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Mixing and asymptotic distribution modulo 1

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Abstract. If μ is a probability measure which is invariant and ergodic with respect to the transformation $x \mapsto qx$ on the circle $\mathbb{R}/\mathbb{Z}$, then according to the ergodic theorem, $\{q^n x\}$ has the asymptotic distribution μ for μ-a.e. x. On the other hand, Weyl showed that when μ is Lebesgue measure, λ, and $\{m_j\}$ is an arbitrary sequence of integers increasing strictly to ∞, the asymptotic distribution of $\{m_j x\}$ is λ for λ-a.e. x. Here, we investigate the asymptotic distributions of $\{m_j x\}$ μ-a.e. for fairly arbitrary $\{m_j\}$ under some strong mixing conditions on μ. The result is a kind of stable ergodicity: the distributions are obtained from simple operations applied to μ. The ideas extend to the situation of a sequence of transformations $x \mapsto q_n x$ where invariance is not present. This gives us information about many Riesz products and Bernoulli convolutions. Finally, we apply the theory to resolve some questions about H-sets.

1. *Introduction*

Suppose that T is a continuous transformation on a compact metric space X. If μ is a T-invariant Borel probability measure on X, then the ergodic theorem says that for all $f \in L^1(\mu)$ and for μ-a.e. x, the limit as $n \to \infty$ of

$$\frac{1}{N} \sum_{n \le N} f(T^n x)$$

exists. If we restrict our attention to $f \in C(X)$, or a countable dense subset thereof, we see that $\{T^n x\}_{n \ge 1}$ has an asymptotic distribution, call it σ_x, for μ-a.e. x: that is, for μ-a.e. x,

$$\frac{1}{N} \sum_{n \le N} f(T^n x) \to \int_X f \, d\sigma_x \qquad \text{for all } f \in C(X). \tag{1}$$

We write $\{T^n x\} \sim \sigma_x$ μ-a.e. Evidently, σ_x is T-invariant and integrating (1) with respect to μ shows that

$$\mu = \int_X \sigma_x \, d\mu(x) \tag{2}$$

in the weak sense. The measure μ is ergodic if and only if $\sigma_x = \mu$ a.e. The Bogoliouboff theory [12] shows that in any case σ_x is ergodic μ-a.e. The integral (2) is thus a convex combination of invariant measures in terms of ergodic measures (the extreme points).

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We are interested in the case where T is the transformation $T_q: x \mapsto qx$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $q \in \mathbb{Z}$, $|q| \geq 2$. The additional structure on the circle interacts with ergodic theory in many interesting ways. We intend to explore here the asymptotic distribution of $\{m_j x\}$ for sequences other than simply $m_j = q^j$. For example, Weyl [**17**, § 7] showed that when μ is Lebesgue measure, λ, and $\{m_j\}$ is an arbitrary sequence of integers increasing strictly to ∞, then $\{m_j x\} \sim \lambda$ for λ-a.e. x. Now given an arbitrary sequence $\{m_j\}$ and measure μ, $\{m_j x\}$ need not possess an asymptotic distribution on a set of non-zero μ-measure. However, there always does exist [**8**] a subsequence $\{m'_j\}$ of $\{m_j\}$, which we denote simply by $\{m'_j\} \subset \{m_j\}$, such that even for any further subsequence $\{m''_j\} \subset \{m'_j\}$ and for μ-a.e. x, the sequence $\{m''_j x\}$ has an asymptotic distribution σ_x. Hence we shall restrict our attention to sequences $\{m_j\}$ already enjoying this property of stability: that is, we assume that there exists $\sigma: \mathbb{T} \to M(\mathbb{T})$ such that for all $\{m'_j\} \subset \{m_j\}$ and for μ-a.e. x $\{m'_j x\} \sim \sigma_x$. We remark that if $\{q^j\}$ itself is to have this property for a q-invariant μ, then it is necessary (though not sufficient) that μ be q-mixing:

$$\forall f, g \in L^2(\mu) \int f(T_q^n x) g(x)\, d\mu(x) \to \left(\int f\, d\mu\right)\left(\int g\, d\mu\right),$$

or, equivalently,

$$\forall a, b \in \mathbb{Z}\ \hat{\mu}(aq^n + b) \to \hat{\mu}(a)\hat{\mu}(b),$$

where

$$\hat{\mu}(k) = \int_{\mathbb{T}} e(-kx)\, d\mu(x),\ e(x) = e^{2\pi i x}.$$

We shall impose an even stronger mixing condition on μ in order to determine σ_x for any (stable) $\{m_j\}$. Now σ_x is determined by its Fourier–Stieltjes coefficients, $\hat{\sigma}_x(r)$, $r \in \mathbb{Z}$. By (1),

$$\frac{1}{J} \sum_{j \leq J} e(-rm'_j x) \to \hat{\sigma}_x(r) \qquad \mu\text{-a.e.}$$

for all $\{m'_j\} \subset \{m_j\}$, whence

$$\forall r \in \mathbb{Z}\ e(-rm_j x) \to \hat{\sigma}_x(r) \text{ weak* in } L^\infty(\mu). \tag{3}$$

(Here, we regard $L^\infty(\mu)$ as the dual of $L^1(\mu)$.) The problem is thus equivalent to determining the simultaneous weak* limits of $e(-rm_j x)$ in $L^\infty(\mu)$.

If we integrate (3) and let Σ be the measure such that $\hat{\Sigma}(r) = \lim_{j\to\infty} \hat{\mu}(-rm_j x)$, then we obtain a formula analogous to (2):

$$\Sigma = \int_{\mathbb{T}} \sigma_x\, d\mu(x). \tag{3a}$$

Another way of viewing (3) and (3a) is given in [**10**].

The reader may find it easier to follow our proofs if he first works out the following set of examples, in which the most important phenomena are present. We take $q = 3$ and μ the Riesz product [**5**, p. 107]

$$\mu = \prod_{k \geq 0} (1 + \mathrm{Re}\,\{\alpha e(3^k x)\}),$$

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where $|\alpha|\le 1$. If $m_j=3^j$, then $\sigma_x=\mu$ a.e. If $m_j=3^{2j}+3^j$, then $\sigma_x=\mu*\mu=\mu^2$ a.e. If $m_j=3^j+1$, then $\sigma_x=\mu*\delta(x)$ a.e., where $\delta(x)$ is the unit mass at x. If $m_j=2\cdot 3^j$, then σ_x is the measure such that $\hat{\sigma}_x(r)=\hat{\mu}(2r)$ a.e. If

$$m_j=\frac{3^j-1}{2}=3^{j-1}+3^{j-2}+\cdots+3+1,$$

then σ_x is the measure such that

$$\hat{\sigma}_x(r)=\begin{cases}\hat{\omega}(r/2) & \text{if } r \text{ is even,}\\ 0 & \text{if } r \text{ is odd,}\end{cases}$$

where $\omega=\mu*\delta(-x)$.

2. *The invariant case*

For any integer q, we let T_q be the operator on $M(\mathbb{T})$ such that

$$(T_q\omega)\hat{}\,(n)=\hat{\omega}(qn)\quad(n\in\mathbb{Z},\,\omega\in M(\mathbb{T})).$$

If $q\neq 0$, we define T_q^{-1} by

$$(T_q^{-1}\omega)\hat{}\,(n)=\begin{cases}\hat{\omega}(n/q) & \text{if } q\mid n,\\ 0 & \text{if } q\nmid n,\end{cases}$$

while if $q=0$, we set $T_0^{-1}\omega=\hat{\omega}(0)\lambda$, where λ is Lebesgue measure. Thus for $q\neq 0$, $T_q\circ T_q^{-1}=\mathrm{id}$. It is easily checked that

$$\omega*T_q^{-1}\omega'=T_q^{-1}[T_q\omega*\omega']. \tag{4}$$

The hypotheses of our first theorem below are immediately seen to be satisfied by Riesz products,

$$\mu=\prod_{k\ge 0}(1+\mathrm{Re}\,\{\alpha e(q^kx)\}),\quad |\alpha|\le 1,\quad |q|\ge 3,$$

and it is not difficult to verify them for Bernoulli convolutions ([**4**, p. 182])

$$\mu=\mathop{*}_{k\ge 1}(p\delta(0)+(1-p)\delta(2^{-k})),\quad 0<p<1$$

(here, $q=2$), for example. After the proof of the theorem, we shall discuss the hypotheses more thoroughly, including their mixing character.

THEOREM 1. *Let μ be a q-invariant probability measure such that*

$$\forall b\in\mathbb{Z}\ \lim_{n\to\infty}\sup_{a\in\mathbb{Z}}|\hat{\mu}(aq^n+b)-\hat{\mu}(a)\hat{\mu}(b)|=0 \tag{5}$$

and

given any sequence $\{e(m_jx)\}_{j\ge 1}$ which does not converge to 0 weak in $L^\infty(\mu)$, there exists a subsequence $\{m_j'\}\subset\{m_j\}$ and integers b, a_j, n_j such that $n_j\to\infty$ and* (6)

$$m_j'=a_jq^{n_j}+b.$$

Then if $|m_j|\to\infty$ is such that (3) *holds, there exist integers r, b, an integer $l\ge 1$, and non-zero integers $s_1,\ldots,s_l$ such that*

$$\sigma_x=T_r^{-1}[\delta(bx)*T_{s_1}\mu*\cdots*T_{s_l}\mu]\qquad \mu\text{-a.e.} \tag{7}$$

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We shall find it handy to use the equivalence of (6) to

$$\text{if } \lim_{n\to\infty}\lim_{j\to\infty} d_n(m_j) \neq \infty, \quad \text{then } e(m_jx)\to 0 \text{ weak* in } L^\infty(\mu), \tag{8}$$

where $d_n(m) = |m - q^n\mathbb{Z}|$, i.e., the distance from m to the multiples of q^n. It is straightforward to show that (6) implies (8). Conversely, suppose that (8) holds and let $e(m_jx) \nrightarrow 0$ weak* in $L^\infty(\mu)$. Then by compactness, there is a subsequence $\{e(m_j''x)\}$ having a non-zero weak* limit. Let $\{m_j'''\} \subset \{m_j''\}$ be such that for all $n \geq 1$, $\lim_{j\to\infty} d_n(m_j''')$ exists. Then $e(m_j'''x) \nrightarrow 0$ and by (8), since $\lim_{j\to\infty} d_n(m_j''')$ is increasing in n, for large enough n, say $n > N$, we have $\lim_{j\to\infty} d_n(m_j''') = \tilde{b} < \infty$. This means that for $n > N$ and for $j \geq J(n)$, we can write $m_j''' = \tilde{a}_j q^n + b$, where $b = \pm\tilde{b}$ and is fixed without loss of generality. In particular, $m'''_{J(n)} = \tilde{a}_{J(n)}q^n + b$ for $n > N$. Thus, $m_k' = m'''_{J(N+k)}$ defines the required sequence.

