SOME PROBLEMS FOR TYPICALLY REAL FUNCTIONS

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1. Many extremal properties of the class of normalized univalent functions are shared by the class of typically real functions each considered in the unit circle. By the class T of typically real functions we mean those functions $f(z)$, regular for $|z| < 1$ with $f(0) = 0$, $f'(0) = 1$, and such that $\Im f(z) > 0$ for $\Im z > 0$, $\Im f(z) < 0$ for $\Im z < 0$. This class was first studied by Rogosinski **(4)** who proved various simple properties for it. Later Robertson **(3)** took up the study and proved the following important representation result. If $f(z) \in T$ there exists a function $\alpha(\theta)$, increasing for $0 \leqslant \theta \leqslant \pi$ with $\alpha(0) = 0$, $\alpha(\pi) = 1$ such that

$$f(z) = \int_0^\pi z(1 - 2\cos\theta z + z^2)^{-1} d\alpha(\theta).$$

Other authors have treated further problems for the class T but none of these problems seems to belong to the class we would characterize as conditional extremal problems. For the class S of normalized univalent functions in the unit circle the best known such problem is of course the problem of Gronwall **(1)** to determine the maxima and the minima of $|f(z)|$ and $|f'(z)|$ for $|z| = r$ when the value of $|f''(0)|$ is assigned. In the present paper we will solve analogous problems for the class T. For the minimum problems an essential role in the solution is played by the Neyman–Pearson Lemma, important in statistical theory. For the maximum problems a different but similar lemma is applied. The author wishes to express his thanks to Ky Fan and Irving Glicksberg for calling his attention to the Neyman–Pearson Lemma and to Seymour Sherman for a useful conversation on the relationship between the mathematical and statistical aspects of the lemma.

2. Because of the difficulty of giving a reference to the Neyman–Pearson Lemma in a simple purely mathematical form we will give here a simple proof in the special case which is sufficient for our purposes.

LEMMA 1. *Let $g(x)$, $h(x)$ be non-negative measurable functions defined on the interval $[a, b]$ such that $g(x)/h(x)$ is continuous and strictly increasing. Let*

$$0 \leqslant k \leqslant \int_a^b g(x)dx.$$

Then the extremal problem given by

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$$\int_a^b f(x)g(x)dx = k$$

$$\int_a^b f(x)h(x)dx = \text{minimum}$$

for functions $f(x)$ *measurable on* $[a, b]$ *and satisfying* $0 \leqslant f(x) \leqslant 1$ *has as a solution the function* $f^*(x)$ *defined by*

$$(1) \qquad \begin{aligned} f^*(x) &= 0 \qquad & a \leqslant x < c \\ f^*(x) &= 1 \qquad & c \leqslant x \leqslant b \end{aligned}$$

where c *is determined by*

$$(2) \qquad \int_c^b g(x)dx = k.$$

If $f(x)$ *is further required to be increasing this solution is unique apart from its definition at the value* $x = c$.

It is clear that the value c is uniquely determined by (2). Then let $f^*(x)$ be the function defined by (1) and $f(x)$ any other admissible function for the extremal problem. Then if $g(c)/h(c) = \lambda$ we have

$$\int_a^b (f - f^*)(\lambda h - g)dx \geqslant 0$$

since on $[a, c)$

$$f(x) - f^*(x) \geqslant 0, \qquad \lambda h(x) - g(x) \geqslant 0$$

and on $[c, b]$

$$f(x) - f^*(x) \leqslant 0, \qquad \lambda h(x) - g(x) \leqslant 0.$$

Since

$$\int_a^b f\, g\, dx = \int_a^b f^* g\, dx = k$$

and $\lambda > 0$, unless $c = a$ in which case the result is evident, we find

$$\int_a^b f\, h\, dx \geqslant \int_a^b f^*\, h dx$$

as stated. Evidently the solution is uniquely determined up to a set of measure zero. When $f(x)$ is required to be increasing the solution of the extremal problem is thus uniquely determined apart from its value at the point $x = c$.

