (X, S) and with (strictly positive, continuous) ceiling function $f : X \to \mathbb{R}$. Let μ be an S-invariant Borel probability measure on X. Consider the map with domain $\mathbb{Q} \otimes \Lambda K(X, S, f)$ and codomain $\mathbb{R}$ defined by $r \otimes x \mapsto r \frac{\tau_\mu(x)}{\tau_\mu(f)}$ for all (r, x) in $\mathbb{Q} \times \Lambda K(X, S, f)$, where $\Lambda K(X, S, f)$ is as in Theorem 2.5. This map is a $\mathbb{Q}$-linear monomorphism with range equal to*

$$\{0\} \cup \{t \in \mathbb{R} \setminus \{0\} : T^{1/t} \text{ is not minimal}\}.$$

Hence the set above is a countable $\mathbb{Q}$-linear subspace of $\mathbb{R}$ isomorphic to $\mathbb{Q} \otimes \Lambda K(X, S, f)$.

Proof   Since Theorem 2.5 says that $\Lambda K(X, S, f)$ is isomorphic to $\Lambda(Y, T)$ via the map defined by $x \mapsto \frac{\tau_\mu(x)}{\tau_\mu(f)}$ for all $x \in \Lambda K(X, S, f)$, we obtain that the map defined by

$$r \otimes x \mapsto r \otimes \frac{\tau_\mu(x)}{\tau_\mu(f)},$$

for all (r, x) in $\mathbb{Q} \times \Lambda K(X, S, f)$, gives an isomorphism of $\mathbb{Q} \otimes \Lambda K(X, S, f)$ with $\mathbb{Q} \otimes \Lambda(Y, T)$. On the other hand, Proposition 1.5 asserts that $\mathbb{Q} \otimes \Lambda(Y, T)$ is isomorphic to the $\mathbb{Q}$-linear space $\{0\} \cup \{t \in \mathbb{R} \setminus \{0\} : T^{1/t} \text{ is not minimal}\}$ via the map defined by $r \otimes \lambda \mapsto r\lambda$ for all $(r, \lambda) \in \mathbb{Q} \times \Lambda(Y, T)$. Combining these two isomorphisms, we get the desired result.   ∎

When we specialize to the case for which f is the constant function 1 (that is, when we consider the standard suspension of S), we can say the following.

Corollary 2.7 *Let (X, S) be a minimal dynamical system. If t is a nonzero real number such that the time-$\frac{1}{t}$ map on the standard suspension of S is not minimal, then there exists $(r_1, r_2) \in \mathbb{Q} \times \mathbb{Q}$ and there exists a Rieffel projection p in $C(X) \rtimes_S \mathbb{Z}$ such that for any S-invariant Borel probability measure μ on X we have $t = r_1 + r_2 \tau_\mu (p)$.*

Proof Denote by (Y, T) the standard suspension flow of (X, S). Since $T^{1/t}$ is not minimal, Proposition 1.5 implies the existence of (r, λ) in $\mathbb{Q} \times \Lambda(Y, T)$ such that $t = r\lambda$. Let n and γ denote the integral and fractional parts of λ, respectively. To avoid trivialities, assume that λ is irrational (else we may choose p to be trivial). Then γ lies in the open interval $(0, 1)$. Observe that $\mathbb{Z} \subset \Lambda(Y, T)$, *cf.* Lemma 1.7. Therefore, as $\Lambda(Y, T)$ is a subgroup of $\mathbb{R}$ (cf. Proposition 1.2), the fractional part γ of λ belongs to $\Lambda(Y, T)^+ \setminus \{0\}$. By Theorem 2.5, there exists a projection p in $M_n(C(X) \rtimes_S \mathbb{Z})$ (in fact $[p] = \Phi([U_\gamma])$) such that $\gamma = \tau_\mu(p) = \tau_{\mu,1}(p)$ for each S-invariant Borel probability measure μ on X. Combining now Theorems 3.2 and 3.4 in [13], we conclude that p is a Rieffel projection. Thus $t = rn + r\tau_\mu(p)$, as desired. ∎