Proof. Let $|m_j| \to \infty$ be such that (3) holds. If $\sigma_x = \lambda$ μ-a.e., then we take $r = 0$ in (7) and we are done. In the contrary case, there is an $r \neq 0$ such that $\hat{\sigma}_x(r) \neq 0$. By a diagonal procedure, we may find a subsequence $\{m_j'\}$ – which we shall take to be the whole sequence without loss of generality – such that for all $n \geq 1$ and all r, $\lim_{j\to\infty} d_n(rm_j)$ exists. Thus by (8), the set

$$E = \left\{ r\colon \lim_{n\to\infty}\lim_{j\to\infty} d_n(rm_j) < \infty \right\}$$

is not just $\{0\}$. We claim that $E = r_0\mathbb{Z}$ for some $r_0 > 0$. It suffices to show that E is a subgroup of $\mathbb{Z}$. But if $r, s \in E$, then $d_n((r-s)m_j) \leq d_n(rm_j) + d_n(sm_j)$, whence $r - s \in E$ and so E is indeed a subgroup. It follows that if r is not a multiple of r_0, then $\hat{\sigma}_x(r) = 0$ μ-a.e., whence there exists ν_x such that $\sigma_x = T_{r_0}^{-1}\nu_x$ μ-a.e.

We must now determine ν_x. Since $r_0 \in E$, we can, by replacing $\{m_j\}$ by a subsequence if necessary, suppose that there are integers a_j, n_j, b_0 such that $r_0 m_j = a_j q^{n_j} + b_0$, $n_j \to \infty$, $q \nmid a_j$, and also that $\{e(ra_jx)\}_j$ has a weak* limit in $L^\infty(\mu)$ for each r. Let $\hat{\Sigma}_1(r) = \lim_{j\to\infty} \hat{\mu}(ra_j)$. We claim that

$$\forall r \in \mathbb{Z}\ \hat{\nu}_x(r) = \hat{\Sigma}_1(r)\, e(-rb_0x) \quad \mu\text{-a.e.} \tag{9}$$

Indeed, for all s, we have

$$\begin{aligned}\int_{\mathbb{T}} \hat{\nu}_x(r)\, e(-sx)\, d\mu(x) &= \int \hat{\sigma}_x(rr_0)\, e(-sx)\, d\mu(x)\\ &= \lim_{j\to\infty} \int e(-rr_0m_jx)\, e(-sx)\, d\mu(x) = \lim_{j\to\infty} \hat{\mu}(rr_0m_j + s)\\ &= \hat{\Sigma}_1(r)\hat{\mu}(rb_0 + s) \quad \text{(by (5))}\\ &= \int \hat{\Sigma}_1(r)\, e(-rb_0x)\, e(-sx)\, d\mu(x).\end{aligned}$$

From (9), it follows that $\nu_x = \delta(b_0x) * \Sigma_1$ μ-a.e. and it remains to determine Σ_1.

Since $\Sigma_1 \neq \lambda$ (otherwise, $\sigma_x = \lambda$ a.e.), we can argue as in the first paragraph to obtain $r_1 > 0$ such that $\Sigma_1 = T_{r_1}^{-1}\Sigma_1'$, where, taking a subsequence of $\{a_j\}$ if necessary as in the second paragraph, we can assume that

$$r_1 a_j = a_j^{(2)} q^{n_j'} + s_1', \; n_j' \to \infty, \quad q \nmid a_j^{(2)} \quad \text{or} \quad a_j^{(2)} = 0,$$

and $\hat{\Sigma}_1'(r)=\hat{\mu}(rs_1')\hat{\Sigma}_2(r)$, where $\hat{\Sigma}_2(r)=\lim_{j\to\infty}\hat{\mu}(ra_j^{(2)})$. Thus, $\Sigma_1=T_{r_1}^{-1}[T_{s_1'}\mu*\Sigma_2]$. Note that $s_1'\neq 0$. We then proceed for Σ_2 as we did for Σ_1, and so on. This process ends if and only if $a_j^{(l+1)}\equiv 0$ for some l. We claim the process must indeed end. Otherwise, for each r, $|\hat{\sigma}_x(r)|$ would be bounded by $[\sup_{n\neq 0}|\hat{\mu}(n)|]^l$ for each l; since $\sigma_x\neq\lambda$, it follows that $\sup_{n\neq 0}|\hat{\mu}(n)|=1$. Thus there exists $\{N_j\}$ such that $|\hat{\mu}(N_j)|\to 1$. By (6), (5), and q-invariance, there is then a $b\neq 0$ such that $|\hat{\mu}(b)|=1$. This, of course, implies that μ has finite support and that $\hat{\mu}$ is periodic, which contradicts (6).

We have thus obtained the expression

$$\sigma_x=T_{r_0}^{-1}[\delta(b_0x)*T_{r_1}^{-1}[T_{s_1'}\mu*T_{r_2}^{-1}[T_{s_2'}\mu*\cdots*T_{r_l}^{-1}T_{s_l'}\mu]]\ldots]\quad \mu\text{-a.e.}$$

Use of (4) l times yields (7) with $r=r_0r_1\cdots r_l$, $b=b_0r_1\cdots r_l$, $s_i=s_i'r_{i+1}r_{i+2}\cdots r_l$ for $1\leq i<l$, and $s_l=s_l'$. □

Recall that if T is a measure-preserving transformation of a Lebesgue space $(X,\mathscr{B},\mu)$, then T (or μ) is called *exact* if the σ-field

$$\text{Tail}(\mathscr{B})\overset{\text{def}}{=}\bigcap_{n\geq 0}T^{-n}\mathscr{B}$$

is trivial, i.e. consists only of sets of measure 0 or 1 [**3**, p. 289]. (This is the same as saying that Kolmogorov's 0–1 law holds.) There are several convenient equivalent conditions which depend on the following notions. If ξ is a partition of X, let $\mathscr{B}(\xi)$ denote the smallest complete sub-σ-field of $\mathscr{B}$ containing those measurable sets which are unions of elements of ξ. We say that ξ is trivial if $\mathscr{B}(\xi)$ is trivial. Let Tail(ξ) denote the partition $\bigwedge_{n\geq 0}\bigvee_{k\geq n}T^{-k}\xi$. For a set A, let Tail$(A)=\bigcup_{n\geq 0}T^{-n}T^nA$. Thus, Tail$(\mathscr{B})=\{\text{Tail}(A): A\in\mathscr{B}\}$. It is not hard to demonstrate that the following conditions are equivalent (see [**3**, pp. 283–4], [**15**], or [**16**, Chap. VII]):

(i) T is exact;

(ii) for any finite partition ξ, Tail(ξ) is trivial;

(iii) for every set A of positive measure, $\mu(\text{Tail}(A))=1$;

(iv) if $\langle f,g\rangle$ denotes $\int f\bar{g}\,d\mu$, then

$$\forall g\in L^2(\mu)\ \lim_{n\to\infty}\sup_{\substack{f\in L^2(\mu)\\ \|f\|_2\leq 1}}|\langle T^nf,g\rangle-\langle f,1\rangle\langle 1,g\rangle|=0; \tag{10}$$

(v) T is K-mixing, i.e. if ξ is any finite partition, then

$$\forall r\geq 1\quad \forall B\in\bigvee_{k\leq r}T^{-k}\xi\quad \lim_{n\to\infty}\sup_{A\in\bigvee_{k\geq n}T^{-k}\xi}|\mu(A\cap B)-\mu(A)\mu(B)|=0. \tag{11}$$

Furthermore, if ξ is some finite generating partition (i.e., $\mathscr{B}(\bigvee_{n\geq 0}T^{-n}\xi)=\mathscr{B}$) and Tail$(\xi)$ is trivial or (11) holds, then T is exact.

In our case, the partition

$$\xi_q=\begin{cases}\left\{\left[\dfrac{i}{q},\dfrac{i+1}{q}\right[:0\leq i<q\right\} & \text{if } q>0,\\[2ex] \left\{\left[\dfrac{1}{|q|+1}+\dfrac{i}{|q|},\dfrac{1}{|q|+1}+\dfrac{i+1}{|q|}\right[:0\leq i<|q|\right\} & \text{if } q<0\end{cases}$$

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is generating for $T=T_q$ since $\bigvee_{n\geq 0} T_q^{-n}\xi_q$ is the discrete partition $\{\{x\}: x\in\mathbb{T}\}$. Evidently, our hypothesis (5) is a bit weaker than (10), i.e., than exactness. By using the partition ξ_q and (11), we immediately deduce the exactness of any Bernoulli convolution

$$\mu = \mathop{*}_{k\geq 1} [p_0\,\delta(0)+p_1\,\delta(q^{-k})+\cdots+p_{|q|-1}\,\delta((|q|-1)q^{-k})], \tag{12}$$

where $0\leq p_i\leq 1$, $p_0+p_1+\cdots+p_{|q|-1}=1$, and $p_i\neq 0$ for at least two i's. (Note that when $q<0$,

$$\begin{aligned}\mu &= \delta\left(\frac{1}{|q|+1}\right) * \mu * \left(\mathop{*}_{k\,\mathrm{odd}} \delta((|q|-1)|q|^{-k})\right)\\ &= \delta\left(\frac{1}{|q|+1}\right) * \left[\mathop{*}_{k\,\mathrm{even}} \sum_{i=0}^{|q|-1} p_i\,\delta(i|q|^{-k})\right] * \left[\mathop{*}_{k\,\mathrm{odd}} \sum_{i=0}^{|q|-1} p_{|q|-1-i}\,\delta(i|q|^{-k})\right].\end{aligned}$$

Also, $(\mathbb{T}, \mu, T_q)$ is metrically isomorphic to $(\mathbb{T}, \nu, T_{|q|})$ via the mapping

$$\sum_{k\geq 1} \varepsilon_k q^k \mapsto \sum_{k\geq 1} \varepsilon_k |q|^{-k}$$

$(0\leq \varepsilon_k<|q|)$, where $\nu = *_{k\geq 1}\sum_{i=0}^{|q|-1} p_i\,\delta(i|q|^{-k})$.) We remark that an approximation argument quickly shows that Riesz products satisfy (10) as well. A stronger result [**11, 13**] is that Riesz products are isomorphic to Bernoulli shifts.

The hypothesis (5) is also equivalent to the following kind of tameness [**4**, Chapter 6] of μ: if $e(a_n q^n x)\to\chi(x)$ weak* in $L^\infty(\mu)$, then χ is constant a.e. We leave this as an exercise.

