Compositio Mathematica **136:** 89–101, 2003.

The Structure of the (Outer) Automorphism Group of a Bieberbach Group

WIM MALFAIT[1] and ANDRZEJ SZCZEPAŃSKI[2]
[1]*Department of Mathematics, Katholieke Universiteit Leuven, Campus Kortrijk, Universitaire Campus, Etienne Sabbelaan 53, B-8500 Kortrijk, Belgium. e-mail: wim.malfait@kulak.ac.be*
[2]*Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland. e-mail: matas@paula.univ.gda.pl*

(Received: 3 May 2001; accepted in final form: 3 January 2002)

Abstract. Referring to Tits' alternative, we develop a necessary and sufficient condition to decide whether the normalizer of a finite group of integral matrices is polycyclic-by-finite or is containing a non-Abelian free group. This result is of fundamental importance to conclude whether the (outer) automorphism group of a Bieberbach group is polycyclic-by-finite or has a non-cyclic free subgroup.

Mathematics Subject Classifications (2000). Primary 20H15, 20F28; secondary 20F34, 20C15, 16U60.

Key words. (outer) automorphism group, Bieberbach group, dicyclic group, finite group of integral matrices, polycyclic-by-finite group, (ir)reducible representation, Schur index, unit group of an order.

1. Introduction

Let M be a flat Riemannian manifold of dimension n and assume that $E = \pi_1(M)$ denotes its fundamental group. Then E is a torsion-free group fitting into an extension $\mathbb{Z}^n \rightarrowtail E \twoheadrightarrow F$ where $\mathbb{Z}^n$ is maximal Abelian in E and F is a finite group. Equivalently, E is a uniform discrete subgroup of $\mathbb{R}^n \rtimes \mathrm{O}(n)$, acting freely on the Euclidean space $\mathbb{R}^n$. We refer to F as the holonomy group of $M = E\backslash\mathbb{R}^n$ and E is said to be a Bieberbach group. The holonomy group F acts on $\mathbb{Z}^n$ by conjugation in E, defining a faithful representation $T\colon F \to \mathrm{Gl}(n, \mathbb{Z})$. The outer automorphism group $\mathrm{Out}(E)$ of E has an important geometric meaning as it is isomorphic to the group of affine diffeomorphisms of M modulo those isotopic to the identity ([4]).

In [13], a necessary and sufficient condition is presented for the outer automorphism group of a Bieberbach group E to be infinite. More precisely, $\mathrm{Out}(E)$ is infinite if and only if the decomposition over $\mathbb{Q}$ of the associated holonomy representation $T\colon F \to \mathrm{Gl}(n, \mathbb{Z})$ has at least two components of the same isomorphism type or there is a $\mathbb{Q}$-irreducible component which is reducible over $\mathbb{R}$. Essentially, the proof reduces to analyzing when the normalizer of a finite group of integral matrices is infinite.

In [14], Tits proved that a finitely generated linear group is either virtually solvable or it contains a noncyclic free subgroup. This motivates, for a faithful representation $T: F \to \mathrm{Gl}(n, \mathbb{Z})$ of a finite group F, to investigate further the normalizer of $T(F)$ in $\mathrm{Gl}(n, \mathbb{Z})$ and to refine the result quoted above. In fact, in this paper we present a necessary and sufficient condition, again in terms of (the decomposition over $\mathbb{Q}$ of) the representation T, for the normalizer of $T(F)$ in $\mathrm{Gl}(n, \mathbb{Z})$ to be polycyclic-by-finite or alternatively, to contain a non-Abelian free group.

This result has nice and straightforward implications towards understanding the (outer) automorphism group of a Bieberbach group E. An equivalent condition, in terms of the associated holonomy representation, for $\mathrm{Aut}(E)$ and $\mathrm{Out}(E)$ to be polycyclic-by-finite (or on the other hand, to admit a noncyclic free subgroup), is obtained. As an application of our approach, we focus on Bieberbach groups with a dicyclic holonomy group, discuss their (outer) automorphism group and classify the dicyclic groups which occur as holonomy group of a Bieberbach group possessing only finitely many outer automorphisms.

2. The Normalizer of a Finite Group of Integral Matrices

Let us start with some notational remarks used throughout this paper. If G is a group, then $\mathrm{Inn}(G)$ denotes the group of inner automorphisms of G, $\mathrm{Aut}(G)$ is the group of automorphisms of G and $\mathrm{Out}(G) = \mathrm{Aut}(G)/_{\mathrm{Inn}(G)}$ is the outer automorphism group of G. If X is a subset in G, then $C_G X$ denotes the centralizer of X in G and $N_G X$ is the normalizer of X in G. In case A is an algebra and $X \subseteq A$, then $C_A X$ is a subalgebra of A, referred to as the commuting algebra of X in A.

Recall that each semisimple finite-dimensional $\mathbb{Q}$-algebra A has a Wedderburn decomposition $A = \bigoplus_{i=1}^{k} A_i$ $(k \in \mathbb{N})$ into simple algebras A_i. Each component A_i can be identified with a full $(f_i \times f_i)$-matrix ring $M_{f_i}(D_i)$ over a finite-dimensional division algebra D_i over $\mathbb{Q}$ $(1 \leqslant i \leqslant k)$. A unital subring Λ of A is a $\mathbb{Z}$-order if it is a finitely generated $\mathbb{Z}$-module such that $\mathbb{Q}\Lambda = A$.