Corollary 2.8 *Let (X, S) be a minimal Cantor system. If t is a nonzero real number such that the time-$\frac{1}{t}$ map on the standard suspension of S is not minimal, then there exists $(r_1, r_2) \in \mathbb{Q} \times \mathbb{Q}$ and there exits an open and closed subset F of X such that for each S-invariant Borel probability measure μ on X we have $t = r_1 + r_2 \mu(F)$.*

Proof By Corollary 2.7, there is $(r_1, r_2) \in \mathbb{Q} \times \mathbb{Q}$ and there is a Rieffel projection p in $C(X) \rtimes_S \mathbb{Z}$ such that for each S-invariant Borel probability measure μ on X we have $t = r_1 + r_2 \tau_\mu(p)$. To avoid trivialities we assume t is irrational (else we may choose F to be trivial). Therefore $0 < \tau_\mu(p) < 1$ for each S-invariant Borel probability measure μ on X. Furthermore, since p is a projection in $C(X) \rtimes_S \mathbb{Z}$, then p can be regarded as a continuous function in $C(X, \mathbb{Z})$, *cf.* [16, Theorem 1.1]. Using [9, Lemma 2.4], we find an open and closed subset F of X such that $\tau_\mu(p) = \mu(F)$ for all S-invariant Borel probability measure μ on X. Thus $t = r_1 + r_2 \mu(F)$ for each S-invariant Borel probability measure μ on X, as wanted. ∎

Extending a result of Rieffel [19], and Pimsner and Voiculescu [19], N. Riedel [18] showed that the range of the unique trace on the K_0-group of a minimal group rotation can be described in terms of the eigenvalues of this group rotation. This result was later proved by other authors using different techniques (see [12, Theorem 2.5] and [4, Theorem IX.11]). For general minimal dynamical systems we get the following.

Corollary 2.9 *Let (X, S) be a minimal dynamical system and let μ be an S-invariant Borel probability measure on X. Then $\Lambda(X, S) \subset \varphi\big(\tau_\mu(K_0(C(X) \rtimes_S \mathbb{Z}))\big)$, where $\varphi \colon \mathbb{R} \to \mathbb{T}^1$ is the canonical map $\mathbb{R} \ni x \mapsto e^{2\pi i x} \in \mathbb{T}^1$.*

Proof   Let (Y, T) be the standard suspension of S. Then

$$
\begin{aligned}
\Lambda(X, S) &= \varphi(\Lambda(Y, T)) && \text{by Lemma 1.7}\\
&\subset \varphi(\tau_{\mu,1}(K_0\left(C(X) \rtimes_S \mathbb{Z}\right))) && \text{by Theorem 2.5}\\
&= \varphi\left(\tau_\mu\left(K_0\left(C(X) \rtimes_S \mathbb{Z}\right)\right)\right) && \text{by definition of } \tau_{\mu,1}. \qquad \blacksquare
\end{aligned}
$$

We show next that the *K*-theory of a minimal Cantor system keeps track of the eigenvalues of every dynamical system in its orbit equivalence class (*cf.* [6, Definition 1.2]).

Corollary 2.10   *Let (X, S) be a minimal Cantor system and let μ be an S-invariant Borel probability measure on X. If (X_1, S_1) is orbit equivalent to (X, S), then*

$$
\Lambda(X_1, S_1) \subset \varphi\left(\tau_\mu(K_0(C(X) \rtimes_S \mathbb{Z}))\right),
$$

where $\varphi \colon \mathbb{R} \to \mathbb{T}^1$ denotes the canonical map $\mathbb{R} \ni x \mapsto e^{2\pi i x} \in \mathbb{T}^1$.