THEOREM 1. *Let* $f(z) \in T$ *and have the expansion about the origin*

$$f(z) = z + A_2 z^2 + \ldots .$$

It is well known that

$$-2 \leqslant A_2 \leqslant 2.$$

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For a fixed value μ, $-2 \leqslant \mu \leqslant 2$, *if*

$$A_2 = \mu$$

then for $0 < r < 1$

(3) $$f(r) \geqslant r(1 - \mu r + r^2)^{-1}$$

(4) $$f'(r) \geqslant (1 - r^2)(1 - \mu r + r^2)^{-2}.$$

Equality is attained only for the function $z(1 - \mu z + z^2)^{-1}$.

Using for $f(z)$ the representation

$$f(z) = \int_0^\pi z(1 - 2\cos\theta z + z^2)^{-1} d\alpha(\theta)$$

where $\alpha(\theta)$ is increasing for $0 \leqslant \theta \leqslant \pi$ with $\alpha(0) = 0$, $\alpha(\pi) = 1$ we have

(5) $$A_2 = \int_0^\pi 2\cos\theta d\alpha(\theta),$$

(6) $$f(r) = \int_0^\pi r(1 - 2\cos\theta r + r^2)^{-1}\, d\alpha(\theta),$$

(7) $$f'(r) = \int_0^\pi (1 - r^2)(1 - 2\cos\theta r + r^2)^{-2}\, d\alpha(\theta).$$

Integrating each of these equalities by parts (compare (**5**; Theorem 4b)) we have

(8) $$A_2 = 2[-\alpha(\pi) - \alpha(0)] + 2\int_0^\pi \alpha(\theta)\sin\theta\, d\theta = -2 + 2\int_0^\pi \alpha(\theta)\sin\theta\, d\theta,$$

(9) $$f(r) = r(1 + 2r + r^2)^{-1} + \int_0^\pi 2r^2\alpha(\theta)\sin\theta(1 - 2r\cos\theta + r^2)^{-2}\, d\theta,$$

(10) $$f'(r) = (1 - r^2)(1 + 2r + r^2)^{-2} + \int_0^\pi 4r(1 - r^2)\alpha(\theta)\sin\theta \, (1 - 2r\cos\theta + r^2)^{-3}\, d\theta.$$

We now apply Lemma 1 with $a, b, g(x), h(x), k$ replaced by $0, \pi, \sin\theta$, $\sin\theta(1 - 2r\cos\theta + r^2)^{-2}$, $\frac{1}{2}(\mu + 2)$. Thus we find $f(r)$ will be minimal when $\alpha(\theta)$ is defined by

$$\alpha(\theta) = 0, \qquad 0 \leqslant \theta < \theta^*$$
$$\alpha(\theta) = 1, \qquad \theta^* \leqslant \theta \leqslant \pi$$

and θ^* is such that

$$2\cos\theta^* = \mu.$$

From this the inequality (3) is immediate. The equality statement follows at once from Lemma 1. Alternatively replacing $h(x)$ by $\sin\theta(1 - 2r\cos\theta + r^2)^{-3}$ we obtain the inequality (4) and the corresponding equality statement.

3. If we consider instead of the minimum problem of Theorem 1 the corresponding maximum problem the Neyman–Pearson Lemma no longer provides a solution since the result corresponding to Lemma 1 would lead to a decreasing rather than an increasing function $\alpha(\theta)$. However we can use instead the following lemma which is more special and essentially confined to the present situation.

LEMMA 2. *Let $g(x)$, $h(x)$ be non-negative measurable functions defined on the interval $[a, b]$ such that $g(x)/h(x)$ is continuous and strictly increasing. Let*

$$0 \leqslant k \leqslant \int_a^b g(x)\, dx.$$

Then the extremal problem given by

$$\int_a^b f(x)g(x)dx = k$$

$$\int_a^b f(x)h(x)dx = \text{maximum}$$

for functions $f(x)$ increasing on $[a, b]$ and satisfying $0 \leqslant f(x) \leqslant 1$ has as a solution the function $f^*(x)$ defined by

$$f^*(x) = k \Big/ \int_a^b g(x)dx, \qquad a \leqslant x \leqslant b.$$

This solution is uniquely determined apart from its values at a and b.