We now turn to the hypothesis (6). Rather than being of a purely mixing character, it links an arbitrary sequence $\{m_j\}$ to the transformation T_q. It too can be thought of as a tameness condition, for if μ satisfies (5) and (6), then μ is 'weakly tame': if $e(m_j x)\to\chi(x)$ weak* in $L^\infty(\mu)$, then $\chi(x)=ce(nx)$ a.e. for some constant c and some integer n. In any case, it is obvious that Riesz products satisfy (6) and this is not hard to see for Bernoulli convolutions (12) with $p_0p_1\neq 0$. Indeed, we shall establish (6) assuming that $gcd\{i-i_0: p_i\neq 0\}=1$, where i_0 is any subscript such that $p_{i_0}\neq 0$. First note that if $e(m_j x)\not\to^{w^*} 0$ in $L^\infty(\mu)$, then for some m,

$$\hat{\mu}(m_j+m)=\int e(-m_j x)\,e(-mx)\,d\mu(x)\not\to 0.$$

By replacing the sequence $\{m_j\}$ with $\{m_j+m\}$, we may assume that $m=0$. Now if $\hat{\mu}(m_j)\not\to 0$, then

$$\prod_{k\geq 1}\left(\sum_i p_i\,e(-im_jq^{-k})\right)\not\to 0 \quad \text{as } j\to\infty.$$

By taking a subsequence if necessary, we may assume that for all i and k, $\lim_{j\to\infty} e(-im_jq^{-k})$ and $\lim_{j\to\infty}\hat{\mu}(m_j)$ exist. It follows that

$$\infty>\lim_{j\to\infty}\sum_{k\geq 1}\left[1-\left|\sum_i p_i\,e(-im_jq^{-k})\right|\right]\geq \sum_{k\geq 1}\left[1-\left|\sum_i p_i \lim_{j\to\infty} e(-im_jq^{-k})\right|\right],$$

so that for some θ_k,

$$\lim_{k\to\infty} e(\theta_k)\sum_i p_i \lim_{j\to\infty} e(-im_jq^{-k})=1.$$

Thus, for $p_i \neq 0$,

$$\lim_{k\to\infty} e(\theta_k) \lim_{j\to\infty} e(-im_j q^{-k}) = 1.$$

If $p_{i_0} \neq 0$, we then have $\lim_{k\to\infty} \lim_{j\to\infty} e(-(i-i_0)m_j q^{-k}) = 1$, so that the hypothesis $gcd\{i-i_0: p_i \neq 0\} = 1$ implies that $\lim_{k\to\infty} \lim_{j\to\infty} e(-m_j q^{-k}) = 1$, which is the same as $\lim_k \lim_j \|m_j q^{-k}\| = 0$, where $\|x\| = |x - \mathbb{Z}|$. We have only to apply the following lemma to be able to conclude (8) and thus (6). (More precise information on sequences $\{m_j\}$ such that $\hat{\mu}(m_j) \to 0$ is given in **[2]** for certain μ.)

LEMMA 2. $\lim_{k\to\infty} \overline{\lim}_{j\to\infty} d_k(m_j) = \infty \Leftrightarrow \overline{\lim}_{k\to\infty} \overline{\lim}_{j\to\infty} \|m_j q^{-k}\| > 0$.

Proof. Since $\|m_j q^{-k}\| = |q|^{-k} d_k(m_j)$, the implication ($\Leftarrow$) is immediate. Conversely, suppose that $\lim_k \overline{\lim}_j d_k(m_j) = \infty$. Then for all N there is a $k = k(N)$ such that $\overline{\lim}_j d_k(m_j) \geq N$. Let $\{j_l\}$ be a sequence such that for all l, $d_k(m_{j_l}) = N_1 \geq N$. Let $n = n(N)$ be the least integer such that $|q|^n/2 \geq N_1$. Then $k \geq n$ and for all l, $d_n(m_{j_l}) = d_k(m_{j_l}) = N_1$ and $\|m_{j_l} q^{-n}\| = |q|^{-n} d_n(m_{j_l}) = |q|^{-n} N_1 > 1/|2q|$. Since $n(N) \to \infty$ as $N \to \infty$, it follows that $\overline{\lim}_k \overline{\lim}_l \|m_{j_l} q^{-k}\| \geq 1/|2q|$. □

Suppose, on the other hand, that the Bernoulli convolution (12) satisfies $gcd\{i-i_0: p_i \neq 0\} = r_0 \neq 1$, $p_{i_0} \neq 0$. We may assume that $i_0 = \min\{i: p_i \neq 0\}$. Consider the measure

$$\mu * \delta\left(-\frac{i_0}{q-1}\right) = \mathop{*}_{k\geq 1} \left[\left(\sum_i p_i\, \delta(iq^{-k})\right) * \delta(-i_0 q^{-k})\right]$$
$$= \mathop{*}_{k\geq 1} \left(\sum_i p_i\, \delta((i-i_0)q^{-k})\right) = T_{r_0}\nu,$$

where

$$\nu = \mathop{*}_{k\geq 1} \sum_i p_i\, \delta\left(\frac{i-i_0}{r_0} q^{-k}\right).$$

By definition of r_0 and what we've just proved, ν satisfies (6) and, of course, (5). Since Theorem 1 applies to ν, we claim that if (3) holds for μ, then there exist $t \in \mathbb{T}$, $r, b \in \mathbb{Z}$, $l \in \mathbb{N}^+$, $s_1, \ldots, s_l \in \mathbb{Z}^*$, and a function $\zeta: \mathbb{T} \to \mathbb{T}$ such that $T_{r_0} \circ \zeta = \mathrm{id}$ and

$$\sigma_x = \delta(t) * T_r^{-1}[\delta(b\zeta(x)) * T_{s_1}\nu * \cdots * T_{s_l}\nu] \quad \mu\text{-a.e.} \tag{13}$$

This follows from the following general observations.

First, if $\mu = \delta(t') * \nu$, $e(-rm_j x) \to \hat{\sigma}_{x,\nu}(r)$ weak* in $L^\infty(\nu)$, and $e(-rm_j x) \to \hat{\sigma}_{x,\mu}(r)$ weak* in $L^\infty(\mu)$, then let $\{m_j'\} \subset \{m_j\}$ be such that $m_j' t' \to t$. It is easy to see that

$$\sigma_{x,\mu} = \delta(t) * \sigma_{x-t',\nu} \quad \mu\text{-a.e.}$$

Second, if $\mu = T_{r_0}\nu$ $(r_0 \neq 0)$, then using the same notation, we claim that

$$\sigma_{x,\mu} = T_{r_0}\sigma_{\zeta(x),\nu} \quad \mu\text{-a.e.}$$

for some function $\zeta: \mathbb{T} \to \mathbb{T}$ with $T_{r_0} \circ \zeta = \mathrm{id}$. For we have $\{m_j x\} \sim \sigma_{x,\nu}$ ν-a.e. without loss of generality; let $E = \{x: \{m_j x\} \sim \sigma_{x,\nu}\}$. Since $\nu E = 1$, we have $\mu T_{r_0} E = 1$. Let $\zeta: \mathbb{T} \to \mathbb{T}$ be any map such that $\zeta(x) \in E$ for $x \in T_{r_0} E$ and $T_{r_0} \circ \zeta = \mathrm{id}$. Then for μ-a.e. x, we have $\zeta(x) \in E$, so that $\{m_j \zeta(x)\} \sim \sigma_{\zeta(x),\nu}$, whence $\{m_j x\} = \{r_0 m_j \zeta(x)\} \sim T_{r_0}\sigma_{\zeta(x),\nu}$, as desired.

Now if ν is q-invariant and satisfies (5) and (6), suppose that $\mu = \delta(t') * T_{r_0}\nu$, $r_0 \neq 0$. With notation as above, we have

$$\sigma_{x,\mu} = \delta(t_1) * \sigma_{x-t', T_{r_0}\nu} = \delta(t_1) * T_{r_0}\sigma_{\zeta_1(x-t'),\nu} \quad \mu\text{-a.e.}$$

for some $t_1 \in \mathbb{T}$ and some ζ_1 with $T_{r_0} \circ \zeta_1 = \text{id}$. By (7), we may then write

$$\sigma_{x,\mu} = \delta(t_1) * T_{r_0}[T_{r'}^{-1}[\delta(b'\zeta_1(x-t')) * T_{s'_1}\nu * \cdots * T_{s'_l}\nu]] \quad \mu\text{-a.e.}$$
$$= \delta(t_1) * T_p T_r^{-1}[\delta(b'\zeta_1(x-t')) * T_{s'_1}\nu * \cdots * T_{s'_l}\nu] \quad \mu\text{-a.e.},$$

where r and p are relatively prime. In this case, T_p and T_r^{-1} commute, so that

$$\sigma_{x,\mu} = \delta(t_1) * T_r^{-1}[\delta(pb'(\zeta(x)+t_2)) * T_{ps'_1}\nu * \cdots * T_{ps'_l}\nu] \quad \mu\text{-a.e.},$$

where t_2 is chosen as any (fixed) point such that $T_{r_0}t_2 = -t'$ and $\zeta(x) = \zeta_1(x-t') - t_2$; we then have $T_{r_0} \circ \zeta = \text{id}$. Therefore

$$\sigma_{x,\mu} = \delta(t) * T_r^{-1}[\delta(b\zeta(x)) * T_{s_1}\nu * \cdots * T_{s_l}\nu] \quad \mu\text{-a.e.},$$

where t is any point such that $T_r(t-t_1) = pb't_2$, $b = pb'$ and $s_i = ps'_i$ $(1 \le i \le l)$. This gives (13).

Other examples of q-invariant measures satisfying (5) and (6) are given by generalized Riesz products: if $P(x)$ is a trigonometric polynomial

$$P(x) = 1 + \text{Re}\left\{\sum_{n=D_1}^{D_2} \alpha_n e(nx)\right\}$$

satisfying $P(x) \ge 0$, $0 < D_2/D_1 \le (|q|-1)/2$, and $\alpha_n = 0$ if $q \mid n$, then $\mu = \prod_{k\ge 0} P(q^k x)$ is seen to be q-invariant and to satisfy (5) and (6). In general, if ν is a q-invariant measure satisfying (5) and (6), then so is $\mu = T_{r_0}^{-1}\nu$ for any r_0 relatively prime to q.

We wonder whether hypothesis (6) can be eliminated from Theorem 1, subject to an appropriate modification of (7).

3. *Products of transformations*

In the non-invariant case, the following kind of phenomenon occurs. Suppose that

$$\mu = \prod_{k\ge 0} (1 + \text{Re}\{\alpha_k e(q^k x)\})$$

and $\alpha_k \to \alpha$; then $\{q^k x\} \sim \rho$ μ-a.e., where $\rho = \prod_{k\ge 0}(1 + \text{Re}\{\alpha e(q^k x)\})$. Although ρ may be singular to μ, nevertheless ρ is clearly closely related to μ. Once we give up invariance, our problem is almost as easy to treat for products of transformations $T_{q_n}T_{q_{n-1}} \cdots T_{q_1}$ as for iterates T_q^n. Thus, we proceed directly to this general case.

Given $|q_n| \ge 2$, $Q_n = q_1 q_2 \cdots q_n$, $Q_0 = 1$, let $\alpha(m)$ be the largest integer α such that $Q_\alpha \mid m$; put $\alpha(0) = 0$. We denote

$$\delta_n(m) = \left| \frac{m}{Q_{\alpha(m)}} - \frac{Q_{n+\alpha(m)}}{Q_{\alpha(m)}}\mathbb{Z} \right|;$$

thus $\delta_n(m) \neq 0$ if $mn \neq 0$.

THEOREM 3. *Let μ be a probability measure, $|q_n| \ge 2$, $\sup_n |q_n| < \infty$, $Q_n = q_1 q_2 \cdots q_n$, $Q_0 = 1$. Suppose that*

$$\forall b \in \mathbb{Z} \lim_{n\to\infty} \sup_{\substack{a\in\mathbb{Z}\\ p\in\mathbb{N}}} |\hat{\mu}(aQ_{n+p} + bQ_p) - \hat{\mu}(aQ_{n+p})\hat{\mu}(bQ_p)| = 0, \tag{14}$$

$$\text{if } \lim_{n\to\infty}\lim_{j\to\infty}\delta_n(m_j)=\infty, \quad \text{then } e(m_jx)\overset{w^*}{\to}0 \quad \text{in } L^\infty(\mu), \tag{15}$$

and

$$\overline{\lim_{n\to\infty}}\,|\hat{\mu}(n)|<1. \tag{16}$$

Then if $|m_j|\to\infty$ *is such that* (3) *holds, there exist* $r, b\in\mathbb{Z}$, $l\in\mathbb{N}^+$, *and* $s_1,\ldots,s_l\in\mathbb{Z}^*$ *such that*

$$\sigma_x=T_r^{-1}\left[\delta(bx)*\left(\mathop{*}_{i=1}^{l} T_{s_i}\nu_i\right)\right] \quad \mu\text{-a.e.}, \tag{17}$$

where for each i, there is a sequence $n_j\uparrow\infty$ *such that*

$$\forall k\in\mathbb{Z} \quad \hat{\mu}(kQ_{n_j})\to\hat{\nu}_i(k). \tag{18}$$

In other words, σ_x *is obtained from the weak* limit points of* $\{T_{Q_n}\mu\}_{n\geq 0}$.

