Canad. Math. Bull. Vol. 17 (5), 1975

A NOTE ON KLEIN'S OSCILLATION THEOREM FOR PERIODIC BOUNDARY CONDITIONS

BY
M. FAIERMAN

Recently Howe [4] has considered the oscillation theory for the two-parameter eigenvalue problem

$$(p_1(x_1)y_1')'+(\lambda A_1(x_1)+\mu B_1(x_1)+q_1(x_1))y_1 = 0, \quad a_1 \le x_1 \le b_1, \quad ' = d/dx_1, \tag{1a}$$

$$(p_2(x_2)y_2')'+(\lambda A_2(x_2)+\mu B_2(x_2)+q_2(x_2))y_2 = 0, \quad a_2 \le x_2 \le b_2, \quad ' = d/dx_2, \tag{1b}$$

subjected to the boundary conditions

$$y_1(a_1) = y_1(b_1), \qquad y_1'(a_1) = y_1'(b_1), \tag{2a}$$

$$y_2(a_2) = y_2(b_2), \qquad y_2'(a_2) = y_2'(b_2), \tag{2b}$$

where for $i=1, 2$, $-\infty<a_i<b_i<\infty$, p_i, p_i', A_i, B_i, and q_i are real-valued, continuous functions in $[a_i, b_i]$, p_i is positive in $[a_i, b_i]$, and $p_i(a_i)=p_i(b_i)$. Furthermore, it is also assumed that $(A_1B_2-A_2B_1)\neq 0$ for all values of x_1 and x_2 in their respective intervals.

It is our opinion that some of the arguments used by Howe are incomplete and require further modification. In particular, we refer to the proofs of his lemma 1 and theorem 4, and also to his assumption concerning the existence of a continuously turning tangent for a curve which may possess singular points (see his theorem 1). Hence it has been felt worthwhile to present here a new proof of the oscillation theorem for the system (1, 2). Moreover, by subjecting the differential equations (1) to the further boundary conditions

$$y_1(a_1) = y_1(b_1), \qquad y_1'(a_1) = y_1'(b_1), \tag{3a}$$

$$y_2(a_2) = -y_2(b_2), \qquad y_2'(a_2) = -y_2'(b_2), \tag{3b}$$

and

$$y_1(a_1) = -y_1(b_1), \qquad y_1'(a_1) = -y_1'(b_1), \tag{4a}$$

$$y_2(a_2) = -y_2(b_2), \qquad y_2'(a_2) = -y_2'(b_2), \tag{4b}$$

the same arguments can be employed to prove the oscillation theorem for each of the systems (1, 2), (1, 3), and (1, 4).

Before proceeding further, we first wish to make the following definitions concerning the system (1, 2). Analogous definitions also hold for each of the systems (1, 3) and (1, 4). By an eigenvalue of the system (1, 2) we shall mean a pair of numbers, (λ^*, μ^*), such that for $\lambda=\lambda^*$ and $\mu=\mu^*$, (1a) (resp. (1b)) has a non-trivial solution satisfying (2a) (resp. (2b)). If $y_1(x_1, \lambda^*, \mu^*)$ (resp. $y_2(x_2, \lambda^*, \mu^*)$) denotes such a solution, then the product, $\prod_{i=1}^2 y_i(x_i, \lambda^*, \mu^*)$, will be called an eigenfunction of the system (1, 2) corresponding to (λ^*, μ^*). Let I_2 denote the Cartesian product of the intervals $a_1 \leq x_1 \leq b_1$, $a_2 \leq x_2 \leq b_2$; and denote by L^2_Δ the Hilbert space of those Lebesgue measurable functions which are square-integrable in I_2 with respect to the weight function $\Delta = |A_1B_2 - A_2B_1|$ (we note that here the inner product is always taken with respect to the weight Δ). Then the eigenfunctions corresponding to an eigenvalue of the system (1, 2) generate a subspace in L^2_Δ, which we shall refer to as the corresponding eigenspace. By the multiplicity of an eigenvalue, we shall mean the dimension of the corresponding eigenspace in L^2_Δ. Also, in the sequel, when we speak of linearly independent eigenfunctions, we shall mean linearly independent as elements of L^2_Δ. Similarly, we shall say that two eigenfunctions are orthogonal in L^2_Δ if their inner product is zero.