Let $T_0: F \to \mathrm{Gl}(n, \mathbb{Q})$ be a $\mathbb{Q}$-irreducible representation of a finite group F. Then T_0 decomposes as a direct sum of absolutely irreducible constituents and in this decomposition, all components occur with equal multiplicity. The multiplicity of such a component is by definition the Schur index $m_{\mathbb{Q}}(T_0)$ of T_0 over $\mathbb{Q}$. Equivalently, if D is the commuting algebra of $T_0(F)$ in $M_n(\mathbb{Q})$, then $m_{\mathbb{Q}}(T_0)^2$ is equal to the dimension of D over its center $Z(D)$ (which is a number field).

Now let $T: F \to \mathrm{Gl}(n, \mathbb{Z})$ be a faithful representation of a finite group F. In [13], the following necessary and sufficient condition was obtained for the finiteness of the normalizer of $T(F)$ in $\mathrm{Gl}(n, \mathbb{Z})$. It depends on how T decomposes when the coefficient domain is enlarged from $\mathbb{Z}$ to $\mathbb{Q}$, and then to $\mathbb{R}$.

THEOREM 2.1. *Let $T: F \to \mathrm{Gl}(n, \mathbb{Z})$ be a faithful representation of a finite group F. Then $N_{\mathrm{Gl}(n,\mathbb{Z})} T(F)$ is finite if and only if in the $\mathbb{Q}$-decomposition of T, all $\mathbb{Q}$-irreducible components have multiplicity one and are also $\mathbb{R}$-irreducible.*

An alternative proof of this theorem can be deduced from [3], using the language of $\mathbb{Z}$-orders. Essential in this context is a result due to Siegel ([12]) and Brown ([1]), providing a criterion for the unit group of a $\mathbb{Z}$-order in a finite-dimensional division algebra over $\mathbb{Q}$ to be finite. More generally, it follows that the unit group of a $\mathbb{Z}$-order in a semisimple finite-dimensional $\mathbb{Q}$-algebra A is finite if and only if each Wedderburn component of A is either the rationals, an imaginary quadratic extension of $\mathbb{Q}$ or definite quaternions over $\mathbb{Q}$.

An analogue approach will now be used to study when, for a faithful representation $T: F \to \mathrm{Gl}(n, \mathbb{Z})$ of a finite group F, the normalizer of $T(F)$ in $\mathrm{Gl}(n, \mathbb{Z})$ is virtually solvable. In this perspective, the following theorem of H. Zassenhaus ([16]) will be of key importance:

THEOREM 2.2. *The unit group of a $\mathbb{Z}$-order in a semisimple finite-dimensional $\mathbb{Q}$-algebra A is virtually solvable if and only if the Wedderburn components of A are number fields or definite quaternions over $\mathbb{Q}$.*

For any representation $T: F \to \mathrm{Gl}(n, \mathbb{Z})$ of a finite group F, it is known that the normalizer $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ is finitely generated. Therefore, one can apply a theorem of J. Tits ([14]) to conclude that this normalizer is either virtually solvable or has a non-Abelian free subgroup. In fact, since one is working with integral matrices, in the first case, $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ is known to be polycyclic-by-finite. The following theorem (which extends Theorem 2.1) establishes a criterion to decide which one of the two alternatives occurs for $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$:

THEOREM 2.3. *Assume that $T: F \to \mathrm{Gl}(n, \mathbb{Z})$ is a faithful representation of a finite group F. Then $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ is polycyclic-by-finite if and only if in the $\mathbb{Q}$-decomposition of T, all components occur with multiplicity one and the Schur index over $\mathbb{Q}$ of each $\mathbb{R}$-reducible component is equal to one.*

Proof. Obviously, because F is finite, the centralizer $C_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ is of finite index in $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ and it is, hence, sufficient to prove the statement for $C_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$. This is exactly the unit group of the commuting ring of $T(F)$ in $M_n(\mathbb{Z})$. Moreover, $C_{M_n(\mathbb{Z})}T(F)$ is a $\mathbb{Z}$-order in the semisimple finite-dimensional $\mathbb{Q}$-algebra $C_{M_n(\mathbb{Q})}T(F)$.

For a suitable basis of $\mathbb{Q}^n$, T decomposes into a direct sum of $\mathbb{Q}$-irreducible components T_i. Without loss of generality, one can assume that in this decomposition, equivalent components are identical, write f_i for the multiplicity of T_i and hence $T = \bigoplus_{i=1}^{k} f_i T_i$ $(k \in \mathbb{N})$. The Wedderburn decomposition of $C_{M_n(\mathbb{Q})}T(F)$ is then

$$C_{M_n(\mathbb{Q})}T(F) \cong \bigoplus_{i=1}^{k} M_{f_i}(D_i)$$

where D_i can be chosen as the commuting algebra of $T_i(F)$ $(1 \leqslant i \leqslant k)$.