Proof   Since (X_1, S_1) is orbit equivalent to (X, S), there is a homeomorphism $F \colon X_1 \to X$ carrying the S_1-invariant Borel probability measures on X_1 onto the S-invariant Borel probability measures on X. Furthermore, this homeomorphism induces an order isomorphism of

$$
\frac{K_0(C(X_1) \rtimes_{S_1} \mathbb{Z})}{\mathrm{Inf}(K_0(C(X_1) \rtimes_{S_1} \mathbb{Z}))} \quad \text{with} \quad \frac{K_0(C(X) \rtimes_S \mathbb{Z})}{\mathrm{Inf}(K_0(C(X) \rtimes_S \mathbb{Z}))},
$$

cf. [6, Theorem 2.2]. Let ν be the S_1-invariant measure corresponding to μ under F. Then

$$
\begin{aligned}
\tau_\nu\left(K_0(C(X_1) \rtimes_{S_1} \mathbb{Z})\right) &= \tau_\nu\left(\frac{K_0(C(X_1) \rtimes_{S_1} \mathbb{Z})}{\mathrm{Inf}(K_0(C(X_1) \rtimes_{S_1} \mathbb{Z}))}\right)\\
&= \tau_\mu\left(\frac{K_0(C(X) \rtimes_S \mathbb{Z})}{\mathrm{Inf}(K_0(C(X) \rtimes_S \mathbb{Z}))}\right)\\
&= \tau_\mu(K_0(C(X) \rtimes_S \mathbb{Z})).
\end{aligned}
$$

Combining the above equality with Corollary 2.10 we obtain

$$
\begin{aligned}
\Lambda(X_1, S_1) &\subset \varphi\left(\tau_\nu(K_0(C(X_1) \rtimes_{S_1} \mathbb{Z}))\right)\\
&= \varphi(\tau_\mu(K_0(C(X) \rtimes_S \mathbb{Z}))),
\end{aligned}
$$

as desired. $\blacksquare$

3 Examples

In this section we include some examples illustrating our results. We start by showing dynamical systems for which every irrational time map on its standard suspensions is minimal. To contrast these, we also exhibit a dynamical system for which some irrational time map on its standard suspension is not minimal. Our next examples will present dynamical systems (X, S), one of which has group $\Lambda K(X, S, 1)$ (*cf.* Theorem 2.5) equal to $K_0(C(X) \rtimes_S \mathbb{Z})$ and one for which $\Lambda K(X, S, 1)$ is different from $K_0(C(X) \rtimes_S \mathbb{Z})$. We will also give an example of two strong orbit equivalent minimal Cantor systems which have different eigenvalues. Our last example exhibits a minimal suspension flow (Y, T) satisfying the following: the map T^t is minimal for every nonzero real number t.

Proposition 3.1   *Let (X, S) be a minimal dynamical system. Suppose there is an S-invariant Borel probability measure μ on X such that $\tau_\mu(K_0(C(X) \rtimes_S \mathbb{Z}))$ is contained in $\mathbb{Q}$. Then for each real number t, the time-t map on the standard suspension of S is minimal if and only if t is irrational.*

Proof   Suppose that t is a real number such that the time-t map on the standard suspension of (X, S) is not minimal. We show that t is rational. If t is zero, there is nothing to prove. If t is nonzero, then by Theorem 2.6 there is (r, x) in $\mathbb{Q} \times \Lambda K(Y, T, 1)$ such that $\frac{1}{t} = r\tau_\mu(x)$. As $\Lambda K(X, S, 1)$ is a subgroup of $K_0(C(X) \rtimes_S \mathbb{Z})$ and $\tau_\mu(K_0(C(X) \rtimes_S \mathbb{Z}))$ is contained in $\mathbb{Q}$, we conclude that $\tau_\mu(x)$ is rational and so t is rational, as wanted. The converse is Lemma 1.6.   ∎

Example 3.2   The rotation $R_t \colon z \mapsto ze^{2\pi it}$ on S^1 is minimal if and only if t is irrational.