We shall first establish

LEMMA 4. *Let* $\sup|q_n|<\infty$ *and* $\{m_j\}$ *be a sequence of integers such that for all* $n\geq 1$ *and all r,* $\lim_{j\to\infty}\delta_n(rm_j)$ *exists. Then*

$$E=\left\{r\colon \lim_{n\to\infty}\lim_{j\to\infty}\delta_n(rm_j)<\infty\right\} \tag{19}$$

is a subgroup of $\mathbb{Z}$.

Proof. If $E=\{0\}$, there is nothing to prove. Otherwise, let $r, s\in E$, $r\neq s$. Since $\lim_{j\to\infty}\delta_n(rm_j)$ is constant for large n, we may write

$$\forall n\ \forall^e j \quad rm_j=a_{j,n}Q_{n+\alpha(rm_j)}+bQ_{\alpha(rm_j)}, \tag{20}$$

where '$\forall^e j$' means 'for all but a finite number of j' (cf. [6]). Likewise, we may write

$$\forall n\ \forall^e j \quad sm_j=a'_{j,n}Q_{n+\alpha(sm_j)}+b'Q_{\alpha(sm_j)}. \tag{21}$$

Furthermore, by taking a subsequence of $\{m_j\}$ if necessary, we may assume that either $\forall j\ \alpha(rm_j)=\alpha(sm_j)$ or $\forall j\ \alpha(rm_j)>\alpha(sm_j)$, and that either $\alpha(rm_j)-\alpha(sm_j)\to\infty$ or $\{\alpha(rm_j)-\alpha(sm_j)\}$ is bounded.

Suppose first that $\alpha(rm_j)>\alpha(sm_j)$. Then $\alpha((r-s)m_j)=\alpha(sm_j)$ and

$$\forall n\ \forall^e j \quad (r-s)m_j=a''_{j,n}Q_{n+\alpha(sm_j)}+b''Q_{\alpha(sm_j)},$$

where $b''=-b'$ if $\alpha(rm_j)-\alpha(sm_j)\to\infty$ and $\forall^e j\ b''=bQ_{\alpha(rm_j)}Q^{-1}_{\alpha(sm_j)}-b'$ if $\alpha(rm_j)-\alpha(sm_j)$ is bounded. Hence $r-s\in E$.

Now suppose that $\alpha(rm_j)=\alpha(sm_j)$. We claim that $b\neq b'$. For if $b=b'$, then multiplying (20) by s, (21) by r, and subtracting, we obtain that

$$\forall n\ \forall^e j \quad Q_{n+\alpha(sm_j)}\,|\,(r-s)bQ_{\alpha(sm_j)}.$$

This contradicts the fact that $r\neq s$ and $b\neq 0$. Since

$$\forall n\ \forall^e j \quad (r-s)m_j=(a_{j,n}-a'_{j,n})Q_{n+\alpha(sm_j)}+(b-b')Q_{\alpha(sm_j)},$$

it follows that $r-s\in E$. □

The proof now proceeds essentially as for Theorem 1 and we restrict ourselves to its outline.

Proof of Theorem 3. We take $\sigma_x \neq \lambda$ and assume that $\lim_{j\to\infty} \delta_n(rm_j)$ exists for all n and r. Then the set E in (19) is equal to $r_0\mathbb{Z}$ for some $r_0 > 0$. By (15), there exists ν_x such that $\sigma_x = T_{r_0}^{-1}\nu_x$ μ-a.e.

We may suppose that $r_0 m_j = a_j Q_{n_j+\alpha(r_0 m_j)} + s_0 Q_{\alpha(r_0 m_j)}$, that $n_j \to \infty$, that $\{e(ra_j Q_{n_j+\alpha(r_0 m_j)} x)\}_j$ has a weak* limit in $L^\infty(\mu)$ for each r, and that $\{\alpha(r_0 m_j)\}$ is either constant or tends to ∞. If $\{\alpha(r_0 m_j)\}$ is constant, let

$$\hat{\Sigma}_1(r) = \lim_{j\to\infty} \hat{\mu}(ra_j Q_{n_j+\alpha(r_0 m_j)}) \quad \text{and} \quad b_0 = s_0 Q_{\alpha(r_0 m_j)};$$

otherwise, let $\hat{\Sigma}_1(r) = \lim_{j\to\infty} \hat{\mu}(rr_0 m_j)$ and $b_0 = 0$. Then $\nu_x = \delta(b_0 x) * \Sigma_1$ μ-a.e. by (14).

Define $m_j^{(1)} = r_0 m_j - b_0$; thus, $\hat{\Sigma}_1(r) = \lim_{j\to\infty} \hat{\mu}(rm_j^{(1)})$. We can argue as in the first paragraph to write $\Sigma_1 = T_{r_1}^{-1}\Sigma_1'$, where, without loss of generality, we may assume that

$$r_1 m_j^{(1)} = a_j^{(2)} Q_{n_j+\alpha(r_1 m_j^{(1)})} + s_1' Q_{\alpha(r_1 m_j^{(1)})}$$

and that $\Sigma_1' = T_{s_1'}\nu_1 * \Sigma_2$, where $\hat{\nu}_1(r) = \lim_{j\to\infty} \hat{\mu}(rQ_{\alpha(r_1 m_j^{(1)})})$. Note that $s_1' \neq 0$. We then define

$$m_j^{(2)} = r_1 m_j^{(1)} - s_1' Q_{\alpha(r_1 m_j^{(1)})}$$

and proceed for Σ_2 as we did for Σ_1, etc. This process ends since $\sigma_x \neq \lambda$ and (16) holds.

We thus obtain the expression

$$\sigma_x = T_{r_0}^{-1}[\delta(b_0 x) * T_{r_1}^{-1}[T_{s_1'}^{-1}\nu_1 * T_{r_2}^{-1}[T_{s_2'}\nu_2 * \cdots * T_{r_l}^{-1} T_{s_l'}\nu_l]] \cdots] \quad \mu\text{-a.e.},$$

which is reduced to (17) by use of (4). □

Remark. Even if (16) does not hold, we may still conclude that

$$\sigma_x = T_r^{-1}[\delta(bx) * T_s\nu * \Sigma] \quad \mu\text{-a.e.},$$

where ν has the form (18) and $\hat{\Sigma}(k) = \lim_j \hat{\mu}(kl_j)$ for some sequence $\{l_j\}$ (not necessarily tending to ∞).

The most obvious example of a measure satisfying the hypotheses of Theorem 3 is a Riesz product

$$\mu = \prod_{k\geq 0} (1 + \mathrm{Re}\,\{\alpha_k\, e(Q_k x)\})$$

with $|\alpha_k| \leq 1$ arbitrary, $Q_k \mid Q_{k+1}$, $|Q_{k+1}/Q_k| \geq 3$, and $\sup_k |Q_{k+1}/Q_k| < \infty$. In this case, the measures ν_i of (17) are also Riesz products $\prod_{k\geq 0}(1 + \mathrm{Re}\,\{\beta_k\, e(P_k x)\})$, with each β_k a limit point of $\{\alpha_j\}$, $P_0 = 1$, $P_k \mid P_{k+1}$, and each P_{k+1}/P_k a limit point of $\{Q_{j+1}/Q_j\}$.

Consider next the measure

$$\mu = \mathop{*}_{k\geq 1} [p_{0,k}\,\delta(0) + p_{1,k}\,\delta(Q_k^{-1}) + \cdots + p_{|q_k|-1,k}\,\delta((|q_k|-1)Q_k^{-1})],$$

where

$$|q_k| \geq 2, \quad \sup |q_k| < \infty, \quad Q_k = q_1 q_2 \cdots q_k, \quad p_{i,k} \geq 0, \quad \text{and} \quad \sum_{i=1}^{|q_k|-1} p_{i,k} = 1.$$

We claim that μ satisfies (14), (15) and (16) if $gcd\{i - i_0 \colon i \in I\} = 1$ for some set I and some $i_0 \in I$, where I satisfies the property that $\exists \varepsilon > 0\ \exists K\ \forall k_0\ \exists k \in [k_0, k_0 + K[\ \forall i \in I\ p_{i,k} \geq \varepsilon$; here, we interpret $p_{i,k} = 0$ if $i \geq |q_k|$. (This is the case in particular for

$$\mu = \mathop{*}_{k\geq 1} [p_k\,\delta(0) + (1 - p_k)\,\delta(2^{-k})]$$

if $\exists \varepsilon > 0\ \exists K\ \forall k_0\ \exists k \in [k_0, k_0 + K[\ \min\{p_k, (1 - p_k)\} \geq \varepsilon$.

This example will be important later.) Now since μ is continuous, (14) is proved just as (5) is proved for ordinary invariant Bernoulli convolutions (12) (that is, by 'lifting' to a product measure). To prove (15), suppose that $\hat{\mu}(m_j)$ has a non-zero limit. By taking an appropriate subsequence of $\{m_j\}$, we may assume that all the limits encountered below exist. For certain $\theta_{k,j}$, we have

$$\infty > \lim_{j\to\infty} \sum_{k\geq 1} \left[1 - \left|\sum_{i=0}^{|q_k|-1} p_{i,k}\, e(-im_jQ_k^{-1})\right|\right]$$
$$= \lim_j \sum_{k\geq 1} \left[1 - \sum_i p_{i,k}\, e(-im_jQ_k^{-1})\, e(\theta_{k,j})\right]$$
$$= \lim_j \sum_{k\geq 1} \sum_i p_{i,k}[1 - e(-im_jQ_k^{-1})\, e(\theta_{k,j})]$$
$$\geq \lim_j \sum_{k\geq 1} \sum_i p_{i,k+\alpha(m_j)} \operatorname{Re}\{1 - e(-im_jQ^{-1}_{k+\alpha(m_j)})\, e(\theta_{k+\alpha(m_j),j})\}.$$

Now for $j\geq 1$ and $l\geq 0$, $\exists k = k(l,j)\in [lK+\alpha(m_j), (l+1)K+\alpha(m_j)[$ such that $\forall i \in I$ $p_{i,k(l,j)}\geq \varepsilon$. Hence for $i\in I$,

$$\infty > \lim_j \sum_{l\geq 0} \varepsilon \operatorname{Re}\{1 - e(-im_jQ^{-1}_{k(l,j)})\, e(\theta_{k(l,j),j})\}$$
$$\geq \varepsilon \sum_{l\geq 0} \lim_j \operatorname{Re}\{1 - e(-im_jQ^{-1}_{k(l,j)})\, e(\theta_{k(l,j),j})\}.$$