THEOREM 1. *The eigenvalues of the system* (1, 2) *form a countably infinite subset of* E_2 (*Euclidean 2-space*) *having no finite cluster points. Each eigenvalue has multiplicity one, two, or four, that is to say, to each eigenvalue there corresponds one, two, or four linearly independent eigenfunctions. Eigenfunctions corresponding to distinct eigenvalues are orthogonal in* L^2_Δ. *Let* m_1, m_2 *be any non-negative integers which are either zero or even. For* m_1 *and* m_2 *both greater than zero there exists precisely four eigenvalues of the system* (1, 2) (*multiple eigenvalues being repeated*) *such that if* $\prod_{i=1}^2 y_{i,j}(x_i)$, $j=1, \ldots, 4$, *denote the corresponding* (*linearly independent*) *eigenfunctions, then* $y_{i,j}(x_i)$ *has exactly* m_i *zeros in* $[a_i, b_i)$, $i=1, 2$, $j=1, \ldots, 4$. *If either* $m_1=0$ *and* $m_2>0$ *or* $m_1>0$ *and* $m_2=0$, *then there exists precisely two eigenvalues of* (1, 2) (*multiple eigenvalues being repeated*) *with the corresponding* (*linearly independent*) *eigenfunctions satisfying the above oscillation properties. If* $m_1=m_2=0$, *then there exists exactly one* (*simple*) *eigenvalue of* (1, 2) *with the corresponding eigenfunction having the above oscillation properties. Finally, considering all such possible tuples* (m_1, m_2), *the corresponding eigenvalues, each repeated as many times as its multiplicity indicates, constitute the totality of the eigenvalues of the system* (1, 2).

THEOREM 2. *If in theorem* 1 *we now let* m_1, m_2 *be any non-negative integers such that* m_1 *is either zero or even and* m_2 *is odd, then, with obvious modifications, the results of theorem* 1 *are valid for the system* (1, 3).

THEOREM 3. *If in theorem* 1 *we now let* m_1, m_2 *be any positive odd integers, then, with obvious modifications, the results of theorem* 1 *are valid for the system* (1, 4).

We shall limit ourselves to the proof of theorem 1. Moreover, since it is quite straightforward to verify that eigenvalues of the system (1, 2) necessarily belong

https://doi.org/10.4153/CMB-1974-135-2 Published online by Cambridge University Press

to E_2, we refer to [4, theorem 2] for such a proof. Similarly, by slightly modifying the arguments given in the reference just cited, it is easy to show that eigenfunctions corresponding to distinct eigenvalues are orthogonal in L^2_Δ. Also, by arguing as in [1, theorem 9.4.1, p. 151], it is clear that by introducing an orthogonal transformation in the parameters λ and μ, if necessary, we can from now on assume that B_1, B_2, and $(A_1B_2-A_2B_1)$ are all positive for all values of x_1 and x_2 in their respective intervals.

Proof of theorem 1. We shall from now on assume that λ is real and for fixed λ denote by $\nu_i(\lambda)$, $i\geq 0$, the eigenvalues of (1a) with boundary conditions $y_1(a_1)=y_1(b_1)=0$; here the eigenfunction corresponding to $\nu_i(\lambda)$ has exactly i zeros in (a_1, b_1) and hence $i+2$ zeros in $[a_1b_1]$. Similarly, we denote by $\rho_i(\lambda)$, $i\geq 0$, the eigenvalues of (1a) with boundary conditions $y_i'(a_1)=y_i'(b_1)=0$ and where the eigenfunction corresponding to $\rho_i(\lambda)$ has exactly i zeros in (a_1, b_1) and hence i zeros in $[a_1, b_1]$.

