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Constructions of some families of smooth Cauchy transforms

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Abstract. For a given Beurling–Carleson subset E of the unit circle $\mathbb{T}$ which has positive Lebesgue measure, we give explicit formulas for measurable functions supported on E such that their Cauchy transforms have smooth extensions from $\mathbb{D}$ to $\mathbb{T}$. The existence of such functions has been previously established by Khrushchev in 1978, in non-constructive ways by the use of duality arguments. We construct several families of such smooth Cauchy transforms and apply them in a few related problems in analysis: an irreducibility problem for the shift operator, an inner factor permanence problem. Our development leads to a self-contained duality proof of the density of smooth functions in a very large class of de Branges–Rovnyak spaces. This extends the previously known approximation results.

1 Introduction

Let E be a closed subset of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ of the complex plane $\mathbb{C}$, and let the notation dm stand for the Lebesgue measure, normalized by the condition $m(\mathbb{T}) = 1$. The following question has been studied by Khrushchev in [12]. What conditions on the set E guarantee the existence of a nonzero measurable function h supported on E for which the *Cauchy transform*, or *Cauchy integral*,

$$C_{h1_E}(z) := \int_{\mathbb{T}} \frac{h(\zeta)1_E(\zeta)}{1 - z\overline{\zeta}} dm(\zeta) = \int_E \frac{h(\zeta)}{1 - z\overline{\zeta}} dm(\zeta), \quad z \in \mathbb{D}, \tag{1.1}$$

which is an analytic function in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, can be smoothly extended to the closed disk $\overline{\mathbb{D}}$? In the above formula, and throughout the article, 1_E denotes the indicator function of the set E.

For the question to be interesting, the set E should contain no arc A of $\mathbb{T}$. Indeed, if E contains an arc A, then certainly any function $s : \mathbb{T} \to \mathbb{C}$ in C^∞ with support on A will be transformed into a function C_s which is a member of $\mathcal{A}^\infty$. Here, $\mathcal{A}^\infty$ denotes the algebra of analytic functions in $\mathbb{D}$ for which the derivatives of any order extend continuously to $\overline{\mathbb{D}}$. The containment $C_s \in \mathcal{A}^\infty$ follows in this case readily from the rapid rate of decay of Fourier coefficients $\{s_n\}_n$ of the smooth function s, and the fact that $C_s(z) = \sum_{n=0}^\infty s_n z^n$.

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By $\mathcal{A}$, we will denote the class of analytic functions in $\mathbb{D}$ which admit a continuous extension to $\overline{\mathbb{D}}$, and by $\mathcal{A}^n$ we denote those functions for which the nth derivative admits such an extension, that is, $f^{(n)} \in \mathcal{A}$. Thus, $\mathcal{A}^\infty = \cap_{n\geq 1}\mathcal{A}^n$. Khrushchev in [12] has solved the existence part of the above stated problem in full. For a general closed set E, he establishes the existence of a nonzero measurable function h, with support only on E, such that C_h given by (1.1) is in the class $\mathcal{A}$. Moreover, he proves that there exists a nonzero measurable function h supported on E for which the transform (1.1) is a function in $\mathcal{A}^\infty$ essentially if and only if E contains a *Beurling–Carleson set* of positive Lebesgue measure. A set E is a Beurling–Carleson set if it is closed and if the condition

$$\sum_{n=1} |A_n| \log(1/|A_n|) < \infty \tag{1.2}$$

is satisfied, where $\{A_n\}_n$ is the system of disjoint open subarcs of $\mathbb{T}$ union of which equals the complement $\mathbb{T}\backslash E$, and $|A|$ denotes the length of the arc A. The class of Beurling–Carleson sets has a rich history, and appears notably in the solution of boundary zero set problems for smooth analytic functions, and zero set problems for Bergman spaces (see, for instance, Carleson's paper [5] and Korenblum's paper [14]). In [18], the present authors found that Beurling–Carleson sets play an important role in smooth approximation theory in de Branges–Rovnyak spaces, another classical and well-studied family of Hilbert spaces of analytic functions.

A notable feature of the proofs of the abovementioned results of Khrushchev in [12] is that they are non-constructive. One of the aims of this article is to show that, in the case in which Beurling–Carleson sets and the class $\mathcal{A}^\infty$ are involved, the theorem of Khrushchev can be obtained in a rather elementary and explicit way by using modifications of other known constructions.

Theorem A *Let E be a Beurling–Carleson set of positive Lebesgue measure. Then there exists an explicit formula for a measurable function $h = h1_E$ for which the Cauchy transform given by* (1.1) *belongs to $\mathcal{A}^\infty$.*

The above result is established in Section 3 as a special case of Proposition 3.1, which deals with a slightly more general situation, and which will be useful for our further applications. We remark that the very interesting problem of giving an explicit formula for h supported on any given closed set E such that C_{h1_E} is in $\mathcal{A}$ remains open, and the approach presented here is not applicable.

Our ultimate application of the construction of smooth Cauchy transforms is to the approximation theory in de Branges–Rovnyak spaces $\mathcal{H}(b)$. For background on the theory of $\mathcal{H}(b)$-spaces, see [6, 7, 21]. A basic problem in the theory is to identify what functions are contained in the space $\mathcal{H}(b)$ and how this depends on the structure of the symbol b, which is any analytic function mapping the disk $\mathbb{D}$ into itself. It has been established by Sarason (see [21]) that the analytic polynomials are contained and norm-dense in the space $\mathcal{H}(b)$ if and only if the weight

$$\Delta(\zeta) := 1 - |b(\zeta)|^2, \quad \zeta \in \mathbb{T}, \tag{1.3}$$

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has an integrable logarithm on $\mathbb{T}$:

$$\int_{\mathbb{T}} \log(\Delta)\, dm > -\infty. \tag{1.4}$$

Moreover, it is also known that any $\mathcal{H}(b)$-space contains a dense subset of functions in $\mathcal{A}$ (see [2]). In particular, considering the usual inner–outer factorization of $b = \theta b_0$ into an inner function θ and an outer function b_0, the inner factor θ plays no role in the context of approximations by analytic polynomials or functions continuous up to the boundary. The situation is different in the context of approximations by functions in the class $\mathcal{A}^\infty$ or $\mathcal{A}^n$, or even the Hölder classes. A combination of results in [16, 18] shows that the functions in the class $\mathcal{A}^n$ will be dense in the space $\mathcal{H}(b)$ if the outer factor of b is "good" and the "bad" part of the singularities of the inner factor of b is appropriately located on $\mathbb{T}$, with respect to the outer factor. More precisely, it was found in [2] that if weight Δ appearing above is of the form

$$\Delta = \sum_{n=1}^{\infty} w_n 1_{E_n}, \tag{1.5}$$

where each set E_n is a Beurling–Carleson set of positive Lebesgue measure, and each w_n is a nonnegative weight satisfying

$$\int_{E_n} \log(w_n)\, dm > -\infty, \tag{1.6}$$

then the functions in $\mathcal{A}^n$ are dense in $\mathcal{H}(b)$ if b is outer. Note that the two conditions above say something about the "good" structure of the support set of the weight Δ (being a union of "good" sets satisfying the Beurling–Carleson condition), and something about Δ not being too small on the support. In [18], examples are highlighted in which bad support and small size of Δ both independently prohibit such approximations in $\mathcal{H}(b)$, not only by functions in $\mathcal{A}^\infty$, but even by functions in the Hölder classes. In the presence of a nontrivial inner factor $\theta = BS_\nu$ of b, where B is a Blaschke product and S_ν is a singular inner function, results of [16, 18] show that what matters is the location on $\mathbb{T}$ of the support of a certain part of the singular measure ν. To describe this mechanism, we will need to introduce a simple decomposition of the measure ν, which has appeared already in a similar context in [17] and also in work of Roberts in [20]. Namely, the measure ν can be expressed as a sum

$$\nu = \nu_{\mathcal{C}} + \nu_{\mathcal{K}}, \tag{1.7}$$

where the two measures are mutually singular, there exists an increasing sequence of Beurling–Carleson sets of Lebesgue measure zero $\{F_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} \nu_{\mathcal{C}}(F_n) = \nu_{\mathcal{C}}(\mathbb{T}),$$

and

$$\nu_{\mathcal{K}}(F) = 0$$

for any Beurling–Carleson set F of Lebesgue measure zero. The part $\nu_{\mathcal{C}}$ plays no role in our approximation problem. However, the support of $\nu_{\mathcal{K}}$ must necessarily be located on the support of Δ for approximations by smooth functions to be possible. Moreover, if the conditions (1.5) and (1.6) are satisfied and the mass of $\nu_{\mathcal{K}}$ is located appropriately in the sense that

$$\nu_{\mathcal{K}}(\cup_n E_n) = \nu_{\mathcal{K}}(\mathbb{T}), \tag{1.8}$$

then indeed functions in the class $\mathcal{A}^n$ are dense in $\mathcal{H}(b)$. This was established by a duality argument in [18], using also the results of [16]. We will sharpen this result by proving density of functions in $\mathcal{A}^\infty$. Thus, we will prove (using duality) the following approximation result, which is the strongest that we are aware of.

Theorem B *Let $b : \mathbb{D} \to \mathbb{D}$ be an analytic function with singular inner factor S_ν such that the weight Δ given by* (1.3) *has the form* (1.5) *for some sequence $\{E_n\}_n$ of Beurling–Carleson sets of positive measure and satisfies* (1.6) *and such that the part $\nu_{\mathcal{K}}$ in the decomposition* (1.7) *of ν satisfies* (1.8). *Then $\mathcal{A}^\infty \cap \mathcal{H}(b)$ is norm-dense in $\mathcal{H}(b)$.*

A more detailed exposition of why this approximation result is close to the best possible also appears in [18].