Therefore for $i\in I$,

$$\lim_{l\to\infty}\lim_{j\to\infty} \operatorname{Re}\{1 - e(-im_jQ^{-1}_{k(l,j)})\, e(\theta_{k(l,j),j})\} = 0,$$

whence

$$\lim_l \lim_j e(-im_jQ^{-1}_{k(l,j)})\, e(\theta_{k(l,j),j}) = 1.$$

The hypothesis $gcd(I-i_0)=1$ implies finally that

$$\lim_l \lim_j e(m_jQ^{-1}_{k(l,j)}) = 1. \tag{22}$$

Now if (15) were not true, in other words, if $\lim_k \lim_j \delta_k(m_j)=\infty$, then for all N there would be a $k_0=k_0(N)$ such that $N_1 = \lim_j \delta_{k_0}(m_j)\geq N$. Let $\delta_{k_0}(m_j)=N_1$ for $j\geq j_0 = j_0(N)$ and, for $j\geq j_0$, let $n_j = n_j(N)$ be the least integer such that $|Q_{n_j+\alpha(m_j)}Q^{-1}_{\alpha(m_j)}|/2\geq N_1$. Since $n_j < 2+\log_2 N_1$, we may choose an infinite sequence $\mathcal{J}=\mathcal{J}(N)\subset [j_0,\infty[$ such that n_j is equal to a fixed n for $j\in\mathcal{J}$. Let $l=l(N)$ be the least integer such that $lK\geq n$ and consider any $j\in\mathcal{J}$. We have

$$\|m_jQ^{-1}_{k(l,j)}\| = |Q^{-1}_{k(l,j)}Q_{\alpha(m_j)}|\delta_{k(l,j)-\alpha(m_j)}(m_j).$$

Now $k_0\geq n$; for $k_0\geq k\geq n$, we have $\delta_k(m_j)=N_1$, while for $k>k_0$, we have $\delta_k(m_j)\geq N_1$. Since $k(l,j)-\alpha(m_j)\geq lK\geq n$, it follows that

$$\|m_jQ^{-1}_{k(l,j)}\| \geq |Q^{-1}_{n+\alpha(m_j)}Q_{\alpha(m_j)}N_1|\,|Q_{n+\alpha(m_j)}Q^{-1}_{k(l,j)}|.$$

By choice of n, the first term on the right is greater than $1/(2q)$, where $q=\sup_k |q_k|$. In addition, since

$$k(l,j)-n-\alpha(m_j) < 2K + (l-1)K - n < 2K,$$

the second term on the right is at least q^{-2K+2}. Therefore $\|m_jQ^{-1}_{k(l,j)}\| > q^{-2K+1}/2$. Since this is true for $j\in\mathcal{J}$ and since $l(N)\to\infty$ as $N\to\infty$, it follows that

$$\lim_l \lim_j \|m_jQ^{-1}_{k(l,j)}\| \geq \tfrac{1}{2}q^{-2K+1},$$

which contradicts (22). This proves (15). Finally, to establish (16), we will show that for $n \neq 0$,

$$|\hat{\mu}(n)| \leq 1 - 4\varepsilon q^{-2K}. \tag{23}$$

Given $n \neq 0$, let $k = \alpha(n) + 1$. Then $nQ_k^{-1} = N + rq_k^{-1}$ for some integers N and r, $0 < r < |q_k|$. Since $gcd(I - i_0) = 1$, there is some $i_1 \in I$ such that $(i_1 - i_0)nQ_k^{-1} \notin \mathbb{Z}$, whence $\|(i_1 - i_0)nQ_k^{-1}\| \geq |q_k|^{-1}$. Furthermore, for some $l \in [k, k+K[$, $p_{i_0,l} \geq \varepsilon$ and $p_{i_1,l} \geq \varepsilon$. We have

$$\begin{aligned}|\hat{\mu}(n)| &\leq \left| \sum_{i=0}^{|q_l|-1} p_{i,l}\, e(-inQ_l^{-1}) \right| \\ &= \left| \sum_{i=0}^{|q_l|-1} p_{i,l}\, e(-(i - i_0)nQ_l^{-1}) \right| \\ &\leq |p_{i_0,l} + p_{i_1,l}\, e(-(i_1 - i_0)nQ_l^{-1})| + 1 - p_{i_0,l} - p_{i_1,l}.\end{aligned}$$

Now for real x, y and θ, we have

$$\begin{aligned}|x + y\, e(\theta)| &= [(x+y)^2 - 4xy \sin^2 \pi\theta]^{1/2} \leq [(x+y)^2 - 16xy\|\theta\|^2]^{1/2} \\ &\leq (x+y) - 8\frac{xy}{x+y}\|\theta\|^2.\end{aligned}$$

Therefore

$$|\hat{\mu}(n)| \leq 1 - 8\frac{p_{i_0,l}\, p_{i_1,l}}{p_{i_0,l} + p_{i_1,l}} \|(i_1 - i_0)nQ_l^{-1}\|^2.$$

Our choice of l ensures (23).

Theorem 3 admits a ready, if somewhat ungainly, extension to the case of unbounded q_n.

THEOREM 5. *Let μ be a probability measure, $|q_n| \geq 2$, $Q_n = q_1 q_2 \cdots q_n$, and $Q_0 = 1$. Suppose that*

$$\forall U \in \mathbb{N} \quad \forall b_0, \ldots, b_U \in \mathbb{Z}$$

$$\lim_{n\to\infty} \sup_{\substack{a \in \mathbb{Z} \\ p \in \mathbb{N}}} \left| \hat{\mu}\left(aQ_{n+p} + \sum_{u=0}^{U} b_u Q_{u+p} \right) - \hat{\mu}(aQ_{n+p})\hat{\mu}\left(\sum_{u=0}^{U} b_u Q_{u+p} \right) \right| = 0, \tag{24}$$

if $e(m_j x) \not\to 0$ weak* in $L^\infty(\mu)$, then there exist $\{m_j'\} \subset \{m_j\}$, U, $n_j \in \mathbb{N}$, $a_j, b_0, \ldots, b_U \in \mathbb{Z}$ such that $n_j \to \infty$ and

$$m_j' = a_j Q_{n_j + \alpha(m_j')} + \sum_{u=0}^{U} b_u Q_{u+\alpha(m_j')}, \tag{25}$$

and

$$\overline{\lim_{n\to\infty}} |\hat{\mu}(n)| < 1. \tag{26}$$

If $|m_j| \to \infty$ is such that (3) *holds, then there exist $r, b \in \mathbb{Z}$ and $l \in \mathbb{N}^+$ such that*

$$\sigma_x = T_r^{-1}\left[\delta(bx) * \left(\mathop{*}_{i=1}^{l} \nu_i \right) \right] \quad \mu\text{-a.e.}, \tag{27}$$

where for each i, there is a sequence $n_j\uparrow\infty$, $U\in\mathbb{N}$, and $b_0,\ldots,b_U\in\mathbb{Z}$ such that

$$\forall k\in\mathbb{Z}\quad \hat{\mu}\left(k\sum_{u=0}^{U} b_u Q_{u+n_j}\right)\to\hat{\nu}_i(k). \tag{28}$$

The proof is exactly parallel; we shall only remark that the analogue of Lemma 4 is the following:

LEMMA 6. *Let $\{m_j\}$ be a sequence of integers such that for all $r\in\mathbb{Z}$, either $\{rm_j\}$ is of the form*

$$rm_j = a_j Q_{n_j+\alpha(rm_j)}+\sum_{u=0}^{U} b_u Q_{u+\alpha(rm_j)}\quad (n_j\to\infty),$$

or $\{rm_j\}$ has no subsequence of this form. Then the set E of $r\in\mathbb{Z}$ with $\{rm_j\}$ of the above form is a subgroup of $\mathbb{Z}$.

As an application, we consider any Riesz product

$$\mu=\prod_{k\geq 0}(1+\mathrm{Re}\,\{\alpha_k\, e(Q_k x)\})$$

with $Q_k\,|\,Q_{k+1}$ and $|Q_{k+1}/Q_k|\geq 3$. The hypotheses (24)-(26) are evidently satisfied and it remains to identify the measures ν_i of (28). Fix integers $U, b_0,\ldots,b_U$, and $n_j\uparrow\infty$, let $B_j=\sum_{u=0}^{U} b_u Q_{u+n_j}$, and assume that ν is the weak* limit of $T_{B_j}\mu$. Clearly, we may assume that $\sum_{u=0}^{U'} b_u Q_{u+n_j}\neq 0$ for $U'\leq U$ and all j. Furthermore, we may assume that for all $k\geq 0$, $\{Q_{k+n_j}Q_{n_j}^{-1}\}_j$ has a finite or infinite limit and that $\{\alpha_{k+n_j}\}_j$ has a limit. There are two possibilities: either for all $k\geq 1$, $\lim_{j\to\infty} Q_{k+n_j}Q_{n_j}^{-1}$ is finite or not. The former case is easily handled: $\nu=T_s\nu'$, where

$$s=\lim_{j\to\infty} B_j Q_{n_j}^{-1},\ \nu'=\prod_{k\geq 0}(1+\mathrm{Re}\,\{\tilde{\alpha}_k\, e(P_k x)\}),$$

$$P_k=\lim_{j\to\infty} Q_{k+n_j}Q_{n_j}^{-1},\quad\text{and}\quad \tilde{\alpha}_k=\lim_{j\to\infty}\alpha_{k+n_j}.$$

On the other hand, we claim that in the latter case, the spectrum of ν is finite (which is enough for our purposes in § 5). Let k_0 be the smallest integer such that $\lim_{j\to\infty} Q_{k_0+n_j}Q_{n_j}^{-1}$ is infinite. Set $n_0=\lim_{j\to\infty}|Q_{k_0-1+n_j}Q_{n_j}^{-1}|$ and define $B_j'=\sum_{u=0}^{\min(k_0-1,U)} b_u Q_{u+n_j}$, $B_j''=B_j-B_j'$. If $\hat{\nu}(n)\neq 0$, then for sufficiently large j, $\hat{\mu}(nB_j)\neq 0$, which means that nB_j has the representation

$$nB_j=\sum_{u\geq 0}\varepsilon_u^{(j)}Q_{u+n_j},\quad \varepsilon_u^{(j)}=0,\pm 1.$$

Since $Q_{k_0+n_j}$ divides nB_j'' and $\sum_{u\geq k_0}\varepsilon_u^{(j)}Q_{u+n_j}$, it follows that for sufficiently large j, $nB_j'=\sum_{u=0}^{k_0-1}\varepsilon_u^{(j)}Q_{u+n_j}$. Hence

$$|nQ_{n_j}|\leq|nB_j'|\leq\sum_{u=0}^{k_0-1}|Q_{u+n_j}|<(3/2)|Q_{k_0-1+n_j}|,\quad\text{whence } |n|<(3/2)n_0.$$

That is, the spectrum of ν is contained in the finite set $]-(3/2)n_0,(3/2)n_0[$, as desired.