Now let us consider the eigenvalue problem (1a, 2a). From [2, pp. 213–218] we know that for fixed λ the eigenvalues of (1a) with (2a), $\mu_i(\lambda)$, $i\geq 0$, form a sequence such that

$$-\infty < \rho_0(\lambda) \leq \mu_0(\lambda) < \nu_0(\lambda) < \mu_1(\lambda) \leq \nu_1(\lambda) \leq \mu_2(\lambda) < \nu_2(\lambda) < \cdots$$
$$\cdots < \nu_{2i}(\lambda) < \mu_{2i+1}(\lambda) \leq \nu_{2i+1}(\lambda) \leq \mu_{2i+2}(\lambda) < \nu_{2i+2}(\lambda) < \cdots.$$

At $\mu_0(\lambda)$ there is a unique eigenfunction having no zeros in $[a_1, b_1]$. If $\mu_{2i+1}(\lambda) < \mu_{2i+2}(\lambda)$ for some $i\geq 0$, then there is a unique eigenfunction at $\mu_{2i+1}(\lambda)$ having exactly $2i+2$ zeros in $[a_1, b_1)$ and a unique eigenfunction at $\mu_{2i+2}(\lambda)$ having exactly $2i+2$ zeros in $[a_1, b_1)$. If, however, $\mu_{2i+1}(\lambda)=\mu_{2i+2}(\lambda)$, then there are two linearly independent eigenfunctions at $\mu_{2i+1}(\lambda)$ each having exactly $2i+2$ zeros in $[a_1, b_1)$. Moreover, from [2, pp. 219–220] and [3, subsection 2.1], we know that $\rho_i(\lambda)$, $\nu_i(\lambda)$ and $\mu_i(\lambda)$, $i\geq 0$, are all continuous in $-\infty<\lambda<\infty$.

Analogous results hold for (1b) with boundary conditions $y_2'(a_2)=y_2'(b_2)=0$, $y_2(a_2)=y_2(b_2)=0$, and (2b), respectively. We denote the corresponding eigenvalues by $\{\rho_i^*(\lambda)\}_{i=0}^\infty$, $\{\nu_i^*(\lambda)\}_{i=0}^\infty$, and $\{\mu_i^*(\lambda)\}_{i=0}^\infty$, respectively.

It now follows that the above eigenvalues, as functions of λ, determine continuous curves in the (λ, μ)-plane. For $i\geq 0$ denote by C_i (resp. C_i^*) the curve in the (λ, μ)-plane determined by $\nu_i(\lambda)$ (resp. $\nu_i^*(\lambda)$), and by S_i (resp. S_i^*) the curve determined by $\mu_i(\lambda)$ (resp. $\mu_i^*(\lambda)$). It is clear that the eigenvalues of the system (1, 2) are precisely the points of intersection, if any, of the curves S_i with the curves S_j^*. Moreover, if d_i (resp. d_i^*) is the distance of C_i (resp. C_i^*) to the origin of the (λ, μ)-plane, then we may argue as in [5, section 2] to prove that d_i (resp. d_i^*) tends to infinity with i. This shows that any bounded subset of the (λ, μ)-plane can intersect only a finite number of the S_i (resp. S_i^*). Hence it follows that the proof of theorem 1 will be completed once we have shown that S_i intersects S_j^* in precisely one point for $i=m_1$, $i=m_1-1$ if $m_1>0$, $j=m_2$, and $j=m_2-1$ if $m_2>0$.

Moreover, we shall also show that at the point of intersection, the curves S_i, S_j^* cannot touch (in a sense which will be made precise in the sequel).