Because of Theorem 2.2, if $f_I > 1$ for some i $(1 \leqslant i \leqslant k)$, then $C_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ contains a non-Abelian free group. Therefore, assume that all components in the $\mathbb{Q}$-decomposition of T have multiplicity one (for all i, $1 \leqslant i \leqslant k, f_i = 1$). Then the commuting algebras D_i of the $\mathbb{Q}$-irreducible components T_i of T are exactly the Wedderburn components of $C_{M_n(\mathbb{Q})}T(F)$. In case T_i is $\mathbb{R}$-reducible, realize that D_i is a number field if and only if $m_{\mathbb{Q}}(T_i) = 1$. Moreover, if T_i is $\mathbb{R}$-irreducible, then its commuting algebra D_i is either $\mathbb{Q}$, an imaginary quadratic extension of $\mathbb{Q}$ or a definite quaternion algebra over $\mathbb{Q}$. Applying Theorem 2.2 now finishes the proof.

3. Finite Groups with Schur Index One Over $\mathbb{Q}$

Let us devote special attention to finite groups F having Schur index equal to one for all $\mathbb{Q}$-irreducible representations of F. In that case, for a faithful representation $T: F \to \mathrm{Gl}(n, \mathbb{Z})$, $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ is polycyclic-by-finite if and only if all $\mathbb{Q}$-irreducible components of T occur with multiplicity one (Theorem 2.3).

In fact, if F is an Abelian finite group, then for all $\mathbb{Q}$-irreducible representations T of F, $m_{\mathbb{Q}}(T) = 1$. More generally, for a finite nilpotent group F and a $\mathbb{Q}$-irreducible representation $T: F \to \mathrm{Gl}(n, \mathbb{Q})$, it is known that $m_{\mathbb{Q}}(T) \leqslant 2$. In particular, for the generalized quaternion groups Q_{2^α},

$$Q_{2^\alpha} = \langle a, b \mid a^{2^{\alpha-1}} = 1, b^2 = a^{2^{\alpha-2}}, b^{-1}ab = a^{-1} \rangle, \quad \alpha \geqslant 3,$$

for each faithful $\mathbb{Q}$-irreducible representation T of Q_{2^α}, $m_{\mathbb{Q}}(T) = 2$. Conversely, if $m_{\mathbb{Q}}(T) = 2$ for a $\mathbb{Q}$-irreducible representation $T: F \to \mathrm{Gl}(n, \mathbb{Q})$ of a finite nilpotent group F, then the Sylow 2-subgroup of F is a generalized quaternion group ([10]). In particular, note that if F is a p-group with $p \neq 2$, then $m_{\mathbb{Q}}(T)$ is always equal to one.

If F is a finite solvable group such that all its Sylow subgroups are elementary Abelian, then $m_{\mathbb{Q}}(T) = 1$ for all $\mathbb{Q}$-irreducible representations T of F ([10]). Moreover, also for all dihedral groups $\mathcal{D}_{2n}$ (of order $2n$), $m_{\mathbb{Q}}(T) = 1$ for all $\mathbb{Q}$-irreducible representation T of $\mathcal{D}_{2n}$ ([11]). Another interesting class of solvable groups are the dicyclic groups:

$$Q_{4m} = \langle a, b \mid a^{2m} = 1, b^2 = a^m, b^{-1}ab = a^{-1} \rangle, \quad m \geqslant 2.$$

As for the generalized quaternion groups, for each faithful $\mathbb{Q}$-irreducible representation T of Q_{4m}, $m_{\mathbb{Q}}(T) = 2$. We come back to this specific class of finite solvable groups in the final section of this paper.

Let us apply the above observations to all finite groups of order <16. First note the following fact ([16]):

LEMMA 3.1. *Let F be a finite group and T a $\mathbb{Q}$-irreducible representation of F. Then $m_{\mathbb{Q}}(T)$ divides the exponent of F and $m_{\mathbb{Q}}(T)^2$ divides the order of F. Moreover, if 2^r divides $m_{\mathbb{Q}}(T)$, then 2^{r+1} divides the exponent of F.*

Of course, for all finite groups of order 2, 3, 4, 5, 7, 9, 11 and 13, the Schur index is always equal to one. Because of the above lemma, this also holds for all finite groups of order 6, 10, 14 and 15. The only non-Abelian groups of order 8 are $\mathcal{D}_8$ (with Schur

https://doi.org/10.1023/A:1022698228140 Published online by Cambridge University Press

index always 1) and Q_8 (which can have Schur index 2). The non-Abelian groups of order 12 are $\mathcal{D}_{12}$ (always having Schur index 1), Q_{12} (possibly with Schur index 2) and the alternating group $\mathcal{A}_4$. The Schur index of $\mathcal{A}_4$ is always equal to one because of Lemma 3.1 (note that its exponent is 6).