The proof of Theorem B presented here is long, but it is self-contained and the constructive proof of the theorem of Khrushchev mentioned above plays a crucial role in our development. In [18], a duality approach to the smooth approximation problem in $\mathcal{H}(b)$ is presented, and it is based on a connection with two problems in analysis which are of independent interest: an operator *irreducibility* problem and an *inner factor permanence* problem. We will need to study both of these problems in detail to prove Theorem B.

In Section 5, we deal with the irreducibility problem. Let X be some space of functions defined on a domain in the complex plane which contains the analytic polynomials and is invariant under the *forward shift operator* $M_z : f(z) \mapsto zf(z)$, where z is the coordinate function (or identity function) of the complex plane. We denote by D the closure of the analytic polynomials in X. In many important cases, the functions in X live on the closed unit disk $\overline{\mathbb{D}}$, the operator M_z is a contraction (in the sense that $\|M_z f\|_X \le \|f\|_X$ holds for all $f \in X$), and a question or assumption which appears in several contexts (see, for instance, [2, 3, 15]) is related to the existence of invariant subspaces of the operator $M_z : D \to D$ on which it acts as an isometry. In the particular case $X = L^2(\mu)$, where μ is a positive Borel measure compactly supported in the complex plane, the closure of analytic polynomials is usually denoted by $\mathcal{P}^2(\mu)$. If μ is a positive measure of the form

$$d\mu = dA + 1_E dm \tag{1.9}$$

(dA and dm being the area measure of $\mathbb{D}$ and Lebesgue measure of $\mathbb{T}$, respectively), then the condition that M_z is completely nonisometric on the closure of polynomials $D := \mathcal{P}^2(\mu)$ is precisely the condition which ensures that $\mathcal{P}^2(\mu)$ can be identified with a genuine space of analytic functions in $\mathbb{D}$. If not, then $\mathcal{P}^2(\mu)$ will contain as a subset a space of the form $L^2(1_F dm)$, for some measurable subset F of E, on which M_z obviously acts as an isometry. In the context of $\mathcal{P}^2(\mu)$-spaces, the nonexistence of a subspaces of the type $L^2(1_F dm)$ goes under the name of *irreducibility* (see [3, 24]). It is known that if E is a Beurling–Carleson set, then the corresponding shift operator will be completely nonisometric on $\mathcal{P}^2(\mu)$. This follows essentially from Khrushchev's work in [12]. However, if we replace dA by a weighted version $\rho\, dA$ in (1.9), where ρ is some function which decays rapidly to zero near the boundary of $\mathbb{D}$, or if we replace E by a set more complicated than a Beurling–Carleson set, then it might very well happen that M_z admits an invariant subspace on which it acts as an isometry (see [15] and in particular [12] for details). In Section 5, we construct a special family of smooth Cauchy transforms and employ it in a functional analytic argument to establish that M_z is completely nonisometric on a wide range of Hilbert spaces of analytic functions which are structurally similar to the $\mathcal{P}^2(\mu)$-spaces discussed here, but much bigger. More precisely, we will work with spaces which we denote below by $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ and which we will equip with a norm defined on an analytic polynomial $p(z) = \sum_k p_k z^k$ by

$$\|p\|^2_{\mathcal{D}(\boldsymbol{\alpha}^{-1},w)} := \sum_k \frac{|p_k|^2}{\alpha_k} + \int_E |p|^2 w \, dm, \tag{1.10}$$

for some rapidly increasing positive sequence $\boldsymbol{\alpha} = \{\alpha_k\}_k$ and a Beurling–Carleson set E with weight w. Our development in particular implies the abovementioned results for $\mathcal{P}^2(\mu)$-spaces, and even their extensions from [16], but the method of proof is completely different, arguably much more straightforward, and the result actually reaches further. We remark that a wealth of information on the behavior of $\mathcal{P}^2(\mu)$-spaces which are spaces of analytic functions can be found in [3].

In Section 6, we will study the inner factor permanence problem. In the problem setting, we let $\mathcal{H}$ be a space of analytic $\mathbb{D}$ which includes at least H^∞, the algebra of bounded analytic functions. Assume that $\mathcal{H}$ carries a norm (or at least some other type of topological structure) and we have a convergent sequence of the form

$$\lim_{n\to\infty} \|\theta f_n - f\|_{\mathcal{H}} = 0,$$

where θ is an inner function, and all other appearing functions are bounded and analytic in $\mathbb{D}$. Then, in particular, f admits an inner–outer factorization $f = IU$ into an inner function I and an outer function U. We ask: is I divisible by θ? In other words, does the inner factor θ get passed onto the limit $f \in H^\infty$ in the metric induced by the norm $\|\cdot\|_{\mathcal{H}}$? We will call this property *permanence of an inner function* θ in the corresponding metric. The problem is only interesting for singular inner functions, since a Blaschke product B will be passed onto the limit in any reasonable norm defined on analytic functions. In the context of the usual L^2-norm computed on the circle, it is of course well known that any inner function θ satisfies the permanence property, but for many other metrics, a more interesting situation occurs. Here, a principal set of examples consists of the weighted L^2-metrics on the unit disk $\mathbb{D}$. A singular inner function has the form

$$S_\nu(z) = \exp\Big(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\nu(\zeta) \Big), \quad z \in \mathbb{D}, \tag{1.11}$$

where ν is a finite positive singular Borel measure on $\mathbb{T}$. Recall the decomposition (1.7) above, which induces according to (1.11) a factorization $S_\nu = S_{\nu_{\mathcal{C}}} S_{\nu_{\mathcal{K}}}$. The part $S_{\nu_{\mathcal{C}}}$ is passed onto the limit under convergence of bounded functions in the weighted Bergman spaces norms with polynomially decreasing weights. That is, if $\nu = \nu_{\mathcal{C}}$ in

(1.7), then for $\theta = S_\nu$, we have that

$$\lim_{n\to\infty} \int_{\mathbb{D}} |\theta f_n - f|^2 (1-|z|^2)^C dA(z) = 0$$
$$\Rightarrow f/\theta \in H^\infty$$

whenever f_n, f are all bounded analytic functions, and $C > -1$. In contrast, $S_{\nu_{\mathcal{K}}}$ can vanish under the same circumstances. A proof for the first claim appears in [17], whereas the second is a consequence of a deep cyclicity theorem for inner functions which was independently established by Roberts in [20] and Korenblum in [14]. In Section 6, a carefully constructed family of smooth Cauchy transforms will help us to implement a functional analytic argument and establish this inner factor permanence for a very large class of singular inner functions and a range of spaces $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ mentioned above. The result is a technical extension of the one appearing in the article [16], in which the present authors investigated this principle for the topologies induced by the abovementioned $\mathcal{P}^2(\mu)$-spaces.

Our results on the spaces $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ are used in Section 7 in which we implement the duality approach from [18] to prove Theorem B.

2 Construction of an analytic "cutoff" function

We start off by presenting the constructing of a certain analytic function with strong decay properties near a given Beurling–Carleson set. The reason for calling it a *cutoff function*, as in the name of the section, will become clear from the proof of the coming application in Proposition 3.1. Our construction is a straightforward adaptation of a technique from [10], more precisely from Lemma 7.11 of that work. We could have also followed the ideas of [19] or [23]. The proof is included for the reader's convenience and because the construction is crucial for our development.

Lemma 2.1 *Let E be a Beurling–Carleson set, of either zero or positive Lebesgue measure. There exists an analytic function $g : \mathbb{D} \to \mathbb{D}$ that extends analytically across $\mathbb{T}\backslash E$, and if $G : \mathbb{T} \to \mathbb{C}$ is defined by*

$$G(e^{it}) = g(e^{it}) 1_{\mathbb{T}\backslash E}(e^{it}), \tag{2.1}$$

where $1_{\mathbb{T}\backslash E}$ denotes the indicator function of the set $\mathbb{T}\backslash E$, then G is a smooth function on $\mathbb{T}$, and we have the estimate

$$|G^{(m)}(e^{it})| = o(\operatorname{dist}(e^{it}, E)^N), \quad e^{it} \to E, \tag{2.2}$$

for each pair of nonnegative integer N and m. Here, $G^{(m)}$ denotes the mth derivative of G with respect to the variable t, and dist$(\cdot,\cdot)$ denotes the distance between two closed sets.

Proof Let $\cup_{n\in\mathbb{N}} A_n = \mathbb{T}\backslash E$ be the complement of E with respect to $\mathbb{T}$. For each subarc A_n, we perform the classical *Whitney decomposition* $A_n = \cup_{k\in\mathbb{Z}} A_{n,k}$. More precisely, let $A_{n,0}$ be the arc with the same midpoint as A_n but having one-third of the length of A_n. For this choice of the length, we have $|A_{n,0}| = \operatorname{dist}(A_{n,0}, E)$. The arcs $A_{n,-1}$ and $A_{n,1}$ should be chosen adjacent to $A_{n,0}$ from the left and the right, respectively, and their lengths should be chosen, again, such that $|A_{n,-1}| = \operatorname{dist}(A_{n,-1}, E)$ and

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$|A_{n,1}| = \operatorname{dist}(A_{n,1}, E)$. It is easy to see that the correct choice is $|A_{n,1}| = |A_{n,-1}| = \frac{|A_n|}{6}$. Proceeding in this manner, we will obtain a decomposition

$$\mathbb{T}\backslash E = \cup_n A_n = \cup_{n,k} A_{n,k},$$

where, for each arc $A_{n,k}$, we have

$$|A_{n,k}| = \frac{|A_n|}{3\cdot 2^{|k|}} = \operatorname{dist}(A_{n,k}, E). \tag{2.3}$$

A straightforward computation based on (2.3) will show that

$$\sum_{n,k} |A_{n,k}| \log(1/|A_{n,k}|) < \infty.$$

Let $\{B_j\}_j$ be a relabeling of the arcs $\{A_{n,k}\}_{n,k}$ and $\{\lambda_j\}_j$ a positive sequence tending to infinity such that

$$\sum_j \lambda_j |B_j| \log(1/|B_j|) < \infty.$$

Now, let $r_j = 1 + |B_j|$, $b_j \in \mathbb{T}$ be the midpoint of the arc B_j, and consider the function

$$h(z) = \sum_j h_j(z) = \sum_j \frac{\lambda_j b_j |B_j| \log(1/|B_j|)}{r_j b_j - z}, \quad z \in \mathbb{D}. \tag{2.4}$$