Introducing angle into the (λ, μ)-plane in the usual way, we shall denote by Θ the angle which a ray emanating from the origin makes with the positive λ-axis. Let G_1 (resp. G_2) be the infimum (resp. supremum) of $(-A_1(x_1)/B_1(x_1))$ in $a_1 \le x_1 \le b_1$ and let G_1^* (resp. G_2^*) be the infimum (resp. supremum) of $(-A_2(x_2)/B_2(x_2))$ in $a_2 \le x_2 \le b_2$. Put $\Theta_1 = \tan^{-1} G_1$, $\Theta_2 = \tan^{-1} G_2$, $\Theta_1^* = \tan^{-1} G_1^*$, and $\Theta_2^* = \tan^{-1} G_2^*$, where the principal branch of the inverse tangent is taken. We note that $-\pi/2 < \Theta_1 \le \Theta_2 < \Theta_1^* \le \Theta_2^* < \pi/2$. Now choose ε so that

$$0 < 4\varepsilon < \min\{(\Theta_1^* - \Theta_1), (\Theta_2^* - \Theta_2), (\Theta_1 + \pi/2), (-\Theta_2^* + \pi/2)\}.$$

From [3, theorem 1] we know that for each $i \ge 0$ there exists a positive number depending upon i and ε, and which we denote by $\lambda_1(i, \varepsilon)$, such that $(\lambda, \nu_i(\lambda))$ lies in the sector $\Theta_1 - \varepsilon < \Theta < \Theta_1 + \varepsilon$ for $\lambda \ge \lambda_1(i, \varepsilon)$. Using arguments similar to those used in the reference just cited, we can also show that there is a positive number $\lambda_2(i, \varepsilon)$ such that $(\lambda, \nu_i(\lambda))$ lies in the sector $\Theta_2 + \pi - \varepsilon < \Theta < \Theta_2 + \pi + \varepsilon$ for $\lambda \le -\lambda_2(i, \varepsilon)$. Hence if $\lambda(i, \varepsilon) = \max\{\lambda_1(i, \varepsilon), \lambda_2(i, \varepsilon)\}$, then $(\lambda, \nu_i(\lambda))$ lies in the sector $\Theta_1 - \varepsilon < \Theta < \Theta_1 + \varepsilon$ for $\lambda \ge \lambda(i, \varepsilon)$ and in the sector $\Theta_2 + \pi - \varepsilon < \Theta < \Theta_2 + \pi + \varepsilon$ for $\lambda \le -\lambda(i, \varepsilon)$. A similar result holds for $\rho_0(\lambda)$. Analogously, for each $i \ge 0$ there is a positive number $\lambda^*(i, \varepsilon)$ such that $(\lambda, \nu_i^*(\lambda))$ lies in the sector $\Theta_1^* - \varepsilon < \Theta < \Theta_1^* + \varepsilon$ for $\lambda \ge \lambda^*(i, \varepsilon)$ and in the sector $\Theta_2^* + \pi - \varepsilon < \Theta < \Theta_1^* + \pi + \varepsilon$ for $\lambda \le -\lambda^*(i, \varepsilon)$. A similar result holds for $\rho_0^*(\lambda)$.

Returning to theorem 1, assume now that both m_1 and m_2 are positive. Then from the above remarks we see that there is a positive number $\lambda^\dagger$ such that both $(\lambda, \nu_{m_1}(\lambda))$ and $(\lambda, \nu_{m_1-2}(\lambda))$ lie in the sector $\Theta_1 - \varepsilon < \Theta < \Theta_1 + \varepsilon$ for $\lambda \ge \lambda^\dagger$ and in the sector $\Theta_2 + \pi - \varepsilon < \Theta < \Theta_2 + \pi + \varepsilon$ for $\lambda \le -\lambda^\dagger$, while both $(\lambda, \nu_{m_2}^*(\lambda))$ and $(\lambda, \nu_{m_2-2}^*(\lambda))$ lie in the sector $\Theta_1^* - \varepsilon < \Theta < \Theta_1^* + \varepsilon$ for $\lambda \ge \lambda^\dagger$ and in the sector $\Theta_2^* + \pi - \varepsilon < \Theta < \Theta_2^* + \pi + \varepsilon$ for $\lambda \le -\lambda^\dagger$. Thus for $\lambda \ge \lambda^\dagger$,

$$\nu_{m_1-2}(\lambda) < \mu_{m_1-1}(\lambda) \le \mu_{m_1}(\lambda) < \nu_{m_1}(\lambda) < \nu_{m_2-2}^*(\lambda)$$
$$< \mu_{m_2-1}^*(\lambda) \le \mu_{m_2}^*(\lambda) < \nu_{m_2}^*(\lambda),$$

while for $\lambda \le -\lambda^\dagger$,

$$\nu_{m_2-2}^*(\lambda) < \mu_{m_2-1}^*(\lambda) \le \mu_{m_2}^*(\lambda) < \nu_{m_2}^*(\lambda) < \nu_{m_1-2}(\lambda)$$
$$< \mu_{m_1-1}(\lambda) \le \mu_{m_1}(\lambda) < \nu_{m_1}(\lambda).$$