4. *Perturbed Riesz products*

The above ideas can be used to analyze the asymptotic distribution of sequences relative to Riesz products based on a set of perturbed frequencies. The simplest example is

$$\mu=\prod_{k\geq 0}(1+\operatorname{Re}\{\alpha_k\, e[(q^k+d)x]\},$$

with $\alpha_k\to\alpha$. Set

$$\rho_t=\prod_{k\geq 0}(1+\operatorname{Re}\{\alpha\, e(dt)\, e(q^kx)\})$$

for $t\in\mathbb{T}$. Then if (3) holds, we can write

$$\sigma_t=T_r^{-1}[\delta(bt)*T_{s_1}\rho_t*\cdots*T_{s_l}\rho_t]\quad \mu\text{-a.e.}$$

In fact, we shall treat the following more general case. Let

$$\mu=\prod_{k\geq 0}(1+\operatorname{Re}\{\alpha_k\, e[(Q_k+d_k)x]\}),\tag{29}$$

where $|\alpha_k|\leq 1$, $Q_k\,|\,Q_{k+1}$, $|Q_{k+1}/Q_k|\geq 3$, and $|d_k|$ is bounded. Set $\mu'=\prod_{k\geq 0}(1+\operatorname{Re}\{\alpha_k\, e(Q_kx)\})$. The fundamental observation is that $e(m_jt)\not\to 0$ weak* in $L^\infty(\mu)$ if and only if $e(m_jt)\not\to 0$ weak* in $L^\infty(\mu')$. Indeed, $e(m_jt)\not\to 0$ weak* in $L^\infty(\mu)$ iff there are a subsequence $\{m_j'\}\subset\{m_j\}$, a sequence $\{u_j\}$ tending to ∞ and $\varepsilon_{u,j}=0,\pm 1$ such that

$$m_j'=\sum_{u\geq u_j}\varepsilon_{u,j}(Q_u+d_u)+O(1)\tag{30}$$

and

$$\liminf_{j\to\infty}\left|\prod_{u\geq u_j}(\tfrac{1}{2}\alpha_u)^{(\varepsilon_{u,j})}\right|>0,\tag{31}$$

where we denote, for complex z,

$$z^{(\varepsilon)}=\begin{cases} z & \text{if } \varepsilon=1,\\ \bar z & \text{if } \varepsilon=-1,\\ 1 & \text{if } \varepsilon=0.\end{cases}$$

But by (31), $\sum_{u\geq u_j}|\varepsilon_{u,j}|=O(1)$, whence (30) is equivalent (under (31)) to $m_j'=\sum_{u\geq u_j}\varepsilon_{u,j}Q_u+O(1)$; i.e. (30) is then independent of $\{d_k\}$. This establishes our claim.

Our next observation is that an analogue of mixing occurs. Let $U\geq 0$, a_j, $b_0,\ldots,b_U\in\mathbb{Z}$, $p_j\in\mathbb{N}$, and $n_j\to\infty$. Write $B_j=\sum_{u=0}^{U}b_uQ_{u+p_j}$ and assume that the following weak* limits exist in $L^\infty(\mu)$:

$$h(t)=\lim e(-(a_jQ_{n_j+p_j}+B_j)t),$$

$$f(t)=\lim e(-a_jQ_{n_j+p_j}t),\quad g(t)=\lim e(-B_jt).$$

Then

$$h=f\cdot g.\tag{32}$$

To prove this, we may, by taking a subsequence if necessary, assume that the above weak* limits also exist in $L^\infty(\mu')$; denote them by $\tilde h$, $\tilde f$, and $\tilde g$ respectively. We may also assume that the limits to appear below exist. It is clear that $\tilde h=\tilde f\cdot\tilde g$. If $\tilde h\neq 0$, then $\tilde f\neq 0$ and $\tilde g\neq 0$, whence we may write

$$a_jQ_{n_j+p_j}=\sum_{u\geq n_j}\varepsilon_{u,j}Q_{u+p_j},$$

$$B_j=\sum_{u=0}^{U'}\varepsilon'_{u,j}Q_{u+p_j},$$

$$\varepsilon_{u,j},\ \varepsilon'_{u,j}\in\{0,\pm 1\},$$

$$\sum_{u\geq n_j}|\varepsilon_{u,j}|=O(1).$$

Therefore

$$a_jQ_{n_j+p_j}+B_j=\sum_{u\ge n_j}\varepsilon_{u,j}(Q_{u+p_j}+d_{u+p_j})+C_j,$$

where

$$C_j=\sum_{u=0}^{U'}\varepsilon'_{u,j}(Q_{u+p_j}+d_{u+p_j})-\sum_{u\ge n_j}\varepsilon_{u,j}\,d_{u+p_j}-\sum_{u=0}^{U'}\varepsilon'_{u,j}\,d_{u+p_j}.$$

Noting that $\tilde{f}$ is a constant and that the last two sums in C_j are bounded, we see that $h=\tilde{f}G$, where

$$G(t)=w^*-\lim e(-C_jt)\quad \text{in } L^\infty(\mu).$$

But evaluation of G gives

$$G(t)=g(t)\,e(Dt),$$

where $D=\lim\sum_{u\ge n_j}\varepsilon_{u,j}\,d_{u+p_j}$. Therefore $h=\tilde{f}g\,e(Dt)=fg$. On the other hand, if $\tilde{h}=0$, then since $\tilde{f}$ is a constant, either $\tilde{f}$ or $\tilde{g}$ is 0. Therefore $h=0$ and f or g is 0, whence (32) again holds.

We are now in a position to imitate our preceding proofs in order to determine σ_t of (3). If $\sigma_t\not\equiv\lambda$ μ-a.e., then there is an $r\ne0$ such that $\hat{\sigma}_t(r)\not\equiv0$ μ-a.e., whence $e(-rm_jt)\not\to0$ weak* in $L^\infty(\mu')$. The subgroup E described in Lemma 6 therefore has a least positive element, r_0; let $\sigma_t=T_{r_0}^{-1}\nu_t$ μ-a.e.

We may suppose that

$$r_0m_j=a_jQ_{n_j+\alpha(r_0m_j)}+\sum_{u=0}^{U}b_uQ_{u+\alpha(r_0m_j)},$$

$n_j\to\infty$, and $\{\alpha(r_0m_j)\}$ is either constant or tends to ∞. If $\alpha(r_0m_j)$ is constant, set $b'=\sum_{u=0}^{U}b_uQ_{u+\alpha(r_0m_j)}$; otherwise, set $b'=0$. Put $m_j'=r_0m_j-b'$. Application of (32) to the sequence $\{rr_0m_j\}$ shows that

$$\hat{\nu}_t(r)=\hat{\Sigma}_{1,t}(r)\,e(-rb't)\quad \mu\text{-a.e.},$$

where $e(-rm_j't)\to\hat{\Sigma}_{1,t}(r)$ weak* in $L^\infty(\mu)$. Thus $\nu_t=\delta(b't)*\Sigma_{1,t}$.

We may argue as above to write $\Sigma_{1,t}=T_{r_1}^{-1}\Sigma'_{1,t}$ and, without loss of generality,

$$r_1m_j'=a_j'Q_{n_j'+\alpha(r_1m_j')}+\sum_{u=0}^{U'}b_u'Q_{u+\alpha(r_1m_j')},$$

$n_j'\to\infty$. We now have $\alpha(r_1m_j')\to\infty$. By (32), $\Sigma'_{1,t}=\nu'_{1,t}*\Sigma_{2,t}$, where $\hat{\nu}'_{1,t}(r)$ is the weak* limit in $L^\infty(\mu)$ of $e(-r\sum_{u=0}^{U'}b_u'Q_{u+\alpha(r_1m_j')}t)$ and

$$\hat{\Sigma}_{2,t}(r)=w^*-\lim e(-ra_j'Q_{n_j'+\alpha(r_1m_j')}t).$$

We proceed for $\Sigma_{2,t}$ as for $\Sigma_{1,t}$: we have $\Sigma_{2,t}=T_{r_2}^{-1}(\nu'_{2,t}*\Sigma_{3,t})$, and so on. Since $\overline{\lim}\,|\hat{\mu}(n)|\le\frac{1}{2}<1$, this process ends in a finite number of steps. As before, we conclude that

$$\sigma_t=T_r^{-1}\left[\delta(bt)*\left(\mathop{*}_{i=1}^{l}\nu_{i,t}\right)\right]\quad \mu\text{-a.e.,}\tag{33}$$

where each $\nu_{i,t}$ is a measure ω_t of the form

$$\forall k\in\mathbb{Z}\quad e\left(-k\sum_{u=0}^{U}b_uQ_{u+n_j}t\right)\to\hat{\omega}_t(k)\text{ weak* in }L^\infty(\mu),\quad n_j\to\infty.\tag{34}$$

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We now identify such measures ω_t. We base this on the result for μ'. We may assume that $\sum_{u=0}^{U'} b_u Q_{u+n_j} \neq 0$ for all $U' \leq U$ and all j. Let $B_j = \sum_{u=0}^{U} b_u Q_{u+n_j}$ and assume that for all $k \geq 0$, $\{Q_{k+n_j} Q_{n_j}^{-1}\}_j$ has a finite or infinite limit, P_k, and that $\{\alpha_{k+n_j}\}_j$ and $\{d_{k+n_j}\}_j$ have limits, call then $\tilde{\alpha}_k$ and $\tilde{d}_k$, respectively. If for some $k \geq 1$, P_k is infinite, then we know that $e(-kB_j t) \to 0$ weak* in $L^\infty(\mu')$ for all but a finite number of k; the same is true in $L^\infty(\mu)$, so that the spectrum of ω_t is contained in a finite set (independent of t). On the other hand, if P_k is finite for all k, then $\omega_t = T_s \omega'_t$, where $s = \lim_{j\to\infty} B_j Q_{n_j}^{-1}$ and $e(-rQ_{n_j} t) \to \hat{\omega}'_t(r)$ weak* in $L^\infty(\mu)$. We claim that $\omega'_t = \rho_t$ μ-a.e., where

$$\rho_t = \prod_{k\geq 0} (1 + \mathrm{Re}\,\{\tilde{\alpha}_k\, e(\tilde{d}_k t)\, e(P_k x)\}). \tag{35}$$

For if we define

$$\rho' = \prod_{k\geq 0} (1 + \mathrm{Re}\,\{\tilde{\alpha}_k\, e(P_k x)\}),$$

then $e(-rQ_{n_j} t) \to \hat{\rho}'(r)$ weak* in $L^\infty(\mu')$. Thus, if $\hat{\omega}'_t(r) = 0$ μ-a.e., we have $\hat{\rho}'(r) = 0$, which implies that $\hat{\rho}_t(r) = 0$ for all t. On the other hand, if $\hat{\omega}'_t(r) \not\equiv 0$, then $\hat{\rho}'(r) \neq 0$, so that we can write $r = \sum_{k\geq 0} \varepsilon_k P_k$, $\varepsilon_k = 0, \pm 1$. Therefore, interpreting limits as weak* in $L^\infty(\mu)$, we have

$$\begin{aligned}
\hat{\omega}'_t(r) &= \lim_{j\to\infty} e\left(-\sum_{k\geq 0} \varepsilon_k P_k Q_{n_j} t\right) \\
&= \lim_{j\to\infty} e\left(-\sum_{k\geq 0} \varepsilon_k Q_{k+n_j} t\right) \\
&= [\lim e(-\sum \varepsilon_k (Q_{k+n_j} + d_{k+n_j}) t]\, e(\sum \varepsilon_k \tilde{d}_k t) \\
&= \left[\prod_{k\geq 0} \tfrac{1}{2} \tilde{\alpha}_k^{(\varepsilon_k)}\right] e(\sum \varepsilon_k \tilde{d}_k t) = \prod_{k\geq 0} [\tfrac{1}{2}\tilde{\alpha}_k\, e(\tilde{d}_k t)]^{(\varepsilon_k)} \\
&= \hat{\rho}_t(r).
\end{aligned}$$

This shows that $\hat{\omega}'_t(r) = \hat{\rho}_t(r)$ for all r, whence the claim.