It is not hard to see that the real part of $h(z)$ is positive in $\mathbb{D}$. In fact, the real part of the jth term in the sum is

$$\operatorname{Re} h_j(z) = \lambda_j |B_j| \log(1/|B_j|) \frac{\operatorname{Re}(r_j - \overline{z} b_j)}{|r_j b_j - z|^2} > 0,$$

where the last inequality follows from $\operatorname{Re}(r_j - \overline{z} b_j) > 0$, which is a consequence of the inequalities $r_j > 1$ and $|\overline{z} b_j| < 1$. It follows that

$$g(z) := \exp(-h(z)) \tag{2.5}$$

is bounded by 1 in modulus for $z \in \mathbb{D}$. Moreover, the series defining $h(z)$ converges also for $z \in B_j$, and h extends analytically across each B_j, because the poles $\{r_j b_j\}_j$ of h cluster only at the set E. For $z \in B_j$, we have that the quantities $|r_j b_j - z|$ and $\operatorname{Re}(r_j - \overline{z} b_j)$ are both approximately equal to $|B_j|$, and so

$$|g(z)| \le \exp(-\operatorname{Re} h_j(z)) \le \exp(-c\lambda_j \log(1/|B_j|)) = |B_j|^{c\lambda_j}$$

for some positive constant c. Since $|B_j|$ equals the distance from B_j to E, for $z \in B_j$, we obtain that

$$|g(z)| \le C \operatorname{dist}(z, E)^{c\lambda_j}$$

for some positive constant $C > 0$ independent of j. Note that as z tends to E along the complement $\mathbb{T}\backslash E$, it needs to pass through infinitely many intervals B_j. Since λ_j tends to infinity, we obtain that

$$|g(z)| = o(\operatorname{dist}(z, E)^N) \tag{2.6}$$

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as $z \to E$ along the complement of E on $\mathbb{T}$, for any choice of positive integer N. Now, for $e^{it} \in \mathbb{T} \backslash E$, the derivatives $G^{(m)}(e^{it})$ have the form $H(e^{it})G(e^{it})$, where H is a linear combination of products of derivatives of $h(e^{it})$ with respect to t. But a glance at (2.4) shows that such a product cannot grow faster than a constant multiple of $\operatorname{dist}(e^{it}, E)^{-n}$ for $e^{it} \in \mathbb{T} \backslash E$, for some positive integer $n = n_m$ depending only on the number of derivatives m taken. Now, according to the definition of G in (2.1), it follows that the estimate (2.2) holds whenever $e^{it} \in \mathbb{T} \backslash E$. Using this, it is now also evident from the definition in (2.1) that G is continuous on $\mathbb{T}$. In fact, it is straightforward to verify that whenever $e^{it} \in E$, the function G is differentiable as many times as we wish, with $G^{(m)}(e^{it}) = 0$ for any positive integer m. This shows that G is smooth on all of $\mathbb{T}$ and thus completes the proof of this lemma. ∎

Note the fact that the proof above gives an explicit computable formula for the cutoff function g. It is given in terms of the Beurling–Carleson set E and is presented in equations (2.4) and (2.5).

3 A constructive proof of Khrushchev's theorem

3.1 Smooth Cauchy transforms

As before, let E be a Beurling–Carleson set of positive measure. Lemma 2.1 will allow us to construct, and give explicit formulas for, measurable functions supported on E which have a smooth Cauchy transform. Thus, we will now give the constructive proof of the theorem of Khrushchev from his seminal work [12].

Proposition 3.1 (Construction of smooth Cauchy transforms) *Let E be a Beurling–Carleson set of positive measure such that $E \neq \mathbb{T}$, and let w be a bounded positive measurable function with support on E which satisfies $\int_E \log(w)\,dm > -\infty$. Let W be the outer function*

$$W(z) = \exp\Big(\int_E \frac{\zeta + z}{\zeta - z} \log(w(\zeta))\,dm(\zeta)\Big), \tag{3.1}$$

and let g be the function associated with E, which is given by Lemma 2.1. Consider the set

$$K = \Big\{ s = \overline{\zeta p g W} : p \text{ analytic polynomial} \Big\} \tag{3.2}$$

consisting of functions on $\mathbb{T}$, where ζ is the coordinate function on $\mathbb{T}$. Then the Cauchy transform

$$C_{s1_E}(z) := \int_E \frac{s(\zeta)}{1 - z\overline{\zeta}}\,dm(\zeta) \tag{3.3}$$

is a nonzero function in $\mathcal{A}^\infty$ for each nonzero $s \in K$, the restrictions to E of elements of the set K form a dense subset of $L^2(1_E dm)$, and the set

$$C_E K := \{ C_{s1_E} : s \in K \} \tag{3.4}$$

is dense in H^2.

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Certainly, our more general form of the theorem, together with the density statements, is obtainable by Khrushchev's methods from [12]. We therefore emphasize that our main contribution in this context are the explicit formulas for the measurable functions supported on E for which the Cauchy transform is an analytic function in $\mathcal{A}^\infty$. More precisely, the formulas for the functions in K are given by the equations (2.4), (2.5), and (3.1).

The density statements in Proposition 3.1 will be useful for our further applications. It is not our point to prove these density statements constructively. In this part of the proof, we will use the following well-known theorem.

Lemma 3.2 **(Beurling–Wiener theorem)** *Let $M_{\overline{\zeta}} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ be the operator of multiplication by $\overline{\zeta}$. The closed $M_{\overline{\zeta}}$-invariant subspaces of $L^2(\mathbb{T})$ are of the form*

$$L^2(1_F dm) = \{f \in L^2(\mathbb{T}) : f = 0 \text{ almost everywhere on } \mathbb{T}\backslash F\},$$

where F is a measurable subset of $\mathbb{T}$, or of the form

$$U\overline{H^2} = \{U\overline{f} : f \in H^2\},$$

where U is a unimodular function.

For a proof of the Beurling–Wiener theorem, see, for instance, [11].

Proof of Proposition 3.1 Since s is a conjugate analytic and satisfies $\int_{\mathbb{T}} s dm = 0$, we have

$$\int_{\mathbb{T}} \frac{s(\zeta)}{1 - z\overline{\zeta}} dm(\zeta) = 0$$

for each $z \in \mathbb{D}$. This implies that

$$C_{s1_E}(z) = \int_E \frac{s(\zeta)}{1 - z\overline{\zeta}} dm(\zeta) = -\int_{\mathbb{T}\backslash E} \frac{s(\zeta)}{1 - z\overline{\zeta}} dm(\zeta). \tag{3.5}$$

Consider now the function $S(e^{it}) := s(e^{it})1_{\mathbb{T}\backslash E}(e^{it}) = \overline{e^{it}p(e^{it})G(e^{it})W(e^{it})}$. From the formula (3.1) for W, it is clear that this function extends analytically across $\mathbb{T}\backslash E$, and a simple differentiation argument shows that the derivatives in the variable t of the function $W(e^{it})$ admit a bound

$$\left|\frac{\partial^m}{\partial t^m} W(e^{it})\right| \le C_m \cdot \operatorname{dist}(e^{it}, E)^{-2m} \tag{3.6}$$

for $e^{it} \in \mathbb{T}\backslash E$. Thus, by (2.2) of Lemma 2.1 and the definition in (3.2), the derivatives of any order of S tends to zero as e^{it} tends to E along $\mathbb{T}\backslash E$, and it is not hard to see that the derivatives of S vanish on E. Thus, $S \in C^\infty$. It follows that the Fourier coefficients S_n of S satisfy $|S_n| \le C|n|^{-M}$ for each positive integer M and some constant $C = C(M) > 0$. Obviously, then, (3.5) implies that the function $C_{s1_E}(z) = -\sum_{n=0}^\infty S_n z^n$ is in $\mathcal{A}^\infty$. It is nonzero if s is nonzero because the positive Fourier coefficients cannot vanish for the function $s1_E$ which is identically zero on the set $\mathbb{T}\backslash E$ of positive Lebesgue measure.

The density in $L^2(1_E dm)$ of the restrictions to E of elements of the set K is an easy consequence of the invariance of K under multiplication by $\overline{\zeta}$ and the Beurling–Wiener theorem (Lemma 3.2). Indeed, the restriction to E of an element of K is

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nonzero almost everywhere on E, but obviously zero on $\mathbb{T}\backslash E$. It follows from Lemma 3.2 that the closure of K in $L^2(1_E dm)$ could not be anything else than the full space.

The set $C_E K$ is certainly contained in H^2, and the density in H^2 follows from the classical Beurling theorem for the Hardy spaces. More precisely, the set $C_E K$ is invariant under the backward shift operator

$$f(z) \mapsto \frac{f(z) - f(0)}{z}. \tag{3.7}$$

Indeed, we have that

$$\frac{C_{s1_E}(z) - C_{s1_E}(0)}{z} = \int_E \frac{\overline{\zeta} s(\zeta)}{1 - z\overline{\zeta}} dm(\zeta) = C_{\overline{\zeta} s1_E}(z). \tag{3.8}$$

By Beurling's theorem, the closure of $C_E K$ is either all of H^2, or it coincides with a model space K_θ of functions which have boundary values on $\mathbb{T}$ of the form $\theta\overline{h}, h \in zH^2$, for some nonzero inner function θ. If we would be in the second case, then there would exist a function $k \in zH^2$ such that on the circle $\mathbb{T}$ we would have the equality $s1_E = C_{s1_E} + \overline{k} = \theta\overline{h} + \overline{k}$, and consequently $\overline{\theta} s1_E \in \overline{H^2}$. This is a contradiction, since $\overline{\theta} s1_E$ vanishes on a set of positive measure. ■

3.2 A technical improvement

Sets of the form K as in (3.2) have another useful property, one of which will be employed in the coming applications. The property is that the set $C_E K$ defined in (3.4) is contained in a single Hilbert space consisting purely of functions which are in $\mathcal{A}^\infty$. This applies to many sets similar to K, as we shall see next.