Let $f(x)$ be any admissible function for the extremal problem. There will exist a value c, $a \leqslant c \leqslant b$ such that

$$f(x) \leqslant k \Big/ \int_a^b g(x)dx, \qquad a \leqslant x < c$$

$$f(x) \geqslant k \Big/ \int_a^b g(x)dx, \qquad c < x \leqslant b.$$

Then if $g(c)/h(c) = \lambda$ we have

$$\int_a^b (f - f^*)(\lambda h - g)dx \leqslant 0,$$

since on $[a, c)$

$$f(x) - f^*(x) \leqslant 0, \qquad \lambda h(x) - g(x) > 0$$

and on $(c, b]$

$$f(x) - f^*(x) \geqslant 0, \qquad \lambda h(x) - g(x) < 0.$$

Since

$$\int_a^b f\, g\, dx = \int_a^b f^* g\, dx = k$$

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and $\lambda > 0$, unless $c = a$ in which case the result is evident, we find

$$\int_a^b f\,h\,dx \leqslant \int_a^b f^* h\,dx$$

as stated. Since the solution is uniquely determined up to a set of measure zero the equality statement is immediate from the requirement that $f(x)$ be increasing.

THEOREM 2. *Let $f(z) \in T$ and have the expansion about the origin*

$$f(z) = z + A_2 z^2 + \ldots .$$

For a fixed value μ, $-2 \leqslant \mu \leqslant 2$, if

$$A_2 = \mu$$

then for $0 < r < 1$

(11) $\quad f(r) \leqslant (\tfrac{1}{4}\mu + \tfrac{1}{2})r(1-r)^{-2} + (\tfrac{1}{2} - \tfrac{1}{4}\mu)r(1+r)^{-2}$

(12) $\quad f'(r) \leqslant (\tfrac{1}{4}\mu + \tfrac{1}{2})(1+r)(1-r)^{-3} + (\tfrac{1}{2} - \tfrac{1}{4}\mu)(1-r)(1+r)^{-3}.$

Equality is attained only for the function

$$(\tfrac{1}{4}\mu + \tfrac{1}{2})z(1-z)^{-2} + (\tfrac{1}{2} - \tfrac{1}{4}\mu)z(1+z)^{-2}.$$

As in the proof of Theorem 1, $A_2, f(r), f'(r)$ are represented by the equations (5), (6), (7) or by the partially integrated expressions (8), (9), (10). We apply this time Lemma 2 first with a, b, $g(x)$, $h(x)$, k replaced by 0, π, $\sin\theta$, $\sin\theta(1 - 2r\cos\theta + r^2)^{-2}$, $\tfrac{1}{2}(\mu + 2)$. Then we find that $f(r)$ is maximal for $\alpha(\theta)$ defined by

$$\alpha(\theta) = \tfrac{1}{4}\mu + \tfrac{1}{2}, \qquad 0 < \theta < \pi.$$

By the normalization imposed on $\alpha(\theta)$ we have $\alpha(0) = 0$, $\alpha(\pi) = 1$. Inserting this function in (6) we obtain the bound (11) and in the Stieltjes integral representation of $f(z)$ we find that the unique extremal function is

$$(\tfrac{1}{4}\mu + \tfrac{1}{2})z(1-z)^{-2} + (\tfrac{1}{2} - \tfrac{1}{4}\mu)z(1+z)^{-2}.$$

Alternatively replacing $h(x)$ by $\sin\theta(1 - 2r\cos\theta + r^2)^{-3}$ we obtain the bound (12) and find that the unique extremal function is again the one just given.

Numerous other problems can be solved by the methods presented here. Moreover, Lemma 2 has extensions which allow further applications. In the present cases it is interesting to note that the lower bounds given by (3) and (4) coincide with those obtained in Gronwall's problem while the upper bounds (11) and (12) occur for a function which is not univalent, compare **(2)**.

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