We sum up our results: either σ_t is a (non-negative) trigonometric polynomial multiplying λ,

$$\sigma_t(x) = \left[1 + \mathrm{Re} \sum_{n=1}^{N} \beta_n\, e(r_n t)\, e(nx)\right] \lambda(x),$$

or σ_t has the form

$$\sigma_t = T_r^{-1}\left[\delta(bt) * \left(\mathop{*}_{i=1}^{I} T_{s_i} \rho_{i,t}\right)\right],$$

where each $\rho_{i,t}$ is of the form given in (35).

5. *H-sets*

We turn now to some applications of the preceding theory. For their proper context, we refer the reader to [7; **9**; **18**, Chaps. IX, XII; and **1**, Chaps. XII, XIV]. In the 1920s, Rajchman introduced the following generalization of Cantor's middle-thirds sets.

Definition. A Borel set $E \subset \mathbb{T}$ is called an *H-set* if there exist a sequence $\{m_j\}_{j=1}^{\infty} \subset \mathbb{N}$ tending to ∞ and a non-empty open set $I \subset \mathbb{T}$ such that for every $x \in E$ and all j, $m_j x \notin I$.

Cantor's middle-thirds set is the set $\{x : \forall j \geq 0 \ 3^j x \notin]\frac{1}{3}, \frac{2}{3}[\}$. The connection of H-sets to our preceding discussion is given by the following observation.

PROPOSITION 7. *Let* $\mu \in M(\mathbb{T})$ *be such that whenever* (3) *holds for a sequence* $m_j \to \infty$, supp $\sigma_x = \mathbb{T}$ μ-a.e. *Then* $\mu E = 0$ *for all H-sets E.*

Proof. Let E be an H-set. Let $m_j \to \infty$ and I be a non-empty open set such that $m_j x \notin I$ for $x \in E$. By choosing a subsequence of $\{m_j\}$ if necessary, we may assume that there is a σ_x such that (3) holds and that $\{m_j x\} \sim \sigma_x$ μ-a.e. If $x \in E$, then clearly supp $\sigma_x \subset \mathbb{T} \backslash I$, whence supp $\sigma_x \neq \mathbb{T}$. The hypothesis implies, then, that $\mu(E) = 0$. □

We established in [7] and [9] that hyperlacunary Riesz products,

$$\mu = \prod_{k \geq 0} (1 + \mathrm{Re}\,\{\alpha_k\, e(n_k x)\}),\ n_{k+1}/n_k \to \infty, |\alpha_k| \leq 1,$$

annihilate all H-sets; if we choose $\alpha_k \not\to 0$, then these are examples of measures whose Fourier–Stieltjes coefficients do not vanish at infinity but which annihilate all H-sets nevertheless. This disproved a conjecture of Rajchman. New counter-examples are given by the following theorem.

THEOREM 8. *Let* μ *satisfy the hypotheses of Theorem* 1. *Then* μ *annihilates all H-sets if and only if* supp $\mu = \mathbb{T}$.

Note that if $\mu \neq \lambda$, then by q-invariance, $\hat{\mu}$ does not vanish at ∞.

Proof. The following facts are easily verified: if ω, ω' are positive measures with supp $\omega = \mathbb{T}$ and $r \in \mathbb{Z}$, then supp $T_r\omega =$ supp $T_r^{-1}\omega =$ supp $(\omega * \omega') = \mathbb{T}$. Therefore the measures σ_x of (7) have full support if μ does, and consequently μ annihilates all H-sets.

The converse is trivial. Indeed, if μ is any q-invariant measure whose support misses a non-empty open set I, then by q-invariance, supp μ also misses $T_q^{-j} I$ for all $j \geq 0$. That is, μ is supported on the H-set $\{x : \forall j \quad q^j x \notin I\}$. □

The following extension would be very interesting.

QUESTION. *If* μ *is a q-invariant q-mixing probability measure of full support, does* μ *annihilate all H-sets?*

Of course, Theorems 3 and 5 and the discussion of § 4 permit the statement of several theorems similar to Theorem 8. We shall restrict ourselves to the two main classes of examples, Riesz products and Bernoulli convolutions.

THEOREM 9. *Let* μ *be a Riesz product as in* (29) (*thus,* $Q_k \mid Q_{k+1}$, $|d_k| = \mathrm{O}(1)$). Then μ *annihilates all H-sets.*

Proof. § 4 shows that σ_t is a trigonometric polynomial, which certainly has full support, or is formed from Riesz products. But it is well-known that Riesz products

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have full support. (The proof is simple: if

$$\rho = \prod_{k\geq 0} (1+\mathrm{Re}\,\{\beta_k\, e(l_k t)\}),$$

define

$$P_K = \prod_{k=0}^{K} (1+\mathrm{Re}\,\{\beta_k\, e(l_k t)\}) \quad \text{and} \quad \rho_K = \prod_{k>K} (1+\mathrm{Re}\,\{\beta_k\, e(l_k t)\}).$$

Thus $\rho = P_K \cdot \rho_K$. If $\rho(I)=0$ for some open set I, then $\rho_K(I)=0$ since P_K has at most finitely many zeros and ρ_K is continuous. But since $\rho_K \to \lambda$ weak*, it follows that $\lambda I = 0$.) □

The same ideas apply to generalized Riesz products, of course. It would be very interesting to know whether all Riesz products annihilate all H-sets. Indeed, this question was the original motivation for the present work.

THEOREM 10. *Let μ be a Bernoulli convolution*

$$\mu = \mathop{*}_{k\geq 1} \sum_{i=0}^{|q_k|-1} p_{i,k}\, \delta(iQ_k^{-1}),$$

where $|q_k| \geq 2$, $\sup |q_k| < \infty$,

$$Q_k = q_1 q_2 \cdots q_k, \quad \sum_{i=0}^{q_k-1} p_{i,k} = 1,$$

and for all $|q| \geq 2$ *and all* $i \in [0, |q|-1]$,

$$\liminf_{k\to\infty} \{p_{i,k} \colon q_k = q\} > 0.$$

Then μ annihilates all H-sets.

Proof. It was shown that Theorem 3 is applicable; we only have to show that the weak* limit points of $\{T_{Q_n}\mu\}$ have full support. Let $T_{Q_{n_j}}\mu \to \nu$ weak*. We may assume the existence of the following limits for all $k \geq 1$:

$$\tilde{q}_k = \lim\nolimits_{j\to\infty} q_{n_j+k}, \qquad \tilde{p}_{i,k} = \lim_{j\to\infty} p_{i,n_j+k} \quad (0 \leq i \leq |\tilde{q}_k|-1).$$

If $\tilde{Q}_k = \tilde{q}_1 \cdots \tilde{q}_k$, we see that

$$\nu = \mathop{*}_{k\geq 1} \sum_{i=0}^{|\tilde{q}_k|-1} \tilde{p}_{i,k}\, \delta(i\tilde{Q}_k^{-1}).$$

Since $\tilde{p}_{i,k} > 0$ by hypothesis, $\mathrm{supp}\, \nu = \mathbb{T}$. □

It turns out that the converse of Proposition 7 holds as well. We first establish the following lemma.

LEMMA 11 [7]. *Let μ be a positive measure on a measurable space X without atoms of infinite measure. Let E_n be measurable sets, $\mathbf{1}_n$ the characteristic functions of E_n, and*

$$E = \left\{ t\colon \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_n(t) = 1 \right\}$$

be the set of points lying in almost all the E_n. Then

$$\mu E \leq \sup_{\{n_k\}} \mu\left(\bigcap_{k=1}^{\infty} E_{n_k} \right),$$

where $\{n_k\}$ *runs over all sequences with* $n_k \to \infty$. *In particular, if* $\mu(\bigcap_{k=1}^{\infty} E_{n_k}) = 0$ *for all* $\{n_k\}$, *then* $\mu E = 0$.

Proof. By restricting μ to a subset of E of finite measure, if necessary, it suffices to assume that μ is a probability measure concentrated on E. It follows that

$$1 = \int_X \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_n(t)\, d\mu(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu E_n.$$

Given $\varepsilon > 0$, there exists, therefore, a sequence $n_k \to \infty$ such that $\mu E_{n_k} > 1 - \varepsilon 2^{-k}$. Since $\mu(\bigcap_{k=1}^{\infty} E_{n_k}) > 1 - \varepsilon$, the lemma follows. □

Definition [7]. A Borel set $E \subset \mathbb{T}$ is called an *asymptotic H-set* if there exists a sequence $m_j \to \infty$ and a non-empty open set $I \subset \mathbb{T}$ such that for $x \in E$,

$$\lim_{j\to\infty} \frac{1}{J} \operatorname{card}\{j \le J\colon m_j x \notin I\} = 1.$$

COROLLARY 12 [7]. *A measure annihilates all asymptotic H-sets [resp., those based on any subsequence of* $\{m_j\}$ *and* I] *if and only if it annihilates all H-sets [resp., those based on any subsequence of* $\{m_j\}$ *and* I].

Proof. This follows immediately from Lemma 11 applied to the sets

$$E_j = \{x\colon m_j x \notin I\}.$$ □

We are now able to give the following version of Proposition 7 and its converse.

THEOREM 13. *Let* $\mu \in M(\mathbb{T})$ *and* $m_j \to \infty$ *be such that* (3) *holds. Then* $\operatorname{supp} \sigma_x = \mathbb{T}$ μ*-a.e. if and only if* μ *annihilates all H-sets based on any subsequence of* $\{m_j\}$.

Proof. One direction was shown in the proof of Proposition 7. For the other, suppose that $\operatorname{supp} \sigma_x \ne \mathbb{T}$ on a set of positive $|\mu|$-measure. We may assume that $\{m_j x\} \sim \sigma_x$ μ-a.e. Then there is a set F' of positive measure and an $\eta > 0$ such that $\operatorname{supp} \sigma_x$ misses some arc of length η for every $x \in F'$, whence there is a set F of positive measure and a fixed arc I' of length $\eta/2$ such that $\operatorname{supp} \sigma_x \cap I' = \varnothing$ for all $x \in F$. Let I be a non-empty open arc whose closure is contained in the interior of I'. Then

$$\lim_{J\to\infty} \frac{1}{J} \operatorname{card}(j \le J\colon m_j x \in I\} = 0 \quad \text{for } x \in F,$$

whence F is an asymptotic H-set based on $\{m_j\}$. By Corollary 12, there is a subsequence $\{m_j'\}$ of $\{m_j\}$ such that the H-set $\{x\colon \forall j\ m_j' x \notin I\}$ has positive $|\mu|$-measure. □

Consider now the Cantor–Lebesgue measure

$$\mu = \mathop{\ast}_{k\ge 1} [\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta(2 \cdot 3^{-k})]$$

supported on the Cantor middle-thirds set. (We have chosen an invariant μ for simplicity, not for any essential reason.) One expects intuitively that the *only* H-sets not annihilated by μ are those based on sequences sufficiently similar to $\{3^j\}$. This is true:

THEOREM 14. *Let*

$$\mu = \mathop{*}_{k\geq 1} [\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta(2\cdot 3^{-k})]$$

be the Cantor–Lebesgue measure. If E is an H-set of positive μ-measure corresponding to a sequence $\{m_j\}$, then every subsequence $\{m_j'\}$ of $\{m_j\}$ has a further subsequence $\{m_j''\}$ of the form

$$rm_j'' = s3^{n_j} + b \quad (r, s \in \mathbb{N}^+, b \in \mathbb{Z}, n_j \to \infty). \tag{36}$$

Conversely, if $\{m_j\}$ is of the form (36), *then there is an H-set of positive μ-measure corresponding to a subsequence of $\{m_j\}$.*

Proof. If $E \subset \{x: \forall j\, m_j x \notin I\}$ is an H-set of positive measure and $\{m_j'\} \subset \{m_j\}$, then $\{x: \forall j\, m_j' x \notin I\}$ is also of positive measure since it contains E. Thus, to prove the first half of the theorem, it suffices to prove only that $\{m_j\}$ has a subsequence $\{m_j''\}$ of the form (36).