More precisely, take the function $s_0 := \overline{\zeta g W} \in K$, i.e., the one where $p = 1$ in (3.2). The only property of K that we will use in the proof is that it is of the form

$$\{\overline{p} s_0 : p \text{ analytic polynomial}\}.$$

For an analytic polynomial $p(z) = \sum_{n=0}^{d} p_n z^n$, we let

$$\tilde{p}(z) := \sum_{n=0}^{d} \overline{p_n} z^n \tag{3.9}$$

and define the operator

$$p(L) := \sum_{n=0}^{d} p_n L^n, \tag{3.10}$$

where L is the backward shift operator defined in (3.7). Every other element of $C_E K$ can be expressed as $p(L) C_{s_0 1_E}$ for some analytic polynomial. This claim is a consequence of the formula

$$p(L) C_{s_0 1_E}(z) = \int_E \frac{\overline{\zeta \tilde{p} g W}}{1 - z\overline{\zeta}} dm(\zeta), \tag{3.11}$$

which, in turn, is a consequence of (3.8). Thus, the Taylor coefficients in the family $C_E K$ have similar asymptotic behavior, and we exploit this fact in the following way.

Being a function in $\mathcal{A}^\infty$, the Taylor coefficients $\{S_k\}_{k=0}^\infty$ of $C_{s_0 1_E}$ satisfy

$$\sum_{k=0}^{\infty} k^N |S_k|^2 < \infty \tag{3.12}$$

for all positive integers N. It follows that for each $N \geq 1$, there exists a positive integer $K(N)$ such that

$$\sum_{k=K(N)}^{\infty} k^N |S_k|^2 < \frac{1}{2^N}. \tag{3.13}$$

We may assume that $\{K(N)\}_{N\geq 1}$ is increasing. Set $K(0) = 0$ and define a sequence $\{\alpha_k\}_{k=0}^\infty$ by

$$\alpha_k = k^N, \quad K(N) \leq k < K(N+1).$$

This sequence is increasing, and satisfies

$$\sum_{k=0}^{\infty} \alpha_k |S_k|^2 = \sum_{N=0}^{\infty} \sum_{k=K(N)}^{K(N+1)-1} k^N |S_k|^2 \leq \sum_{N=0}^{\infty} \frac{1}{2^N} < \infty. \tag{3.14}$$

Moreover, since $\alpha_k \geq k^{N+1}$ if $k \geq K(N+1)$, we have that

$$\lim_{k\to\infty} \frac{\alpha_k}{k^N} \geq \lim_{k\to\infty} \frac{k^{N+1}}{k^N} = \infty \tag{3.15}$$

for any positive integer N.

Definition 3.3 A sequence of positive numbers $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^\infty$ is *rapidly increasing* if

$$\lim_{k\to\infty} \frac{\alpha_k}{k^N} = \infty \tag{3.16}$$

holds for each positive integer N.

Thus, we have constructed above a rapidly increasing sequence. In the coming application, we will also need the very mild condition

$$\lim_{k\to\infty} \alpha_k^{1/k} = 1,$$

which we can safely assume. Indeed, by replacing α_k by $\min(\alpha_k, k^{\sqrt{k}})$, we still have a sequence which is rapidly increasing, and moreover

$$1 \leq \lim_{k\to\infty} \alpha_k^{1/k} \leq \lim_{k\to\infty} \exp(\log(k)/\sqrt{k}) = 1.$$

We now make a somewhat trivial observation, which will, however, be important in the sequel. Because the sequence $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^\infty$ is increasing, it also follows that whenever an analytic function f has a Taylor series which satisfies (3.14), then so does the backward shift Lf of this function. Thus, also $p(L)f$ satisfies this property, for all analytic polynomials p (and, in fact, so does appropriately defined $h(L)f$ for any bounded analytic function h; see Proposition 4.1). Using also the formula (3.11), we have proved the following technical result.

Proposition 3.4 *Let s_0 be a measurable function on $\mathbb{T}$ for which the Cauchy transform C_{s_0} is a function in $\mathcal{A}^\infty$. Then there exists a rapidly increasing sequence $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^\infty$ satisfying*

$$\lim_{k\to\infty} a_k^{1/k} = 1$$

and such that

$$\sum_{k=0}^{\infty} \alpha_k |f_k|^2 < \infty$$

for all functions f which are Cauchy transforms $f = C_s$ of a function s from the set

$$\{s = \overline{p} s_0 : p \text{ analytic polynomial}\}.$$

4 Weighted sequence spaces

We will explore the Hilbert spaces implicitly appearing in Proposition 3.4 a little more, and prove a few basic facts about their duality and operators acting on them. The main results of the following Sections 5 and 6 will be stated in the context of these Hilbert spaces. All results in this section are certainly well known, so we include the proofs for completeness.

4.1 Definition and duality

For a sequence of positive numbers $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^\infty$, we define the Hilbert space $X(\boldsymbol{\alpha})$ to consist of formal power series $f(z) = \sum_{k=0}^\infty f_n z^n$ which satisfy

$$\|f\|_{X(\boldsymbol{\alpha})}^2 := \sum_{k=0}^{\infty} \alpha_k |f_k|^2 < \infty. \tag{4.1}$$

It is obvious that if $\boldsymbol{\alpha}$ is rapidly increasing, then $X(\boldsymbol{\alpha}) \subset \mathcal{A}^\infty$. We define the *dual sequence* $\boldsymbol{\alpha}^{-1}$ by the equation

$$\boldsymbol{\alpha}^{-1} := \{\alpha_k^{-1}\}_{k=0}^\infty.$$

The space $X(\boldsymbol{\alpha}^{-1})$ is isometrically isomorphic to the dual space of $X(\boldsymbol{\alpha})$ under the pairing which maps $f \in X(\boldsymbol{\alpha})$, $g \in X(\boldsymbol{\alpha}^{-1})$ to the complex number

$$\langle f, g \rangle := \sum_{k=0} f_k \overline{g_k}, \tag{4.2}$$

where the sequences $\{f_k\}_{k=0}^\infty$ and $\{g_k\}_{k=0}^\infty$ are the coefficients in the formal power series expansions of f and g, respectively.

In fact, for us, the spaces $X(\boldsymbol{\alpha})$ and $X(\boldsymbol{\alpha}^{-1})$ will always be genuine spaces of analytic functions on $\mathbb{D}$. Indeed, a property which ensures this is $\lim_{k\to\infty} \alpha_k^{1/k} = 1$. The sequences appearing in our context will be the ones constructed in Proposition 3.4 and their dual sequences, so we can safely assume below that this assumption is always satisfied. To see indeed that the assumption $\lim_{k\to\infty} \beta_k^{1/k} = 1$ implies that the radius of

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convergence of a formal power series $f \in X(\boldsymbol{\beta})$ is equal to at least 1, we compute

$$\limsup_{k\to\infty} |f_k|^{1/k} = \limsup_{k\to\infty} \frac{(\beta_k|f_k|^2)^{1/2k}}{\beta_k^{1/2k}} \le \limsup_{k\to\infty} \frac{1}{\beta_k^{1/2k}} = 1,$$

where, in the next-to-last step, we used that

$$\lim_{k\to\infty} \beta_k|f_k|^2 = 0,$$

so that

$$\limsup_{k\to\infty} (\beta_k|f_k|^2)^{1/2k} \le 1.$$

This shows that the radius of convergence of $f \in X(\boldsymbol{\beta})$ is indeed equal to at least 1.

Finally, an obvious but important property of the presented duality pairing is that if f and g happen to be functions in H^2, then we have that (4.2) equals

$$\langle f, g\rangle = \int_{\mathbb{T}} f\overline{g}\, dm.$$

In other words, the duality pairing coincides with the usual $L^2(\mathbb{T})$-duality pairing in the case f and g are functions in H^2. We shall often implicitly use this property.

4.2 Toeplitz operators

The usual Toeplitz operator with symbol $h \in L^\infty(\mathbb{T})$ acts on an H^2-function f by the formula

$$T_h f(z) := \int_{\mathbb{T}} \frac{f(\zeta)h(\zeta)}{1-\overline{\zeta}z}\, dm(\zeta).$$

If p is a polynomial, then

$$T_{\overline{p}} f = \tilde{p}(L) f, \tag{4.3}$$

where $\tilde{p}(L)$ is defined according to (3.10) and where $\tilde{p}(z)$ is given by (3.9). If h is in H^∞, then we can equivalently define the operator T_h as the mapping taking the function $f(z), z \in \mathbb{D}$, to the function $h(z)f(z), z \in \mathbb{D}$. We denote by M_h the multiplication operator

$$M_h f(z) = h(z)f(z), \quad z \in \mathbb{D},$$

which acts on the space of all holomorphic functions on $\mathbb{D}$. If h is analytic and $f \in H^2$, then it is well known that $T_h f = M_h f$. We say that the Toeplitz operator is *co-analytic*, or *has a co-analytic symbol*, if it is of the form $T_{\overline{h}}$ for $h \in H^\infty$.

Proposition 4.1 *Let $\boldsymbol{\alpha}$ be a sequence of positive numbers such that the corresponding $X(\boldsymbol{\alpha})$ space consists of analytic functions in $\mathbb{D}$.*

(i) *If $\boldsymbol{\alpha}$ is increasing, then $X(\boldsymbol{\alpha})$ is continuously contained in H^2, and the Toeplitz operators with bounded co-analytic symbols are bounded on $X(\boldsymbol{\alpha})$.*
(ii) *If $\boldsymbol{\alpha}$ is decreasing, then the operators M_h with bounded analytic symbols are bounded on $X(\boldsymbol{\alpha})$.*

In both cases, the corresponding operators have a norm which is less than or equal to the supremum norm of the corresponding symbol.