4. The (Outer) Automorphism Group of a Bieberbach Group

Write $T: F \to \mathrm{Gl}(n, \mathbb{Z})$ for the induced faithful representation of the holonomy group F of a Bieberbach group E. In [4], a commutative diagram was developed to describe $\mathrm{Aut}(E)$ (and $\mathrm{Out}(E)$), taking off from a crucial group action of $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ on the cohomology group $H^2(F, \mathbb{Z}^n)$. If a denotes the 2-cohomology class corresponding to the given extension $\mathbb{Z}^n \rightarrowtail E \twoheadrightarrow F$ and N_a is the stabilizer of a under this action, then

$$Z^1(F, \mathbb{Z}^n) \rightarrowtail \mathrm{Aut}(E) \twoheadrightarrow N_a \quad \text{and} \quad H^1(F, \mathbb{Z}^n) \rightarrowtail \mathrm{Out}(E) \twoheadrightarrow Q = N_a/F$$

are exact. From this, it can be easily deduced that $\mathrm{Out}(E)$ is finite if and only if $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ is finite and, hence, Theorem 2.1 also provides a necessary and sufficient condition for the outer automorphism group of a Bieberbach group to be finite ([13, Thm. A]).

This observation can now be generalized as follows:

THEOREM 4.1. *Let E be a Bieberbach group with holonomy group F (and associated representation $T: F \to \mathrm{Gl}(n, \mathbb{Z})$). Then $\mathrm{Out}(E)$ is polycyclic-by-finite if and only if in the $\mathbb{Q}$-decomposition of T, all components have multiplicity one and the Schur index over $\mathbb{Q}$ of each $\mathbb{R}$-reducible component is equal to one.*

Proof. It is sufficient to realize that $\mathrm{Out}(E)$, N_a and $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ are all polycyclic-by-finite if and only if one of them is polycyclic-by-finite. Theorem 2.3 finishes the proof. $\square$

Remark 4.2. Of course, because a Bieberbach group E is itself polycyclic-by-finite, $\mathrm{Aut}(E)$ is polycyclic-by-finite if and only if $\mathrm{Out}(E)$ is polycyclic-by-finite. And since $\mathrm{Aut}(E)$ and $\mathrm{Out}(E)$ are finitely presented linear groups ([15]), Tits' alternative implies that in the other case, $\mathrm{Aut}(E)$ and $\mathrm{Out}(E)$ contain a non-Abelian free group.

Before illustrating Theorem 4.1, note that it is only from dimension 5 onwards that Bieberbach groups with infinite but polycyclic-by-finite outer automorphism group can occur ([2]).

EXAMPLE 4.3. Consider the following Bieberbach group E:

$$\begin{aligned} E = \langle & a, b, c, d, e, \alpha \mid \\ & \alpha a = d^{-1}\alpha, \alpha b = a\alpha, \alpha c = b\alpha, \alpha d = c\alpha, \alpha e = e\alpha, \alpha^8 = e \rangle \end{aligned}$$

https://doi.org/10.1023/A:1022698228140 Published online by Cambridge University Press

fitting into the extension $\mathbb{Z}^5 \rightarrowtail E \twoheadrightarrow \mathbb{Z}_8$. In the presentation above, trivial commutators were omitted (e.g. the generators a, b, c, d, e all commute). The same convention will be used throughout this paper.

The corresponding holonomy representation is

$$T\colon \mathbb{Z}_8 \to \mathrm{Gl}(5, \mathbb{Z})\colon \alpha \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

decomposing over $\mathbb{Q}$ into a one-dimensional component and an irreducible component of degree 4 which is however $\mathbb{R}$-reducible ([2, p. 232]). It follows that $\mathrm{Out}(E)$ is infinite (Theorem 2.1) but polycyclic-by-finite (Theorem 4.1).

In fact, taking off from the calculation of the normalizer of the 4-dimensional component of T ([2, p. 338]), one can compute that $\mathbb{Z}^4 \rightarrowtail \mathrm{Aut}(E) \twoheadrightarrow \mathbb{Z}_8 \times \mathbb{Z}$ and $\mathrm{Out}(E) \cong \mathbb{Z}$, generated by the outer automorphism corresponding to the following E-automorphism:

$$a \mapsto abd^{-1}, \quad b \mapsto abc, \quad c \mapsto bcd, \quad d \mapsto a^{-1}bcd, \quad e \mapsto e, \quad \alpha \mapsto \alpha.$$

Referring to Theorem 4.1, a non-Abelian free group of (outer) automorphisms of a Bieberbach group can arise in two possible ways. Here we illustrate how this happens when there is a component in the $\mathbb{Q}$-decomposition of the holonomy representation which occurs more than once. The case of a multiplicity-free holonomy representation with an $\mathbb{R}$-reducible component with Schur index >1 will be illustrated in Example 5.8.

EXAMPLE 4.4. Of course, the easiest example of a Bieberbach group with a non-abelian free group of (outer) automorphisms is $\mathbb{Z}^2$, the fundamental group of the 2-torus. The classical non-Abelian free subgroup of $\mathrm{Gl}(2, \mathbb{Z})$ is the group generated by $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$.