This is a well-known fact, see [22, Proposition III.1.4]. We use our results for an alternative proof. As the unit circle S^1 can be regarded as the standard suspension of the one-point dynamical system, the rotation R_t then corresponds to the time-t map of this suspension. Since the range of the trace on the K_0-group of the one-point dynamical system is equal to $\mathbb{Z}$, we apply Proposition 3.1 to get that R_t is minimal if and only if t is irrational.

Let $\{n_i\}_{i=1}^\infty$ be a sequence of integers greater than 2. Let $X = \Pi_{i=1}^\infty \{0, 1, \ldots, n_{i-1}\}$. Recall that an *odometer system* consists of the space X together with the homomorphism S representing addition of $(1, 0, 0, \ldots, 0)$ with carryover to the right.

Example 3.3   Let t be a real number. The time-t map on the suspension of an odometer is minimal if and only if t is irrational.

The image of the trace on the K_0-group of an odometer is contained in $\mathbb{Q}$, *cf.* [16, p. 332]. Therefore Proposition 3.1 applies to get the desired result.

Given two dynamical systems (X_1, S_1) and (X_2, S_2), we say that (X_1, S_1) is an extension of (X_2, S_2) if there exists a continuous surjective map $F \colon X_1 \to X_2$ such that $F \circ S_1 = S_2 \circ F$. In such case we call F an extension map. We say that (X_1, S_1) is

an almost one-to-one extension of (X_2, S_2) if there is an extension map $F\colon X_1 \to X_2$ such that the set $A = \{x \in X_1\colon F^{-1}(F(x)) = \{x\}\}$ is dense in X_1. We call such F an almost one-to-one extension map. It is easy to check that when (X_1, S_1) is an almost one-to-one extension of (X_2, S_2), then S_1 is minimal if and only if S_2 is minimal.

Lemma 3.4   *If (X_1, S_1) is an almost one-to-one extension of (X_2, S_2), then for each real number t the time-t map on the suspension of (X_1, S_1) is minimal if and only if the time-t map on the suspension of (X_2, S_2) is minimal.*

Proof   Let (Y_1, T_1) and (Y_2, T_2) be the standard suspensions of S_1 and S_2, respectively. The lemma will follow once we prove that for each $t \in \mathbb{R}$ the dynamical system (Y_1, T_1^t) is an almost one-to-one extension of (Y_2, T_2^t). Let $F\colon X_1 \to X_2$ be an almost one-to-one extension map. Consider the continuous surjective map with domain $X_1 \times \mathbb{R}$ and codomain Y_2 given by $(x, s) \mapsto [F(x), s]$, for all (x, s) in $X_1 \times \mathbb{R}$. Since for each x in X_1 we have that $[F(x), 1] = [S_2(F(x)), 0] = [F(S_1(x)), 0]$, this map induces a continuous surjective map, which we denote $\widetilde{F} = \widetilde{F}_t\colon Y_1 \to Y_2$, given by the formula $\widetilde{F}([x, s]) = [F(x), s]$, for all (x, s) in $X_1 \times \mathbb{R}$. Since for each (x, s) in $X_1 \times \mathbb{R}$ we have $T_2^t \circ \widetilde{F}([x, s]) = [F(x), s + t] = \widetilde{F} \circ T_1^t([x, s])$, we conclude that $\widetilde{F}$ is an extension map. Moreover, if we denote $A = \{x \in X_1\colon F^{-1}(F(x)) = \{x\}\}$, we have that

$$(3) \qquad \{y \in Y_1\colon \widetilde{F}^{-1}(\widetilde{F}(y)) = \{y\}\} = l\{[x, s] \in Y_1\colon x \in A,\ s \in \mathbb{R}\}.$$

Since A is dense in X_1, then $A \times \mathbb{R}$ is dense in $X_1 \times \mathbb{R}$ and so the right-hand side in (3) is dense in Y_1. Hence the left-hand side in (3) is dense Y_1. This proves that $\widetilde{F}$ is an almost one-to-one extension map from (Y_1, T_1^t) onto (Y_2, T_2^t), as was to be proved.   ∎

Example 3.5   Let t be a real number. The time-t map on the standard suspension of a Toeplitz flow is minimal if and only if t is irrational.