Now let $\{m_j'''\}$ be a subsequence of $\{m_j\}$ such that $e(-km_j'''x) \to \hat{\sigma}_x(k)$ weak* in $L^\infty(\mu)$. By (13), we know that

$$\sigma_x = T_r^{-1}\left[\delta(b\zeta(x)) * \left(\mathop{*}_{i=1}^{l} T_{s_i}\nu\right)\right] \quad \mu\text{-a.e.},$$

where

$$\nu = \mathop{*}_{k\geq 1} [\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta(3^{-k})],$$

with $l = 1$ if and only if $\{m_j'''\}$ has a further subsequence $\{m_j''\}$ of the form (36). Thus, by Theorem 13, the present theorem reduces to showing that $l \neq 1$ if and only if $\operatorname{supp} \sigma_x = \mathbb{T}$ μ-a.e.

Suppose first that $l = 1$. Then $\operatorname{supp} T_{s_1}\nu = \{s_1 x: x \in \operatorname{supp} \nu\}$. Since $\operatorname{supp} \nu$ is a nowhere dense set, so is $\operatorname{supp} T_{s_1}\nu$, and so, therefore, is $\operatorname{supp} \sigma_x$.

Conversely, suppose that $l > 1$. We shall show that $\operatorname{supp}(T_{s_1}\nu * T_{s_2}\nu) = \mathbb{T}$. Since ν is 3-invariant, we may assume that $3 \nmid s_1 s_2$. Now

$$T_s\nu = \mathop{*}_{k\geq 1} [\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta(s3^{-k})],$$

so that

$$\rho \overset{\text{def}}{=} T_{s_1}\nu * T_{s_2}\nu = \mathop{*}_{k\geq 1} [\tfrac{1}{4}\delta(0) + \tfrac{1}{4}\delta(s_1 3^{-k}) + \tfrac{1}{4}\delta(s_2 3^{-k}) + \tfrac{1}{4}\delta((s_1+s_2)3^{-k})] = \xi_K * \omega_K$$

for any $K \geq 1$, where ξ_K is the discrete measure formed by the convolution of the first K terms and ω_K is the probability measure formed by the remainder. Now $\operatorname{supp} \rho = \mathbb{T} \Leftrightarrow \rho I > 0$ for every arc I of the form

$$I = [A - (|s_1| + |s_2|)3^{-K}, A + (|s_1| + |s_2|)3^{-K}],$$

$$A = \sum_{k=1}^{K} a_k 3^{-k}, \quad a_k \in \{0, 1, 2\}, \quad K \geq 1.$$

Given such an arc, we can choose $\varepsilon_K, \varepsilon_K' \in \{0, 1\}$ such that

$$\varepsilon_K s_1 + \varepsilon_K' s_2 \equiv a_K \pmod 3$$

since $3 \nmid s_1 s_2$. We may then choose $\varepsilon_{K-1}, \varepsilon_{K-1}' \in \{0, 1\}$ such that

$$(\varepsilon_{K-1}s_1 + \varepsilon_{K-1}'s_2)3 + (\varepsilon_K s_1 + \varepsilon_K' s_2) \equiv a_{K-1}3 + a_K \pmod{3^2},$$

and so on, until we have chosen ε_k, $\varepsilon'_k \in \{0, 1\}$ $(1 \le k \le K)$ such that

$$\sum_{k=1}^{K} (\varepsilon_k s_1 + \varepsilon'_k s_2) 3^{K-k} \equiv \sum_{k=1}^{K} a_k 3^{K-k} \pmod{3^K},$$

which is the same as

$$\sum_{k=1}^{K} (\varepsilon_k s_1 + \varepsilon'_k s_2) 3^{-k} \equiv A \pmod 1.$$

Therefore $\xi_K(\{A\}) \ge (1/4)^K$; since $\operatorname{supp} \omega_K \subset [-(|s_1|+|s_2|)3^{-K}, (|s_1|+|s_2|)3^{-K}]$, it follows that $\rho I \ge 4^{-K}$. □

We now present a similar example which will be useful in a moment.

LEMMA 15. *Let*

$$\pi = \mathop{\ast}_{\substack{k \ge 1 \\ k \notin \mathcal{N}}} [\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta(2^{-k})],$$

where $\mathcal{N} = \{n_j\}_{j \ge 1}$ *is a sequence such that* $n_{j+1} - n_j \to \infty$. *If* $\{m_j\}$ *is a sequence which corresponds to an H-set of positive* π*-measure,then there is a subsequence* $\{m'_j\}$ *of the form*

$$rm'_j = s2^{n'_j - 1} + b \quad (r, s \in \mathbb{N}^+, b \in \mathbb{Z}, n'_j \in \mathcal{N}).$$

Proof. We may assume that $e(-km_jx) \to \hat{\sigma}_x(k)$ weak* in $L^\infty(\pi)$. If we interpret π as containing the terms $1 \cdot \delta(0) + 0 \cdot \delta(2^{-k})$ for $k \in \mathcal{N}$, then Theorem 3 is applicable by the discussion which followed that theorem. Thus,

$$\sigma_x = T_r^{-1}\left[\delta(bx) * \left(\mathop{\ast}_{i=1}^{l} T_{s_i}\nu_i\right)\right] \quad \pi\text{-a.e.},$$

where each ν_i is a weak* limit point of $\{T_{2^k}\pi\}$. Suppose that $T_{2^{k_j}}\pi \to \nu$ weak*. If $|k_j - \mathcal{N}|$ is unbounded, then it is easy to see that $\nu = \lambda$. If $|k_j - \mathcal{N}|$ is bounded, then without loss of generality, $k_j = n'_j + d$, where $n'_j \in \mathcal{N}$. If $d \ge 0$, then $\nu = \lambda$; if $d < 0$, then

$$\nu = \mathop{\ast}_{\substack{k \ge 1 \\ k \ne -d}} [\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta(2^{-k})].$$

Since every ν_i is of this form and $\operatorname{supp} \sigma_x \ne \mathbb{T}$ for a set of positive π-measure, it follows that $l = 1$ and that for some $\{m'_j\} \subset \{m_j\}$, $rm'_j = s2^{k_j} + b$ with $k_j = n'_j + d$, $n'_j \in \mathcal{N}$, and $d \le -1$. Therefore $(r2^{-d-1})m'_j = s2^{n'_j - 1} + (b2^{-d-1})$, which is the desired form. □

The following generalization of H-sets was introduced by Pjateckiĭ-Šapiro [**14**; **18**, Chap. XII, § 11; **1**, Chap. XIV, § 15; **9**].

Definition. Let $m \in \mathbb{Z}^+$. If

$$V = (v^{(1)}, \dots, v^{(m)}) \in \mathbb{Z}^m, \ \Lambda = (l_1, \dots, l_m) \in \mathbb{Z}^m$$

and $x \in \mathbb{T}$, we write $V \cdot \Lambda = \sum_{i=1}^m v^{(i)} l_i$ and $Vx = (v^{(1)}x, \dots, v^{(m)}x)$. A sequence $\{V_k\}_1^\infty \subset (\mathbb{Z}^+)^m$ of m-tuples of positive integers is called *quasi-independent* if for each fixed $\Lambda \in \mathbb{Z}^m$, Λ not the 0-vector, we have $|V_k \cdot \Lambda| \to \infty$ as $k \to \infty$. A Borel set $E \subset \mathbb{T}$ is called an $H^{(m)}$*-set* if there exist a quasi-independent sequence $\{V_k\} \subset (\mathbb{Z}^+)^m$ and a non-empty open set $I \subset \mathbb{T}^m$ such that for all $x \in E$ and all k, $V_k x \notin I$.

In [7] and [9], we asked whether for each $m \geq 1$, there is a measure supported on an $H^{(m+1)}$-set which annihilates all $H^{(m)}$-sets, in other words, whether $H^{(m+1)}$ is 'much larger' than $H^{(m)}$. Here we show that the answer is 'yes' for $m=1$.

THEOREM 16. *Let π be the measure in Lemma* 15,

$$\rho = \mathop{*}_{j\geq 1} [\tfrac{1}{3}\delta(0)+\tfrac{1}{3}\delta(2^{-n_{2j-1}})+\tfrac{1}{3}\delta(2^{-n_{2j}})], \quad \text{and } \mu = \pi * \rho.$$

Then μ is supported on an $H^{(2)}$-set and annihilates all H-sets.

Proof. Indeed, μ is supported on the 'canonical' $H^{(2)}$-set

$$\{x : \forall j\ (2^{n_{2j-1}-1}x, 2^{n_{2j}-1}x) \notin]\tfrac{1}{2}, 1[\times]\tfrac{1}{2}, 1[\}.$$

Suppose that E were an H-set corresponding to a sequence $\{m_j\}$ with $\mu E > 0$. Since $\mu E = \int_{\mathbb{T}} \pi(E-t)\, d\rho(t)$, it would follow that $\pi(E-t) > 0$ for some t. But $E-t$ is an H-set corresponding to a subsequence $\{m'_j\} \subset \{m_j\}$ (if $\{m'_j\}$ is chosen so that $\{m'_j t\}$ is almost constant, then $\{m'_j x\}$ is not dense for $x \in E-t$). Lemma 15 shows that for a further subsequence $\{m''_j\} \subset \{m'_j\}$, we have

$$rm''_j = s2^{n'_j-1} + b \quad \text{with } n'_j \in \mathcal{N}.$$

Let $e(-km''_j x) \to \hat{\sigma}_x(k)$ and $e(-k2^{n'_j-1}x) \to \hat{\tau}_x(k)$ weak* in $L^\infty(\mu)$. It is not hard to calculate that

$$\tau_x = [\tfrac{2}{3}\delta(0)+\tfrac{1}{3}\delta(2^{-1})] * \left[\mathop{*}_{k\geq 2} (\tfrac{1}{2}\delta(0)+\tfrac{1}{2}\delta(2^{-k}))\right] \quad \mu\text{-a.e.}$$

(This can also be calculated by convolving the weak* limits in $L^\infty(\pi)$ and $L^\infty(\rho)$; see [10].) Of course, $\operatorname{supp} \tau_x = \mathbb{T}$; since $\sigma_x = T_r^{-1}[\delta(bx) * T_s\tau_x]$, we also have $\operatorname{supp} \sigma_x = \mathbb{T}$, which completes the proof by contradicting Theorem 13. □

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