Let us discuss another example. Consider the following three-dimensional Bieberbach group E ([2, p. 62]):

$$E = \langle a, b, c, \alpha \mid \alpha a = a^{-1}\alpha, \alpha b = b\alpha, \alpha c = c^{-1}\alpha, \alpha^2 = b\rangle$$

fitting into $\mathbb{Z}^3 \rightarrowtail E \twoheadrightarrow \mathbb{Z}_2$. The holonomy group $\mathbb{Z}_2$ acts faithfully on $\mathbb{Z}^3$ as follows:

$$T\colon \mathbb{Z}_2 \to \mathrm{Gl}(3, \mathbb{Z})\colon \alpha \mapsto T(\alpha) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and, hence, T has a component of multiplicity 2. Because of Theorem 4.1, this guarantees the existence of a non-Abelian free group of (outer) automorphisms. And indeed, observe that

$$\sigma_1\colon \begin{cases} a \mapsto a \\ b \mapsto b \\ c \mapsto a^2c \\ \alpha \mapsto \alpha \end{cases} \quad \text{and} \quad \sigma_2\colon \begin{cases} a \mapsto ac^2 \\ b \mapsto b \\ c \mapsto c \\ \alpha \mapsto \alpha \end{cases}$$

generate a non-Abelian free subgroup of $\mathrm{Aut}(E)$ (and the induced outer automorphisms define a non-cyclic free subgroup of $\mathrm{Out}(E)$).

5. Dicyclic Groups of Integral Matrices

In this section, we illustrate and apply the results presented above for dicyclic groups of integral matrices and deduce some interesting results concerning the (outer) automorphism group of Bieberbach groups with a finite group of this type as holonomy group.

5.1. THE IRREDUCIBLE COMPLEX REPRESENTATIONS OF A DICYCLIC GROUP

Let us first discuss all possible, inequivalent $\mathbb{C}$-irreducible representations of a dicyclic group

$$Q_{4m} = \langle a, b \mid a^{2m} = 1, b^2 = a^m, b^{-1}ab = a^{-1} \rangle, \quad m \geqslant 2$$

and meanwhile fix some notations used throughout this section.

There are four one-dimensional representations given by

– if m is odd:

$$\chi_1\colon \begin{cases} a \mapsto 1 \\ b \mapsto 1 \end{cases}, \quad \chi_2\colon \begin{cases} a \mapsto 1 \\ b \mapsto -1 \end{cases}, \quad \chi_3\colon \begin{cases} a \mapsto -1 \\ b \mapsto i \end{cases} \quad \text{and} \quad \chi_4\colon \begin{cases} a \mapsto -1 \\ b \mapsto -i \end{cases}.$$

– if m is even, then $a \mapsto \pm 1$, $b \mapsto \pm 1$.

Next, we consider absolutely irreducible representations of degree 2. Put $\omega = \mathrm{e}^{\pi i/m}$, a primitive $2m$th root of unity. Let r be an arbitrary integer and define representations ρ_r of Q_{4m} by setting:

$$\rho_r\colon\ a \mapsto \begin{pmatrix} w^r & 0 \\ 0 & w^{-r} \end{pmatrix} \quad \text{and} \quad b \mapsto \begin{pmatrix} 0 & (-1)^r \\ 1 & 0 \end{pmatrix}.$$

Obviously, ρ_r and ρ_{2m-r} are equivalent and since ρ_0 and ρ_m are reducible over $\mathbb{C}$, we may assume that $0 < r < m$ (the remaining representations are indeed irreducible over $\mathbb{C}$ and pairwise not equivalent).

In fact, the representations of degree 1 and 2 as given above, are the only $\mathbb{C}$-irreducible representations of Q_{4m} (up to equivalence). Indeed, the sum of the squares of their degrees is equal to $4 \times 1 + (m-1) \times 4 = 4m$, which is exactly the order of Q_{4m}. Also note that ρ_r is faithful if and only if r and $2m$ are relative prime. Hence, there are, up to equivalence, exactly $\varphi(2m)/2$ faithful irreducible representations of Q_{4m} over $\mathbb{C}$ (where φ denotes the Euler phi function).

https://doi.org/10.1023/A:1022698228140 Published online by Cambridge University Press

From the above classification of all $\mathbb{C}$-irreducible representations, all possible $\mathbb{Q}$-irreducible representations of Q_{4m} can be deduced. If m is even, then all one-dimensional representations are suitable. If m is odd however, then χ_1, χ_2 and $\chi_3 \oplus \chi_4$ are the only irreducible representations of Q_{4m} over $\mathbb{Q}$ which can be obtained from the absolutely irreducible representations of degree 1. Of course, the Schur index over $\mathbb{Q}$ is always equal to one in these cases.

The Galois group of the $2m$th cyclotomic field $\mathbb{Q}(\omega)$ over $\mathbb{Q}$ (which is of order $\varphi(2m)$) permutes the faithful representations ρ_r transitively and their Schur index is 2. Therefore, there exists a unique faithful irreducible representation ρ of Q_{4m} defined over $\mathbb{Q}$, which is equivalent to $\bigoplus 2\rho_r$ where $0 < r < m$ and r is relative prime to $2m$. Note that the degree of ρ is $2 \cdot 2(\varphi(2m)/2) = 2\varphi(2m)$.