Proof We prove (i). It is clear that $X(\boldsymbol{\alpha})$ is continuously contained in H^2. A direct computation shows that $\boldsymbol{\alpha}$ being increasing implies that the backward shift operator L in (3.7) is a contraction on $X(\boldsymbol{\alpha})$. There certainly exists no subspace of $X(\boldsymbol{\alpha})$ on which L acts as a unitary (or even an isometry), so the Nagy–Foias functional calculus (see [22] for details) allows us to define the operator

$$h(L) : X(\boldsymbol{\alpha}) \to X(\boldsymbol{\alpha})$$

for any bounded analytic function h, in such a way that the definition is consistent with (3.10) for polynomials h, the operator norm of $h(L)$ is at most $\|h\|_\infty$, and if

$$\lim_{n\to\infty} h_n(\zeta) \to h(\zeta)$$

almost everywhere on $\mathbb{T}$ and

$$\sup_n \|h_n\|_\infty < \infty,$$

then $h_n(L)$ converges in the strong operator topology to $h(L)$. In fact, the operators $h(L)$ are co-analytic Toeplitz operators with symbol $\overline{\tilde{h}}$, where $\tilde{h}(z) = \overline{h(\overline{z})}$. To see this, fix $h \in H^\infty$ and let $\{h_n\}_n$ be the Fejér polynomials for h, so that the above properties of the Nagy–Foias functional calculus imply that $h_n(L)f \to h(L)f$ in the norm of $X(\boldsymbol{\alpha})$, for any $f \in X(\boldsymbol{\alpha})$. The same is true in the norm of H^2. Recalling (4.3), for $z \in \mathbb{D}$, we get

$$h(L)f(z) = \lim_{n\to\infty} h_n(L)f(z) = \lim_{n\to\infty} T_{\overline{\tilde{h}_n}} f(z) = T_{\overline{\tilde{h}}} f(z).$$

Thus, $h(L) = T_{\overline{\tilde{h}}}$, and by reversing roles of h and $\tilde{h}$, we see that the Nagy–Foias functional calculus for L on $X(\boldsymbol{\alpha})$ is a bijection onto the co-analytic Toeplitz operators.

Next, we prove (ii). If $\boldsymbol{\alpha}$ is decreasing, then the dual sequence $\boldsymbol{\alpha}^{-1}$ is increasing, so by (i) we can define $T^*_{\overline{h}} : X(\boldsymbol{\alpha}) \to X(\boldsymbol{\alpha})$ as the adjoint of $T_{\overline{h}} : X(\boldsymbol{\alpha}^{-1}) \to X(\boldsymbol{\alpha}^{-1})$ with respect to our duality pairing (4.2) between the spaces. Let $f \in X(\boldsymbol{\alpha})$, $\lambda \in \mathbb{D}$, and $s_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$. Recall that s_λ is an eigenvector of $T_{\overline{h}}$, with eigenvalue $\overline{h(\lambda)}$. We compute

$$T^*_{\overline{h}} f(\lambda) = \langle T^*_{\overline{h}} f, s_\lambda \rangle = \langle f, T_{\overline{h}} s_\lambda \rangle = h(\lambda)\langle f, s_\lambda \rangle = h(\lambda) f(\lambda).$$

Thus, the adjoints of the co-analytic Toeplitz operators on $X(\boldsymbol{\alpha}^{-1})$ are multiplication operators on $X(\boldsymbol{\alpha})$. The operator norm of $T^*_{\overline{h}}$ equals to operator norm of $T_{\overline{h}}$, which is at most $\|h\|_\infty$, as was noted in the proof of part (i). The proof is complete. ∎

5 Completely nonisometric shifts

5.1 A big Hilbert space with a completely nonisometric shift

In the next proposition, we construct a Hilbert space of analytic functions on $\mathbb{D}$ which has desirable properties and which is strictly larger than any space $\mathcal{P}^2(\mu)$ with measure

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μ being of the form

$$d\mu = d\mu_C := (1 - |z|^2)^C dA + w dm$$

and C being any positive number. This Hilbert space will play an important role in the proof of the main result of Section 7. Another application is presented in Corollary 5.3. We note that a similar result certainly can be reached by methods of Khrushchev developed in [12], but our proof below is different, and relies fully on construction of smooth Cauchy transforms.

Proposition 5.1 *Let E be a Beurling–Carleson set of positive Lebesgue measure, and let w be a bounded positive measurable function which is supported on E and satisfies $\int_E \log(w) dm > -\infty$. For a sequence $\boldsymbol{\alpha}$, consider the product space*

$$X(\boldsymbol{\alpha}^{-1}) \oplus L^2(w\, dm)$$

and the norm closure

$$\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$$

of the linear manifold

$$\{(p, p) \in X(\boldsymbol{\alpha}^{-1}) \oplus L^2(w\, dm) : p \text{ analytic polynomial}\}.$$

There exists a rapidly increasing sequence $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^{\infty}$ such that the space $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ has the following property: $f_1 \equiv 0$ implies that $f_2 \equiv 0$, for any tuple $(f_1, f_2) \in \mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$.

Proof The proof is very simple in principle. We will use the set K in (3.2) in combination with the sequence constructed in Proposition 3.4, and this will provide us with enough functionals on $X(\boldsymbol{\alpha}^{-1}) \oplus L^2(wdm)$ to conclude that $f_1 \equiv 0$ implies $f_2 \equiv 0$, by a straightforward duality argument involving the Beurling–Wiener theorem.

For each $s \in K$, consider the functional

$$p \mapsto -\int_{\mathbb{T}} p\overline{C_{s1_E}}\, dm + \int_E p\bar{s}\, dm, \tag{5.1}$$

which we define on the set $\{(p, p) \in X(\boldsymbol{\alpha}^{-1}) \oplus L^2(wdm) : p$ analytic polynomial$\}$. By construction, these functionals are the zero functionals, since the functions C_{s1_E} and $s1_E$ have coinciding Fourier coefficients indexed by nonnegative numbers. Apply Proposition 3.4 to produce a rapidly increasing sequence $\boldsymbol{\alpha}$ such that $C_{s1_E} \in X(\boldsymbol{\alpha})$ for all $s \in K$. The constructed functionals are then continuous with respect to the metric $X(\boldsymbol{\alpha}^{-1}) \oplus L^2(wdm)$. Indeed, we see from (3.2) that $s = wq$ on E, where q is a bounded function, and so

$$\left|\int_E p\bar{s}\, dm\right| \leq \|q\|_{L^2(wdm)} \|p\|_{L^2(wdm)}$$

by Cauchy–Schwarz inequality.

Now, let (f_1, f_2) lie $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ and assume that $f_1 \equiv 0$. Fix a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ such that $(p_n, p_n) \to (f_1, f_2) = (0, f_2)$ in the norm of $X(\boldsymbol{\alpha}^{-1}) \oplus L^2(wdm)$. Then $(0, f_2)$ is annihilated by any functional in (5.1), and so

$$\int_E f_2 \bar{s} dm = 0, \quad s \in K.$$

By the density statements in Proposition 3.1, we conclude that $f_2 \equiv 0$. ■

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Let us take another look at the space $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ appearing above, assuming that it is satisfying the conclusion of Propoition 5.1. If (f, f_1) and (f, f_2) are two tuples in $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ with coinciding first coordinate, then $(0, f_1 - f_2) \in \mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$, and the above result implies that $f_1 \equiv f_2$. In particular, the projection $(f, f_1) \mapsto f$ onto the first coordinate is an injective mapping from such tuples to analytic functions on $\mathbb{D}$. But this means that $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ is in essence a space of analytic functions in which the analytic polynomials are dense.

We make three more very simple but important observations.

Proposition 5.2 *Let $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ be as in Propoition 5.1, and identify it with a space of analytic functions on $\mathbb{D}$ as described above.*

(i) *$M_z : \mathcal{D}(\boldsymbol{\alpha}^{-1}, w) \to \mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ is completely nonisometric.*
(ii) *If $f \in H^2$, then $f \in \mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ and the corresponding tuple equals (f, f), where in the second coordinate f is interpreted in the sense of boundary values of f on $\mathbb{T}$.*
(iii) *Every bounded analytic function h defines a multiplication operator M_h on $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$, with norm at most $\|h\|_\infty$.*

Proof Part (i) follows from the paragraph above. The only way a function $f \in \mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ satisfies $\|M_z f\|_{\mathcal{D}(\boldsymbol{\alpha}^{-1},w)} = \|f\|_{\mathcal{D}(\boldsymbol{\alpha}^{-1},w)}$ is if f vanishes on $\mathbb{D}$, which does not happen by Propoition 5.1.

Part (ii) follows in a similar way. We need to note only that since $\boldsymbol{\alpha}^{-1}$ is decreasing and w is bounded, then for a suitable sequence $\{p_n\}_n$ of Taylor polynomials of $f \in H^2$, the tuples (p_n, p_n) will converge in the norm of $X(\boldsymbol{\alpha}^{-1}) \oplus L^2(w\,dm)$ to (f, f). By part (i), or the discussion in the paragraph above, there is only one tuple in $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ which has f as the first coordinate. So the tuple representing $f \in H^2 \cap \mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ is precisely (f, f).

To prove part (iii), let h be a bounded analytic function and $\{h_n\}_n$ its Fejér means. Let $\{p_n\}_n$ be a sequence of polynomials converging to $f \in \mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$. Then Proposition 4.1 implies that $\{h_n p_n\}_n$ is a norm-bounded sequence in $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$. The weak limit of this sequence equals $hf \in \mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$, and

$$\|hf\|_{\mathcal{D}(\boldsymbol{\alpha}^{-1},w)} \leq \liminf_{n\to\infty} \|h_n p_n\|_{\mathcal{D}(\boldsymbol{\alpha}^{-1},w)} \leq \|h\|_\infty \|f\|_{\mathcal{D}(\boldsymbol{\alpha}^{-1},w)}.$$

The last inequality is again a consequence of Proposition 4.1. ∎

5.2 An uncertainty-type principle

The significance of Propoition 5.1 might not be easy to appreciate. In this section, which is independent of the rest of the article, we want to highlight how such a result can be applied in the theory of $\mathcal{P}^2(\mu)$-spaces and how it relates to a classical result in the theory of Hardy spaces.