Since a Toeplitz flow is an almost one-to-one extension of an odometer (*cf.* [3, Theorem 4]), we combine Lemma 3.4 with Example 3.3 to obtain the desired result.

Example 3.5 together with Theorem 1.5 give us that Toeplitz flows cannot have irrational eigenvalues. This is in contrast with the measure-theoretical case: Toeplitz flows may have measure-theoretical irrational eigenvalues, *cf.* [10]. It is an interesting problem to determine when a measure-theoretical eigenvalue of a dynamical system is a "continuous" eigenvalue (in our sense), *cf.* [2].

In the following two examples we present dynamical systems for which some irrational time maps on its standard suspensions are not minimal.

Example 3.6   Let s be an irrational number and let $R_s\colon z \mapsto ze^{2\pi i s}$ be the rotation by s on the unit circle. For each real number t, the time-$\frac{1}{t}$ map on the standard suspension of R_s is minimal if and only if t does not belong to $\mathbb{Q} + s\mathbb{Q}$.

Let (Y, T) denote the standard suspension flow of R_s. For each $t \in \mathbb{R}$, we claim that (Y, T^t) is conjugate to $(S^1 \times S^1, R_{st} \times R_t)$, where $R_{st} \times R_t$ denotes the homeomorphism of $S^1 \times S^1$ given by $(R_{st} \times R_t)(z_1, z_2) = (e^{2\pi i st} z_1, e^{2\pi i t} z_2)$, for all (z_1, z_2) in $S^1 \times S^1$. Indeed, consider the map with domain $S^1 \times \mathbb{R}$ and codomain $S^1 \times S^1$ defined by $(z, r) \mapsto (e^{2\pi i sr} z, e^{2\pi i r})$, for all (z, r) in $S^1 \times \mathbb{R}$. Since for each $z \in S^1$ we have that $(e^{2\pi i s} z, e^{2\pi i}) = (R_s(z), e^0)$, this map induces a homeomorphism, which we denote $F = F_t \colon Y \to S^1 \times S^1$, given by the formula $F([z, r]) = (e^{2\pi i sr} z, e^{2\pi i r})$ for all (z, r) in $S^1 \times \mathbb{R}$. We check that $F \circ T^t = (R_{st} \times R_t) \circ F$. For this purpose, let (z, r) be an element in $S^1 \times \mathbb{R}$. Then

$$(F \circ T^t)([z, r]) = F([z, r + t]) = (e^{2\pi i s(r+t)} z, e^{2\pi i (r+t)})$$

$$= (R_{st} \times R_t)(e^{2\pi i sr} z, e^{2\pi i r}) = ((R_{st} \times R_t) \circ F)([z, u]).$$

This proves the claim. Now, since $(S^1 \times S^1, R_{st} \times R_t)$ is minimal if and only if 1, st and t are linearly independent over $\mathbb{Q}$ (see, [11, Proposition 1.4.1]). Then, by conjugacy, $(Y, T^{1/t})$ is minimal if and only 1, $\frac{s}{t}$ and $\frac{1}{t}$ are linearly independent over $\mathbb{Q}$. This happens if and only if t is not in $\mathbb{Q} + s\mathbb{Q}$, as was to be proved.