5.2. BIEBERBACH GROUPS WITH A DICYCLIC GROUP AS HOLONOMY

With the knowledgde established above, one easily deduces the following:

PROPOSITION 5.1. *Let $T: Q_{4m} \to \mathrm{Gl}(n, \mathbb{Z})$ be a representation of a dicyclic group Q_{4m} with $m > 3$. If a $\mathbb{Q}$-irreducible component of T is faithful, then $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ contains a non-Abelian free group.*

Proof. It is only possible that ρ (as defined above) is $\mathbb{R}$-irreducible if and only if $2\varphi(2m) = 4$ and $\varphi(2m) = 2$ if and only if $m = 2$ or $m = 3$. Because of Theorem 2.3 (and Tits' alternative), this finishes the proof since $m_{\mathbb{Q}}(\rho) = 2$. □

Hence,

COROLLARY 5.2. *For all faithful representations $T: Q_{2^m} \to \mathrm{Gl}(n, \mathbb{Z})$ of a generalized quaternion group Q_{2^m} with $m > 3$, $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ contains a non-Abelian free group.*

Proof. Since $a^{2^{m-2}}$ is an element of the kernel of all $\mathbb{C}$-irreducible representations of degree 1 and also of all nonfaithful representations ρ_r (r is even), each faithful representation $T: Q_{2^m} \to \mathrm{Gl}(n, \mathbb{Q})$ must have the unique faithful $\mathbb{Q}$-irreducible representation ρ of Q_{2^m} as component (and then apply Proposition 5.1). □

Remark 5.3. The above result does not hold for all Q_{4m} with $m > 3$ (or the occurence of ρ in the $\mathbb{Q}$-decomposition of Q_{4m} is essential in Proposition 5.1). For instance, take the representation $T = T_1 \oplus T_2: Q_{24} \to \mathrm{Gl}(6, \mathbb{Z})$ where

– $T_1: a \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $b \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ ([2, p · 59]),

– $T_2: a \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and $b \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$ ([2, p · 245]).

https://doi.org/10.1023/A:1022698228140 Published online by Cambridge University Press

Since the kernel of T_1 is generated by a^3 and the kernel of T_2 by a^4, it follows that T is faithful. Also note that T_1 and T_2 are both $\mathbb{R}$-irreducible. Hence, the normalizer of $T(Q_{24})$ in $\mathrm{Gl}(6, \mathbb{Z})$ is even a finite group (Theorem 2.1).

The situation for Q_8 and Q_{12} is rather specific:

LEMMA 5.4. *All $\mathbb{Q}$-irreducible representations of Q_8 and Q_{12} are $\mathbb{R}$-irreducible.*

Proof. The $\mathbb{Q}$-irreducible representations of Q_8 are the 1-dimensional representations χ_i $(1 \leqslant i \leqslant 4)$ and the unique faithful representation $\rho_1 \oplus \rho_1$ of degree 4 (which is $\mathbb{R}$-irreducible, [2, p. 245, 32/01]). For Q_{12}, the $\mathbb{Q}$-irreducible representations of degree 1 are χ_1 and χ_2. For degree 2, first note that $\chi_3 \oplus \chi_4$ defines a $\mathbb{Q}$-irreducible representation of Q_{12}, equivalent to the $\mathbb{R}$-irreducible representation given in [2, p. 58, 3/1/1]. Also ρ_2 defines a (nonfaithful) $\mathbb{Q}$-irreducible representation of Q_{12} and is equivalent to the $\mathbb{R}$-irreducible representation of [2, p. 59, 4/2/2]. Finally, there is the unique faithful $\mathbb{Q}$-irreducible representation of Q_{12}, given by $\rho_1 \oplus \rho_1$, which is also $\mathbb{R}$-irreducible ([2, p. 239, 30/01/01]). □

Remark 5.5. It follows immediately from Theorem 2.1 that for a faithful representation T of Q_8, or of Q_{12}, in $\mathrm{Gl}(n, \mathbb{Z})$, the normalizer of $T(Q_8)$, respectively, of $T(Q_{12})$, in $\mathrm{Gl}(n, \mathbb{Z})$ is finite if and only if each $\mathbb{Q}$-irreducible component of T has multiplicity one. If T is not multiplicity-free over $\mathbb{Q}$, then a non-Abelian free subgroup is contained in this normalizer (Theorem 2.3).

Analogously, a Bieberbach group with Q_8 or Q_{12} as holonomy, has only finitely many outer automorphisms if and only if the corresponding holonomy representation is multiplicity-free over $\mathbb{Q}$ ([13, Thm. A]). In the other case, its (outer) automorphism group admits a noncyclic free subgroup (Theorem 4.1).

Referring to Theorem 2.3 and to the discussion on the Schur index of representations of finite groups of small order, we are now able to conclude that

PROPOSITION 5.6. *Let $T: F \to \mathrm{Gl}(n, \mathbb{Z})$ be a faithful representation of a finite group F of order <16. Then $N_{\mathrm{Gl}(n,\mathbb{Z})}T(F)$ is polycyclic-by-finite if and only if all $\mathbb{Q}$-irreducible components of T have multiplicity one.*

Therefore (applying Theorem 4.1)

COROLLARY 5.7. *A Bieberbach group with holonomy group of order <16 has a polycyclic-by-finite (outer) automorphism group if and only if the corresponding holonomy representation is multiplicity-free over $\mathbb{Q}$.*

In this perspective (and because of Corollary 5.2), we devote special attention to faithful, multiplicity-free representations of Q_{16}.