Recall that a square integrable function f on $\mathbb{T}$ which lies in the L^2-closure of the analytic polynomials (that is, in the Hardy space H^2) cannot vanish on a set of positive measure. Thus, the spectral smallness of f (vanishing of negative Fourier coefficients of f) implies that the function cannot be too small. A beautiful exposition of this result, and other manifestations of the uncertainty principle in harmonic analysis, can be found in [9]. A combination of the deep work of Aleman, Richter, and Sundberg in [3] and our Propoition 5.1 will establish the following result of similar nature.

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Corollary 5.3 **(An uncertainty-type principle for a class of $\mathcal{P}^2(\mu)$-spaces)** *Let $C > -1$ and E be a Beurling–Carleson set of positive measure. Let w be a bounded positive measurable function which is supported on E and satisfies $\int_E \log(w) dm > -\infty$. Consider the measure*

$$d\mu = (1 - |z|^2)^C dA + w dm$$

and the classical Lebesgue space $L^2(\mu)$. Let $\mathcal{P}^2(\mu)$ be the closure of analytic polynomials in $L^2(\mu)$. Then we have that $f \neq 0$ almost everywhere with respect to μ, for any nonzero $f \in \mathcal{P}^2(\mu)$.

Proof A computation shows that for $f(z) = \sum_{k=0}^{\infty} f_k z^k$, we have

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^C dA(z) = \sum_{k=0} \beta_k(C) |f_k|^2,$$

where the weights $\beta_k(C)$ satisfy the asymptotics

$$\beta_k(C) \simeq \frac{1}{k^{C+1}}.$$

This means that convergence of polynomials $\{p_n\}_n$ in $\mathcal{P}^2(\mu)$ implies convergence of (p_n, p_n) in the space $X(\boldsymbol{\alpha}^{-1}) \oplus L^2(wdm)$ appearing in Propoition 5.1, and a direct consequence is that $\mathcal{P}^2(\mu)$ contains no nonzero function which vanishes on $\mathbb{D}$. In particular, $\mathcal{P}^2(\mu)$ does not contain the characteristic function of any subset of $\mathbb{T}$ of positive measure, and every element $f \in \mathcal{P}^2(\mu)$ has a unique restriction $f|\mathbb{D}$ to $\mathbb{D}$, which of course is an analytic function. In particular, the space satisfies the assumptions of [3, Theorem A], and the conclusion of that theorem is that for any $f \in \mathcal{P}^2(\mu)$, its restriction $f|\mathbb{D}$ has a nontangential limit almost everywhere with respect to $\mu|\mathbb{T}$, and this limit agrees almost everywhere with $f|\mathbb{T}$. If f would vanish on a set of positive $\mu|\mathbb{T}$-measure, then a classical theorem of Privalov (see, for instance, [13]) can be used to deduce that $f \equiv 0$ throughout $\overline{\mathbb{D}}$. ■

In the above result, we can obviously replace the part $d\mu|\mathbb{T} = wdm$ with a more general weight w which is carried by a countable union $\{E_n\}_n$ of Beurling–Carleson sets of positive measure, and where the weight w is log-integrable on each set E_n separately.

We want to remark also that the use of the very deep and general Aleman–Richter–Sundberg theorem from [3] in the above proof can likely be avoided, and the existence of nontangential limits on E for functions f in $\mathcal{P}^2(\mu)$ of the described form, or even in the space $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$, is likely accessible in a more straightforward way (see the introductory section of the article [3] for an exposition of previously attained special cases of the Aleman–Richter–Sundberg theorem).

6 A permanence principle for inner factors

In this section, we study the inner factor permanence problem for the spaces appearing in the previous sections. We start by formally stating the property which we are investigating.

Definition 6.1 Let $\mathcal{H}$ be a topological space of analytic functions which contains H^∞, and let θ be a given inner function. We say that the pair $(\mathcal{H}, \theta)$ satisfies the *permanence property* if

$$\lim_{n\to\infty} \theta f_n = f,$$

in the sense of the topology of $\mathcal{H}$, implies that f/θ is bounded, whenever f_n, f are bounded analytic functions.

Let us go back to the setting of Propositions 5.1 and 5.2 where the Hilbert space $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ appears. We noted that if $\boldsymbol{\alpha}$ is suitably chosen, then $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ is in fact a space of analytic functions, and it contains H^2. Thus, the above question of inner factor permanence makes sense in the context of the norm on $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$.

Weaker versions of the following results appear in [16], where circumstances allow for statements in much less technical form. This is a consequence of the fact that the sequences $\boldsymbol{\alpha}$ which appear in [16] increase only polynomially. Below, we show that by fixing some singular inner function θ of some particular structure, we can alter the methods in [16] and construct a space $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ where the sequence $\boldsymbol{\alpha}$ is rapidly increasing and in which the permanence principle in Definition 6.1 holds for that given θ. The corresponding results for $\mathcal{P}^2(\mu)$-spaces from [16] are corollaries (we state them in Corollary 6.5), but we will need the full strength of the results established below in the principal application to come.

Lemma 6.2 *Let $\theta = S_\nu$ be a fixed singular inner function for which in the decomposition (1.7) of ν, the part $\nu_{\mathcal{C}}$ is supported on a single Beurling–Carleson set F of Lebesgue measure zero, and $\nu_{\mathcal{K}} \equiv 0$. Then there exists a rapidly increasing sequence $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^\infty$ (which depends on θ) such that the pair $(X(\boldsymbol{\alpha}^{-1}), \theta)$ satisfies the permanence property in Definition 6.1.*

Proof The idea of the proof is as follows: Let u be a function in K_θ, the orthogonal complement of θH^2 in H^2, and let Λ_u be the (in general, unbounded with respect to the norm on $X(\boldsymbol{\alpha}^{-1})$) linear functional

$$\Lambda_u f := \int_{\mathbb{T}} f\overline{u}\, dm, \tag{6.1}$$

which is defined for $f \in H^2 \subset X(\boldsymbol{\alpha}^{-1})$. If Λ_u can be extended to a bounded linear functional on $X(\boldsymbol{\alpha}^{-1})$ for u in a dense subset of K_θ, then $\|\theta f_n - f\|_{X(\boldsymbol{\alpha}^{-1})} \to 0$, with $f_n, f \in H^2$, will imply that f is orthogonal to K_θ in H^2. Indeed, in such a case, we will have

$$\langle f, u\rangle = \lim_{n\to\infty} \langle \theta f_n, u\rangle = 0,$$

for all u in a dense subset of K_θ, and so $f \in (K_\theta)^\perp = \theta H^2$. This of course means that $f/\theta \in H^2$. We will show that such a dense set can be constructed under the stated assumption, for some rapidly increasing sequence $\boldsymbol{\alpha}$. The proof will involve construction of a new set of smooth Cauchy transforms similar to those in (3.4), and an application of Proposition 3.4.

Note that θ extends analytically across the set $\mathbb{T}\backslash F$, and a simple differentiation argument and the formula (1.11) shows that we have the following estimate:

$$\left|\frac{\partial^m}{\partial t^m}\theta(e^{it})\right| \leq C_m \cdot \operatorname{dist}(e^{it}, F)^{-2m}, \quad e^{it} \in \mathbb{T}\backslash F. \tag{6.2}$$

Let $g = g_F$ be the function decaying rapidly near F, which is given by Lemma 2.1. We conclude, similarly to as in the proof of Proposition 3.1, that the set of functions on $\mathbb{T}$ defined by

$$K_1 := \{\theta\overline{\zeta p g_F} : p \text{ analytic polynomial }\}$$

consists of functions in $C^\infty(\mathbb{T})$, and thus the Cauchy transform of any function in this set is in $\mathcal{A}^\infty$. Let P_+ denote the projection operator from $L^2(\mathbb{T})$ to the Hardy space H^2. Then $P_+ f = C_f$, interpreted as functions on the circle. We now verify that the Cauchy transforms of elements of K_1 are members of K_θ. Let $\langle\cdot,\cdot\rangle_{L^2}$ denote the usual inner product for $L^2(\mathbb{T})$. For $s = \theta\overline{\zeta p g_F}$ and any $h \in H^2$, we have

$$\langle\theta h, C_s\rangle_{L^2} = \langle\theta h, P_+ s\rangle_{L^2} = \langle\theta h, s\rangle_{L^2} = \int_{\mathbb{T}} h\zeta p g_F dm = 0,$$

where the last integral vanishes because the integrand represents the boundary function of an analytic function with a zero at the origin. Thus, $C_s \in K_\theta$ for any $s \in K_\theta$. We now verify that this set of Cauchy transforms is dense in K_θ. If $f \in K_\theta$, then $f\overline{\theta} = \overline{\zeta f_0}$ as boundary functions, where $f_0 \in H^2$. Orthogonality of $f \in K_\theta$ to all functions $C_s, s \in K_1$, means that

$$\langle f, C_s\rangle_{L^2} = \int_{\mathbb{T}} \overline{f_0} p g_F dm = 0.$$

Since g_F is outer, the set

$$\{p g_F : p \text{ analytic polynomial }\} \tag{6.3}$$

is dense in H^2, and so the above implies $f_0 \equiv 0$, which means that $f \equiv 0$. We have thus constructed a dense set of functions in K_θ to which Proposition 3.4 applies, and the conclusion is that the Cauchy transforms we constructed are all contained in some space $X(\boldsymbol{\alpha})$ defined by a rapidly increasing sequence $\boldsymbol{\alpha}$. Then the space $X(\boldsymbol{\alpha}^{-1})$ satisfies the permanence principle for θ, by the observation in the first paragraph of this proof. ■

Lemma 6.3 *Let E be a Beurling–Carleson set of positive Lebesgue measure, and let w be a weight supported on E and satisfying $\int_E \log(w)\, dm > -\infty$. Let $\theta = S_\nu$ be a fixed singular inner function for which ν is supported on the set E. There exists a rapidly increasing sequence $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^\infty$ (which depends on E, w, and θ) for which the conclusion of Propoition 5.1 holds, and moreover the pair $(\mathcal{D}(\boldsymbol{\alpha}^{-1}, w), \theta)$ satisfies the permanence property in Definition 6.1.*

The difference from Lemma 6.2 is that Lemma 6.3 also applies to the case when $\nu_{\mathcal{K}}$ in (1.7) is nonzero.