We now give a brief description of Denjoy systems. Recall that an orientation preserving homeomorphism of the unit circle S^1 without periodic points necessarily has an irrational rotation number, *cf.* [11, Proposition 11.1.4]. A Denjoy homeomorphism is an orientation preserving homeomorphism of S^1 with irrational rotation number and which is not conjugate to an irrational rotation of S^1. Denjoy homeomorphisms are uniquely ergodic, *cf.* [11, Theorems 11.2.7 and 11.2.9]. When we restrict a Denjoy homeomorphism S to the support X of its unique invariant measure, we obtain a strictly ergodic Cantor system (X, S) which we call *Denjoy system*. It turns out that a Denjoy system (X, S) with irrational rotation number s is an almost one-to-one extension of the irrational rotation $R_s \colon z \mapsto e^{2\pi i s} z$ of S^1. Furthermore, the conjugacy class of a Denjoy system (X, S) is determined by the rotation number s of S and a countable subset of the circle, denoted by $Q(S)$, which is invariant under rotation by an angle $2\pi i s$. We refer to [17] for a detailed account.

Example 3.7 Let t be a real number. The time-$\frac{1}{t}$ map on the standard suspension of a Denjoy system with irrational rotation number s is minimal if and only if t does not belong to $\mathbb{Q} + s\mathbb{Q}$.

Let (X, S) be a Denjoy system with irrational rotation number s. Since (X, S) is an almost one-to-one extension of the irrational rotation $R_s \colon z \mapsto e^{2\pi i s} z$ on S^1, then by Lemma 3.4, the time-$\frac{1}{t}$ map on the standard suspension of S is minimal if and only if the time-$\frac{1}{t}$ map on the standard suspension of R_s is minimal. Using Example 3.6, this happens if and only if t is not in $\mathbb{Q} + s\mathbb{Q}$.

Our next two examples will show that K_0-group of a minimal dynamical systems might contain more information than just the eigenvalues of its standard suspension, *cf.* Corollary 2.9.

Example 3.8 There are minimal dynamical systems (X, S) with $\Lambda K(X, S, 1)$ strictly contained in $K_0 (C(X) \rtimes_S \mathbb{Z})$

Let s_1 and s_2 be real numbers such that 1, s_1 and s_2 are linearly independent over $\mathbb{Q}$. Consider the Denjoy systems (X, S) with the following conjugacy invariants: the irrational rotation number of (X, S) is s_1 and the set $Q(S)$ is $\{e^{2\pi i n s_1} \colon n \in \mathbb{Z}\} \cup \{e^{2\pi i(s_2 + n s_1)} \colon n \in \mathbb{Z}\}$. By [17, Theorem 5.3], the (unique) normalized trace τ on $C(X) \rtimes_S \mathbb{Z}$ induces an (order) isomorphism between $K_0(C(X) \rtimes_S \mathbb{Z})$ and $\mathbb{Z} + s_1 \mathbb{Z} + s_s \mathbb{Z}$. Let x be the element in $K_0(C(X) \rtimes_S \mathbb{Z})$ for which $\tau(x) = s_2$. Then x does not belong to $\Lambda K(X, S, 1)$, for if it did, then Proposition 2.5 says that $\tau(x) = s_2$ would be an eigenvalue for the standard suspension of (X, S) and so, by Proposition 1.5, the time-$\frac{1}{s_2}$ map on that suspension would not be minimal. Using Example 3.7 we conclude that s_2 belongs to $\mathbb{Q} + s_1 \mathbb{Q}$, contradicting the fact that 1, s_1 and s_2 are linearly independent over $\mathbb{Q}$.

Example 3.9   There are minimal dynamical systems (X, S) with $\Lambda K(X, S, 1)$ equal to $K_0(C(X) \rtimes_S \mathbb{Z})$.