EXAMPLE 5.8. Consider the representation

$$T\colon Q_{16} \to \mathrm{Gl}(8,\mathbb{Z})\colon \begin{cases} \alpha \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \beta \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \end{cases}$$

which is equivalent to the unique faithful $\mathbb{Q}$-irreducible representation $2\rho_1 \oplus 2\rho_3$ of Q_{16} and is hence $\mathbb{R}$-reducible. It follows that $N_{\mathrm{Gl}(8,\mathbb{Z})}T(Q_{16})$ contains a non-Abelian free group (Theorem 2.3).

As in the proof of [8, Thm. 1.6], one can construct an 11-dimensional Bieberbach group E with Q_{16} as holonomy group and a multiplicity-free holonomy representation which is the direct sum of T and a representation of degree 3. It immediately follows that $\mathrm{Aut}(E)$ and $\mathrm{Out}(E)$ contain a non-Abelian free subgroup (Theorem 4.1).

5.3. DICYCLIC $\mathcal{R}_1$-GROUPS

Finite groups which occur as holonomy group of a Bieberbach group with finite outer automorphism group are referred to as $\mathcal{R}_1$-groups. In fact, Q_8 is the only generalized quaternion group which is an $\mathcal{R}_1$-group ([9, Prop. 3.3], see also Corollary 5.2). We now show that Q_{12} and Q_{24} also are $\mathcal{R}_1$-groups.

EXAMPLE 5.9. Consider the following Bieberbach group E (it was checked that E is torsion-free by using [5]):

$$\begin{aligned} E = \langle a, b, c, d, e, f, \alpha, \beta \mid \\ & \alpha a = b^{-1}\alpha,\ \alpha b = ab\alpha,\ \alpha c = d^{-1}\alpha,\ \alpha d = cd\alpha,\ \alpha e = e\alpha,\ \alpha f = f\alpha, \\ & \beta a = c^{-1}d^{-1}\beta,\ \beta b = d\beta,\ \beta c = ab\beta,\ \beta d = b^{-1}\beta,\ \beta e = e\beta,\ \beta f = f^{-1}\beta, \\ & \alpha^6 = e^3 f^2,\ \beta^2 = e^{-2}f^{-1}\alpha^3,\ \beta^{-1}\alpha\beta = e\alpha^{-1}\rangle \end{aligned}$$

fitting into $\mathbb{Z}^6 \rightarrowtail E \twoheadrightarrow Q_{12}$. The holonomy representation is given by

$$T\colon Q_{12} \to \mathrm{Gl}(6,\mathbb{Z})\colon \begin{cases} \alpha \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\ \beta \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \end{cases}$$

Obviously, T is equivalent to $(\rho_1 \oplus \rho_1) \oplus \chi_1 \oplus \chi_2$ (using the notations introduced above). Hence, all $\mathbb{Q}$-irreducible components of T have multiplicity one and are also $\mathbb{R}$-irreducible. If follows that $\mathrm{Out}(E)$ is finite (Theorem 2.1) and Q_{12} is an $\mathcal{R}_1$-group.

EXAMPLE 5.10. Inspired by Remark 5.3, we define the following Bieberbach group E (again [5] was used to verify that E is torsion-free):

$$\begin{aligned} E = \langle a, b, c, d, e, f, g, h, \alpha, \beta \mid \\ &\alpha a = b\alpha, \alpha b = a^{-1}b^{-1}\alpha, \alpha c = e^{-1}\alpha, \alpha d = f\alpha, \alpha e = c\alpha, \alpha f = d^{-1}\alpha, \\ &\beta a = b^{-1}\beta, \beta b = a^{-1}\beta, \beta c = f^{-1}\beta, \beta d = e^{-1}\beta, \beta e = d\beta, \beta f = c\beta, \\ &\alpha g = g\alpha, \alpha h = h\alpha, \beta g = g\beta, \beta h = g^{-1}h^{-1}\beta, \\ &\alpha^{12} = gh^2, \beta^2 = g^{-1}h^{-1}\alpha^6, \beta^{-1}\alpha\beta = \alpha^{-1}\rangle \end{aligned}$$

fitting into an extension $\mathbb{Z}^8 \rightarrowtail E \twoheadrightarrow Q_{24}$ and with holonomy representation $T\colon Q_{24} \to \mathrm{Gl}(8,\mathbb{Z})$ which is equivalent to $T_1 \oplus T_2 \oplus \chi_1 \oplus \chi_2$ (using the notations introduced in Remark 5.3). Then T is multiplicity-free over $\mathbb{Q}$ and all its $\mathbb{Q}$-irreducible components are also $\mathbb{R}$-irreducible, which implies that $\mathrm{Out}(E)$ is finite (Theorem 2.1) and hence Q_{24} is an $\mathcal{R}_1$-group.

In fact, there are no other dicyclic groups which occur as holonomy group of a Bieberbach group with finitely many outer automorphisms.