Proof We follow the same idea as in the proof of Lemma 6.2. From the weight w, we construct the outer function W given by the formula (3.1). For the set E, we construct

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the corresponding function $g = g_E$ as in Lemma 2.1, and we define

$$K_2 := \{\theta\overline{\zeta p g_E W} : p \text{ analytic polynomial }\}.$$

This time, the Cauchy transforms of the functions in K_2 are not necessarily smooth. However, they are again contained and dense in K_θ, as in Lemma 6.2. The only difference in the proof, which we skip, is that the set in (6.3) is replaced by

$$\{p g_E W : p \text{ analytic polynomial }\},$$

which is dense in H^2 by the fact that W and g_E are outer.

For

$$s_0 := \theta\overline{\zeta g_E W} \in K_2,$$

we define the Cauchy transform $u_0 := C_{s_0}$. We can decompose u_0 according to

$$u_0(z) = \int_{\mathbb{T}} \frac{s_0(\zeta)}{1-\overline{\zeta}z}\, dm(\zeta) = \int_{\mathbb{T}\setminus E} \frac{s_0(\zeta)}{1-\overline{\zeta}z}\, dm(\zeta) + \int_E \frac{s_0(\zeta)}{1-\overline{\zeta}z}\, dm(\zeta)$$

$$:= u_1(z) + u_2(z). \tag{6.4}$$

The estimates in (3.6) and (6.2) show that in $s_0 1_{\mathbb{T}\setminus E}$ is a function in C^∞, and thus $u_1 \in \mathcal{A}^\infty$. Consequently, by Proposition 3.4, there exists a rapidly increasing sequence $\boldsymbol{\beta}$ such that $C_{s1_{\mathbb{T}\setminus E}}$ is in $X(\boldsymbol{\beta}^{-1})$, for all $s \in K_2$. Apply now Propoition 5.1 to E and w to obtain another rapidly increasing sequence $\boldsymbol{\gamma}$ such that the conclusion of that proposition holds, and let $\boldsymbol{\alpha}$ be the termwise minimum of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$:

$$\alpha_k = \min\{\beta_k, \gamma_k\}, \quad k \geq 0.$$

Then $\boldsymbol{\alpha}$ is again a rapidly increasing sequence, the conclusion of Propoition 5.1 holds, and $C_{s1_{\mathbb{T}\setminus E}}$ is in $X(\boldsymbol{\alpha})$, for all $s \in K_2$.

Moreover, the linear functional

$$f \mapsto \int_{\mathbb{T}} f\overline{u_2}\, dm,$$

defined on analytic polynomials f, is bounded in the metric of $L^2(w\, dm)$. Indeed, recall that $|W| = w$ on the set E, and so we have

$$\left|\int_{\mathbb{T}} f\overline{u_2}\, dm\right| = \left|\langle f, P_+ s_0 1_E\rangle_{L^2}\right| = \left|\int_E f\overline{s_0}\, dm\right| = \left|\int_E f\overline{\theta}\zeta g_E W\, dm\right| \leq c\|f\|_{L^2(w dm)}$$

for some constant $c > 0$, since $\overline{\theta}\zeta g_E$ is bounded. The same argument shows also that C_{s1_E} defines a bounded linear functional on the analytic polynomials in the metric of $L^2(w\, dm)$, for all $s \in K_2$.

We let $v = C_s$ for $s \in K_2$, $v_1 = C_{s1_{\mathbb{T}\setminus E}}$, and $v_2 = C_{s1_E}$, so that $v = v_1 + v_2$, and go back to the definition of the functional Λ_u in (6.1). We have just verified that we can decompose it according to (6.4)

$$\Lambda_v f := \int_{\mathbb{T}} f\overline{v}\, dm = \int_{\mathbb{T}} f\overline{v_1}\, dm + \int_{\mathbb{T}} f\overline{v_2}\, dm \tag{6.5}$$

in such a way that the first piece defines a continuous linear functional on the analytic polynomials in the metric of $X(\boldsymbol{\alpha}^{-1})$, and the second piece defines a continuous linear functional on the analytic polynomials in the metric of $L^2(w\,dm)$. But then, these functionals extend continuously to $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$, and by the density of $\{C_s : s \in K_2\}$ in K_θ and the argument in the first paragraph of the proof of Lemma 6.2, we conclude that the pair $(\mathcal{D}(\boldsymbol{\alpha}^{-1}, w), \theta)$ satisfies the permanence property. ∎

Corollary 6.4 **(An inner factor permanence principle)** *Let E be a Beurling–Carleson set of positive Lebesgue measure, and let w be a weight supported on E and satisfies $\int_E \log(w)\,dm > -\infty$. Let $\theta = S_\nu$ be a fixed singular inner function such that in the decomposition (1.7), the part $\nu_{\mathcal{C}}$ satisfies $\nu_{\mathcal{C}}(F) = \nu_{\mathcal{C}}(\mathbb{T})$ for a single fixed Beurling–Carleson set F of Lebesgue measure zero, and $\nu_{\mathcal{K}}$ is supported on E. There exists a rapidly increasing sequence $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^{\infty}$ (which depends on E, w, and θ) for which the conclusion of Propoition 5.1 holds, and moreover the pair $(\mathcal{D}(\boldsymbol{\alpha}^{-1}, w), \theta)$ satisfies the permanence property in Definition 6.1.*

Proof The required rapidly increasing sequence $\boldsymbol{\alpha}$ is the one obtained by constructing the termwise minimum of the sequences given by Lemmas 6.2 and 6.3. ∎

The above result is essentially optimal. Indeed, if $\nu = \nu_{\mathcal{K}}$ in (1.7) and $\nu(\mathbb{T}\backslash E) > 0$, then S_ν will be divisible by an inner function which is cyclic in $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ (and so certainly cannot satisfy the permanence property). This can be seen from the corresponding cyclicity result in [16] for the class of $\mathcal{P}^2(\mu)$-spaces appearing in Corollary 5.3. In the other direction, we note that the main inner factor permanence result in [16] is an immediate consequence of Corollary 6.4. Here is the statement.

Corollary 6.5 **(The permanence principle for a class of $\mathcal{P}^2(\mu)$-spaces)** *Let $C > -1$ and E be a Beurling–Carleson set of positive measure. Let w be a bounded positive measurable function which is supported on E and satisfies $\int_E \log(w) dm > -\infty$. Consider the measure*

$$d\mu = (1 - |z|^2)^C dA + w dm$$

and the classical Lebesgue space $L^2(\mu)$. Let $\mathcal{P}^2(\mu)$ be the closure of analytic polynomials in $L^2(\mu)$. If $\theta = S_\nu$ be a singular inner function such that in the decomposition (1.7) the part $\nu_{\mathcal{K}}$ is supported on E, then the pair $(\mathcal{P}^2(\mu), \theta)$ satisfies the permanence property in Definition 6.1.

We skip the proof, which is similar to the proof of Corollary 5.3.

7 Density of smooth functions in extreme $\mathcal{H}(b)$-spaces

This final section is devoted to the proof of density of smooth functions in the class of de Branges–Rovnyak spaces described in Section 1.

7.1 A little background on $\mathcal{H}(b)$

The following construction of the $\mathcal{H}(b)$-space appears in [2].

Proposition 7.1 *Let b be an extreme point of the unit ball of H^∞,*

$$E = \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\},$$

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and let $\Delta = \sqrt{1-|b|^2}$ *be a function on the circle* $\mathbb{T}$*, defined in terms of boundary values of* b *on* $\mathbb{T}$*. For* $f \in \mathcal{H}(b)$*, the equation*

$$P_+\overline{b}f = -P_+\Delta g \tag{7.1}$$

has a unique solution $g \in L^2(E)$*, and the map* $J : \mathcal{H}(b) \to H^2 \oplus L^2(E)$ *defined by*

$$Jf = (f, g)$$

is an isometry. Moreover,

$$J(\mathcal{H}(b))^\perp = \left\{(bh, \Delta h) : h \in H^2\right\}. \tag{7.2}$$

The benefit of the above described way of constructing the $\mathcal{H}(b)$-space (that is, using the embedding J above, and an orthogonal complement) is that it will be particularly easy to implement our duality argument.

We need only one more lemma before going into the final proof.