Let s be an irrational number and consider the irrational rotation $R_s \colon z \mapsto z e^{2\pi i s}$ on S^1. The induced crossed product $C(S^1) \rtimes_{R_s} \mathbb{Z}$ is called an *irrational rotation algebra* [19]. Let (Y, T) denote the standard suspension flow of (S^1, R_s). It is known that $\Lambda(Y, T)$ is equal to the range of the (unique) trace on $K_0(C(S^1) \rtimes_{R_s} \mathbb{Z})$, see [18, p. 155, Example 1], [12, Theorem 2.5] or [4, Example IX.12]. (These results present the image of the trace on $K_0(C(X) \rtimes_S \mathbb{Z})$ as the preimage of the eigenvalues of (S^1, R_s) under the canonical map $\mathbb{R} \ni x \mapsto e^{2\pi i x} \in \mathbb{T}^1$, which are precisely the eigenvalues of the standard suspension of (X, S), *cf.* Lemma 1.7.) Furthermore, by [15, Corollary 2.6], this trace is actually an isomorphism of $K_0(C(S^1) \rtimes_{R_s} \mathbb{Z})$ with $\Lambda(Y, T)$. Hence, since $\Lambda K(S^1, R_s, 1) \subset K_0(C(S^1) \rtimes_{R_s} \mathbb{Z})$, Theorem 2.5 gives us the desired equality $\Lambda K(S^1, R_s, 1) = K_0(C(S^1) \rtimes_{R_s} \mathbb{Z})$.

N. Riedel proved [18, Corollary 3.7] that given two minimal group rotations (X_1, S_1) and (X_2, S_2), their crossed products $C(X_1) \rtimes_{S_1} \mathbb{Z}$ and $C(X_2) \rtimes_{S_2} \mathbb{Z}$ are isomorphic if and only if $\Lambda(X_1, S_1) = \Lambda(X_2, S_2)$. Since (X_1, S_1) and (X_2, S_2) are conjugate if and only if they have the same eigenvalues, [23, Theorem 5.18; Theorem 5.19], R. Gjerde and Ø. Johansen [7, Theorem 6] strengthened Riedel's result by showing that (X_1, S_1) and (X_2, S_2) are conjugate if and only if they are strong orbit equivalent (*cf.* [6, Definition 1.3]). Now, T. Giordano, I. Putnam and C. Skau [6, Theorem 2.1] proved that two minimal Cantor systems are strong orbit equivalent if and only if their crossed products are isomorphic. This might lead one to suspect that the eigenvalues of minimal Cantor systems are an invariant of strong orbit equivalence. That this is not the case is probably known by the expert. However, since we could not find such an example in the literature, we give one next.

Example 3.10   There are two (strong) orbit equivalent dynamical systems (X_1, S_1) and (X_2, S_2) for which $\Lambda(X_1, S_1) \neq \Lambda(X_2, S_2)$.

Let (X_1, S_1) be the Denjoy system of Example 3.8 and let (X_2, S_2) be the Denjoy system with irrational rotation number s_2 and with $Q(S_2) = \{e^{2\pi i n s_2} \colon n \in \mathbb{Z}\} \cup \{e^{2\pi i(s_1 + n s_2)} \colon \in \mathbb{Z}\}$. A combination of [6, Theorem 2.2] and [17, Theorem 5.3] leads us to conclude that (X_1, S_1) and (X_2, S_2) are (strong) orbit equivalent. By Example 3.7, the time-$\frac{1}{s_1}$

on the standard suspension of S_1 is not minimal. Theorem 1.5 tells us then that we can find a (nonzero) rational number r such that rs_1 is an eigenvalue of this standard suspension. But rs_1 is not an eigenvalue for the standard suspension of S_2; for if it were, then using again Theorem 1.5, the time-$\frac{1}{s_1}$ map on the standard suspension of S_2 would not be minimal and so, by Example 3.7, s_1 would belong to $\mathbb{Q} + s_2\mathbb{Q}$, contradicting the fact that 1, s_1 and s_2 are linearly independent over $\mathbb{Q}$.