THEOREM 5.11. *Q_8, Q_{12} and Q_{24} are the only dicyclic $\mathcal{R}_1$-groups.*

Proof. Suppose that Q_{4m} is an $\mathcal{R}_1$-group. Because of Theorem 2.1, observe that the degree of each $\mathbb{Q}$-irreducible component of the holonomy representation $T\colon Q_{4m} \to \mathrm{Gl}(n,\mathbb{Z})$ is at most 4 (otherwise it is $\mathbb{R}$-reducible). Let p be a prime number and let p^α be the highest power of p dividing $2m$. Because of [9, Lemma 2.2(a)], it follows that $p^{\alpha-1}(p-1)$ divides 4. This implies that $2m$ divides 120. In other words, the proof reduces to showing that Q_{16}, Q_{20}, Q_{40}, Q_{48}, Q_{60}, Q_{80}, Q_{120} and Q_{240} are not $\mathcal{R}_1$-groups. Only for Q_{16}, this is already known ([9, Prop. 3.3]).

Note that the only possible absolutely irreducible components of T are the non-faithful representations ρ_r of Q_{4m}, together with the irreducible representations of degree 1 (Proposition 5.1). Recall that the kernel of ρ_r is generated by some a^k with k dividing $2m$. If k divides m, then ρ_r induces a faithful representation of $\mathcal{D}_{2k}$. In the other case (i.e. k is not dividing m and hence k is even), ρ_r defines a faithful representation of Q_{2k} (or of $\mathbb{Z}_4$ if $k = 2$). For dihedral groups $\mathcal{D}_{2n}$, if n is not dividing 12, then each faithful $\mathbb{Q}$-irreducible representation of $\mathcal{D}_{2n}$ is $\mathbb{R}$-reducible ([9, Prop. 5.1]). Similarly, for dicyclic groups, a faithful $\mathbb{Q}$-irreducible representation of Q_{4n} is $\mathbb{R}$-reducible unless $n = 2$ or $n = 3$. We conclude that the only representations ρ_r of Q_{4m} which can be used to build up T are those whose kernel is generated by some a^k with k dividing 12.

If follows that if 6 divides m, then a^{12} always is a nontrivial (since $m \neq 6$!) element of the kernel of the possible ρ_r (and also the one-dimensional representations have a^{12} in their kernel). This applies to Q_{48}, Q_{120} and Q_{240}. For Q_{20}, both ρ_2 and ρ_4 have kernel generated by a^5. Therefore, only representations of degree 1 are allowed. For Q_{40}, the only possible ρ_r is ρ_5, with kernel generated by a^4. Similarly, the admissible representations ρ_r of Q_{60} always have a^6 in their kernel and for Q_{80}, a^4 always belongs to the kernel of the possible ρ_r. In all these cases, T can not be faithful.

Acknowledgements

This work was supported by the Volkswagen-Stiftung (RiP-program at Oberwolfach).

We also would like to thank S. K. Sehgal for drawing our attention to [16].

References

1. Brown, H.: An application of Zassenhaus' unit theorem, *Acta Math.* **20** (1972), 297–298.
2. Brown, H., Bülow, R., Neubüser, J., Wondratscheck, H., and Zassenhaus, H.: *Crystallographic Groups of Four-dimensional Space*, Wiley, New York, 1978.
3. Brown, H., Neubüser, J., and Zassenhaus, H.: On integral groups III: Normalizers, *Math. Comp.* **27** (1973), 167–182.
4. Charlap, L. S. and Vasquez, A. T.: Compact flat Riemannian manifolds III: the group of affinities, *Amer. J. Math.* **95** (1973), 471–494.
5. Eick, B. and Nickel, W.: Polycylic – A GAP [7] package, 2000 (http://www.mathematik.uni-kassel.de/~eick).
6. Fein, B. and Yamada, T.: The Schur index and the order and exponent of a finite group, *J. Algebra* **28** (1974), 496–498.
7. The GAP Group, Aachen, St Andrews: GAP – Groups, algorithms, and programming, Version 4.2, 1999. (http://www-gap.dcs.st-and.ac.uk/~gap).
8. Hiller, H., Marciniak, Z., Sah, C.-H. and Szczepański, A.: Holonomy of flat manifolds with $b_1 = 0$, II, *Quart. J. Math. Oxford* **38** (1987), 213–220.
9. Hiss, G. and Szczepański, A.: Holonomy groups of Bieberbach groups with finite outer automorphism groups, *Arch. Math.* **65** (1995), 8–14.

10. Passman, D. S.: *The Algebraic Structure of Group Rings, Pure Appl. Math*, Wiley, New York, 1977.
11. Serre, J.-P.: *Linear Representations of Finite Groups*, Grad. Texts in Math. 42, Springer, New York, 1977.
12. Siegel, C. L.: Discontinuous groups, *Ann. of Math.* **44** (1943), 674–689.
13. Szczepański, A.: Outer automorphism groups of Bieberbach groups, *Bull. Belg. Math. Soc. Simon Stevin* **3** (1996), 585–593.
14. Tits, J.: Free subgroups of linear groups, *J. Algebra* **20** (1972), 250–270.
15. Wehrfritz, B.: Two remarks on polycyclic groups, *Bull. London Math. Soc.* **26**(6) (1994), 543–548.
16. Zassenhaus, H.: When is the unit group of a Dedekind order solvable? *Comm. Algebra* **6** (1978), 1621–1627.