Lemma 7.2 *Let* $b = \theta u$ *be an extreme point of the unit ball of* H^∞*, where* θ *and* u *are the inner and outer factors of* b*, respectively. Furthermore, let* $\{\theta_n\}_n$ *be a sequence of inner divisors of* θ *such that*

$$\lim_{n\to\infty} \theta_n(z) = \theta(z)$$

for all $z \in \mathbb{D}$*, and let* $\{E_n\}_n$ *be an increasing sequence of subsets of* $\mathbb{T}$ *such that*

$$E := \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\} = \cup_n E_n$$

up to a set of Lebesgue measure zero. For $n \geq 1$*, let* u_n *be the outer function with modulus*

$$|u_n| = 1_{E\setminus E_n} + |b| 1_{E_n}$$

on $\mathbb{T}$*. Set* $b_n = \theta_n u_n$*. Then* $\mathcal{H}(b_n)$ *is contractively contained in* $\mathcal{H}(b)$*, and* $\cup_n \mathcal{H}(b_n)$ *is norm-dense in* $\mathcal{H}(b)$*.*

Proof Let k_b and k_{b_n} be the reproducing kernels of $\mathcal{H}(b)$ and $\mathcal{H}(b_n)$, respectively. Note that the assumptions imply that b_n divides b, in the sense that

$$|b(z)/b_n(z)| \leq 1, \quad z \in \mathbb{D}. \tag{7.3}$$

Then

$$k_b(\lambda, z) - k_{b_n}(\lambda, z) = \frac{\overline{b_n(\lambda)}b_n(z) - \overline{b(\lambda)}b(z)}{1-\overline{\lambda}z} = \overline{b_n(\lambda)}b_n(z)\frac{1-\overline{b/b_n(\lambda)}b/b_n(z)}{1-\overline{\lambda}z}$$

is clearly a positive definite kernel, so by standard theory of reproducing kernel Hilbert spaces (see, for instance, [4] or [1]), it follows that $\mathcal{H}(b_n)$ is contractively contained in $\mathcal{H}(b)$. Moreover, contractivity of the containment means that

$$\|k_{b_n}(\lambda,\cdot)\|^2_{\mathcal{H}(b)} \leq \|k_{b_n}(\lambda,\cdot)\|^2_{\mathcal{H}(b_n)} = \frac{1-|b_n(\lambda)|^2}{1-|\lambda|^2} \leq \frac{1-|b(\lambda)|^2}{1-|\lambda|^2},$$

where in the last step we used (7.3). So, for fixed λ, the functions $k_{b_n}(\lambda,\cdot)$ are norm-bounded in $\mathcal{H}(b)$. It is not hard to see from the usual construction of the outer

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functions that $u_n(z) \to u(z)$ as $n \to \infty$, for every $z \in \mathbb{D}$. Consequently, $b_n(z) \to b(z)$ for each $z \in \mathbb{D}$, and even

$$\lim_{n\to\infty} k_{b_n}(\lambda, z) = k_b(\lambda, z), \quad z, \lambda \in \mathbb{D}.$$

Together with the norm estimate above, this means that for any fixed $\lambda \in \mathbb{D}$, a suitable subsequence of the kernels $k_{b_n}(\lambda, \cdot)$ will converge weakly in $\mathcal{H}(b)$ to $k_b(\lambda, \cdot)$. Elementary functional analysis now ensures that $\cup_n \mathcal{H}(b_n)$ is dense in $\mathcal{H}(b)$. ∎

We remark that a simple consequence of the contractive containment of $\mathcal{H}(b_n)$ in $\mathcal{H}(b)$ is the following: density of $\mathcal{A}^\infty \cap \mathcal{H}(b_n)$ in $\mathcal{H}(b_n)$ for each n implies density of $\mathcal{A}^\infty \cap \mathcal{H}(b)$ in $\mathcal{H}(b)$.

7.2 The density theorem

In the proof below, we use the duality pairing $\langle \cdot, \cdot \rangle$ appearing in Section 4.1. For a set S in either $X(\boldsymbol{\alpha})$ or $X(\boldsymbol{\alpha}^{-1})$, we denote by $S^\perp$ the linear space of elements in the other space which is annihilated by S under the duality. Basic Hilbert space theory says that $(S^\perp)^\perp$ is the norm-closure of S.

For convenience, we restate the Theorem B of Section 1.

Theorem 7.3 ($\mathcal{A}^\infty$**-density theorem in** $\mathcal{H}(b)$**-spaces**) *Let $b = \theta u$ be an extreme point of the unit ball of H^∞ satisfying the following assumptions.*

(i) *There exists an increasing sequence $\{E_n\}_n$ of Beurling–Carleson sets of positive measure such that, up to a set of Lebesgue measure zero, we have the equality*

$$E := \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\} = \cup_n E_n$$

and

$$\int_{E_n} \log(1 - |b|^2)\, dm > -\infty, \quad \textit{for all } n.$$

(ii) *If $\theta = BS_\nu$, where B is a Blaschke product and ν is the measure defining the singular inner factor as in (1.11), then in the decomposition (1.7), the part $\nu_{\mathcal{K}}$ which vanishes on Beurling–Carleson sets of Lebesgue measure zero satisfies $\nu_{\mathcal{K}}(\cup_n E_n) = \nu_{\mathcal{K}}(\mathbb{T})$, where $\{E_n\}_n$ are the sets in (i).*

Then $\mathcal{A}^\infty \cap \mathcal{H}(b)$ is norm-dense in $\mathcal{H}(b)$.

Before going into the proof, we remind the reader of what was remarked in Section 1, that the condition $\nu_{\mathcal{K}}(\cup_n E_n) = \nu_{\mathcal{K}}(\mathbb{T})$ appearing in (ii) above is essentially necessary, and that some type of structure condition on the set E, and a size condition on the weight $(1 - |b|^2)$, also are necessary (examples are mentioned in [18]).

The proof below is essentially the same as the one given in the context of $\mathcal{P}^2(\mu)$-spaces in [18].

Proof By Lemma 7.2 and the remark following it, we can assume that E is a Beurling–Carleson set of positive measure, and the inner factor has a factorization

$$\theta = BS_\nu = BS_{\nu_{\mathcal{C}}} S_{\nu_{\mathcal{K}}},$$

where $\nu_{\mathcal{C}}$ is supported on a single Beurling–Carleson set F of Lebesgue measure zero.

We set $w := \Delta^2 = (1-|b|^2)1_E$, and apply Corollary 6.4 to the data $E, F, S_{\nu_{\mathcal{C}}}, S_{\nu_{\mathcal{K}}}$ to obtain a rapidly increasing sequence $\boldsymbol{\alpha}$ and a space $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ which satisfies the conclusion of Corollary 6.4. Since $\boldsymbol{\alpha}$ is rapidly increasing, the space $X(\boldsymbol{\alpha})$ consists of functions in $\mathcal{A}^\infty$. We will show that $X(\boldsymbol{\alpha}) \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$.

Assume that $f \in \mathcal{H}(b)$ is orthogonal to all functions in $X(\boldsymbol{\alpha}) \cap \mathcal{H}(b)$. In terms of the embedding J appearing in Proposition 7.1, this means that the tuple $Jf := (f, g)$ is orthogonal in $H^2 \oplus L^2(E)$ to all tuples in $J(X(\boldsymbol{\alpha}) \cap \mathcal{H}(b))$. By the definition of the duality pairing appearing in Section 4.1, this means that

$$(f, g) \in X(\boldsymbol{\alpha}^{-1}) \oplus L^2(E)$$

annihilates

$$J(X(\boldsymbol{\alpha}) \cap \mathcal{H}(b)) \subset X(\boldsymbol{\alpha}) \oplus L^2(E).$$

Now, from (7.2), we see that the set $J(X(\boldsymbol{\alpha}) \cap \mathcal{H}(b)) \subset X(\boldsymbol{\alpha}) \oplus L^2(E)$ can be expressed as the pre-annihilator. Then

$$J(X(\boldsymbol{\alpha}) \cap \mathcal{H}(b)) = \{(bh, \Delta h) \in X(\boldsymbol{\alpha}^{-1}) \oplus L^2(E) : h \in H^2\}^\perp. \tag{7.4}$$

We are in a Hilbert space setting, so it follows from the duality remarks above that there exists a sequence $\{h_n\}_n$ of functions in H^2, such that $(bh_n, \Delta h_n)$ converges to (f, g) in the norm of $X(\boldsymbol{\alpha}^{-1}) \oplus L^2(E)$. By passing to a subsequence, we may assume that the convergence of Δh_n to g happens also pointwise almost everywhere on E. Multiplying the second coordinate by the bounded function b, we read that the elements $(bh_n, \Delta bh_n)$ converge to (f, bg) in the norm $X(\boldsymbol{\alpha}^{-1}) \oplus L^2(E)$, and in particular the tuples $(bh_n, \Delta bh_n)$ form a Cauchy sequence. In fact, since $w = \Delta^2$, this is equivalent to

$$\lim_{n,m\to\infty} \|bh_n - bh_m\|_{X(\boldsymbol{\alpha}^{-1})} + \|bh_n - bh_m\|_{L^2(w\,dm)} = 0,$$

so the sequence (bh_n, bh_n) converges in the space $\mathcal{D}(\boldsymbol{\alpha}^{-1}, w)$ which is a space of analytic functions. The limit function must be $f \in H^2$, by the above. Now, we apply part (ii) of Proposition 5.2, which tells us that $bh_n \to f$ pointwise on the set E (it is precisely at this point where M_z being completely nonisometric on $D(\boldsymbol{\alpha}^{-1}, w)$ is crucial, else the sequence $\{bh_n\}_n$ could potentially converge to something else than f). Thus, we have

$$b(\zeta)g(\zeta) = \lim_{n\to\infty} \Delta(\zeta)b(\zeta)h_n(\zeta) = \Delta(\zeta)f(\zeta)$$

for almost every $\zeta \in E$. Thus, $g = \Delta f / b$ on E. All in all, we have identified the tuple Jf as

$$Jf = (f, \Delta f / b).$$

Moreover, by the permanence property of $(\mathcal{D}(\boldsymbol{\alpha}^{-1}, w), \theta)$, f is divisible by the inner factor θ of b.

Note that on $\mathbb{T}$, we have

$$f/b = (|b|^2 + \Delta^2) f/b = \overline{b} f + \Delta g.$$

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Since the right-hand side is square-integrable, so is the left. Since the inner factor of f is divisible by the inner factor of b, we conclude that f/b is a function in the Smirnov class of the unit disk which has square-integrable boundary values, and so $f/b \in H^2$ (see [8, Chapter 2]). But by (7.1), we get

$$f/b = P_+(f/b) = P_+(\overline{b}f + \Delta g) = 0.$$

So $f/b \equiv 0$, and hence $f \equiv 0$. The proof is complete. ∎

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