Proposition 3.11   *Let (X, S) be a minimal dynamical system with at least two distinct S-invariant Borel probability measures. Assume that all normalized traces on $C(X) \rtimes_S \mathbb{Z}$ agree on $K_0(C(X) \rtimes_S \mathbb{Z})$. Then there exists a strictly positive continuous function $f : X \to \mathbb{R}$ such that for any nonzero real number t, the time-t map on the suspension with base S and ceiling function f is minimal.*

Proof   Let μ_0 and μ_1 be two distinct S-invariant Borel probability measures on X. For each t in $[0, 1]$, put $\mu_t = (1 - t)\mu_0 + t\mu_1$. By hypothesis we have that

$$(4) \qquad \tau_{\mu_t}(K_0(C(X) \rtimes_S \mathbb{Z})) = \tau_{\mu_0}(K_0(C(X) \rtimes_S \mathbb{Z}))$$

for all $t \in [0, 1]$. As $\mu_0 \neq \mu_1$, there is a continuous function $f \in C(X)$ such that $\tau_{\mu_0}(f) \neq \tau_{\mu_1}(f)$, *cf.* [23, Theorem 6.2]. Replacing f by a constant plus the imaginary or real part of f if necessary, we may assume that f is a strictly positive real valued function. Let (Y, T) be the suspension with base S and ceiling f. For each t in $[0, 1]$ we then get:

$$\Lambda(Y, T) \subset \frac{1}{\tau_{\mu_t}(f)} \, \tau_{\mu_t}(K_0(C(X) \rtimes_S \mathbb{Z})) \quad \text{by Theorem 2.5}$$

$$= \frac{1}{\tau_{\mu_t}(f)} \tau_{\mu_0}(K_0(C(X) \rtimes_S \mathbb{Z})) \quad \text{by (4).}$$

Thus

$$\Lambda(Y, T) \subset \bigcap_{t \in [0,1]} \frac{1}{\tau_{\mu_t}(f)} \, \tau_{\mu_0}(K_0(C(X) \rtimes_S \mathbb{Z})).$$

As $K_0(C(X) \rtimes_S \mathbb{Z})$ is countable and $\{\tau_{\mu_t}(f)\}_{t \in [0,1]}$ takes all possible values between $\tau_{\mu_0}(f)$ and $\tau_{\mu_1}(f)$, the right-hand side of the above inclusion must be equal to $\{0\}$. Thus $\Lambda(Y, T) = \{0\}$. Now using Proposition 1.5, we conclude that the time-t map on Y is minimal for all nonzero real number t.   ∎

Example 3.12   There is a dynamical system (X, S) and there is a strictly positive continuous function $f : X \to \mathbb{R}$ such that for any nonzero real number t, the time-t map on the suspension with base S and ceiling f is minimal.[1]

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[1] I am grateful to T. Katsura and J. Packer for the discussions we had after I presented the main results of this paper at the Great Plains Operator Theory Symposium 2004. Example 3.12 was inspired by these discussions.

Given $\theta \in [0, 1] \setminus \mathbb{Q}$, a continuous function $\xi \colon S^1 \to \mathbb{R}$, and $n \in \mathbb{Z} \setminus \{0\}$, we define $S = S_{\theta,\xi} \colon S^1 \times S^1 \to S^1 \times S^1$ to be the inverse of the homeomorphism

$$(z_1, z_2) \mapsto (e^{2\pi i\theta} z_1, e^{2\pi i\xi(z_1)} z_1^n z_2).$$

The homeomorphism S is called a Furstenberg transformation. It is known that Furstenberg transformations are minimal (see the remark after the Theorem 2.1 in [5, §2.3]). It is also known that there are Furstenberg transformations which are not uniquely ergodic [5, p. 585]. Let S be a Furstenberg transformation which is not uniquely ergodic. In [14, Example 4.9] it is shown that every normalized trace on $C(S^1 \times S^1) \rtimes_S \mathbb{Z}$ agree on $K_0(C(S^1 \times S^1) \rtimes_S \mathbb{Z})$. Hence, by Proposition 3.11, there is a strictly positive function $f \colon S^1 \times S^1 \to \mathbb{R}$ such that for each nonzero real number t, the time-t map on the suspension with base S and ceiling f is minimal.

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