Hence it follows that S_i intersects S_j^*, $i = m_1 - 1, m_1, j = m_2 - 1, m_2$.

By arguing as above, analogous results can also be proved for the cases $m_1 = 0$ or $m_2 = 0$.

It remains only to show that if i and j are any non-negative integers, then S_i intersects S_j^* in precisely one point. To this end let us examine the curve S_i in greater detail. Let ϕ_1 and ψ_1 be solutions of (1a) satisfying $\phi_1(a_1, \lambda, \mu)=p_1(a_1)\psi_1'(a_1, \lambda, \mu)=1$, $\phi_1'(a_1, \lambda, \mu)=\psi_1(a_1, \lambda, \mu)=0$. If we put

$$F(\lambda, \mu) = (\phi_1(b_1, \lambda, \mu)+p_1(b_1)\psi_1'(b_1, \lambda, \mu)-2)$$

and consider the curve in E_2 whose equation is $F(\lambda, \mu)=0$, then the following statements are easily deduced from the results given in [2, pp. 213–220] and [6, pp. 55–61]. Every point of S_i is a point of the curve $F=0$, and conversely, each point of $F=0$ must lie on at least one of the curves S_k, $k=0, 1, \ldots$. If $(\lambda^\dagger, \mu_i(\lambda^\dagger))$ is an ordinary point of $F=0$ (and where we note that every point of S_i is an ordinary point if $i=0$), then

$$d\mu_i(\lambda^\dagger)/d\lambda = -\left(\int_{a_1}^{b_1} A_1(x_1)H(x_1, \lambda^\dagger, \mu_i(\lambda^\dagger))\, dx_1\right) \times \left(\int_{a_1}^{b_1} B_1(x_1)H(x_1, \lambda^\dagger, \mu_i(\lambda^\dagger))\, dx_1\right)^{-1},$$

where not indicating $\lambda^\dagger$ and $\mu_i(\lambda^\dagger)$ explicitly,

$$\begin{aligned} H(x_1) = {} & \psi_1^2(x_1)p_1(b_1)\phi_1'(b_1) \\ & +\psi_1(x_1)\phi_1(x_1)(\phi_1(b_1)-p_1(b_1)\psi_1'(b_1))-\phi_1^2(x_1)\psi_1(b_1). \end{aligned}$$

Since $H(x_1, \lambda^\dagger, \mu_i(\lambda^\dagger))$ is of one sign in $[a_1, b_1]$, it therefore follows that the slope of the tangent at an ordinary point assumes a value lying in the interval $[\tan \Theta_1, \tan \Theta_2]$.

Before proceeding further let us make the following definition. We shall say that a singular point of the curve $F=0$ is isolated if there is an open disc in E_2 with this point as centre containing no other singular point of the curve; a singular point which is not isolated will be called a non-isolated singular point.

Assume now that i is a positive even integer and that there are points of S_i which are not ordinary points of $F=0$. Then the singular points of $F=0$ lying on S_i coincide with those lying on S_{i-1} and are precisely the points of intersection of S_i and S_{i-1}. Furthermore, each such singular point is also a point of intersection of the curves C_{i-1} and γ_i, where, for $k\geq 0$, γ_k denotes the curve $\{(\lambda, \rho_k(\lambda)) \mid -\infty<\lambda<\infty\}$. Since both $\nu_{i-1}(\lambda)$ and $\rho_i(\lambda)$ are analytic in $-\infty<\lambda<\infty$ [3, subsection 2.1], it therefore follows that C_{i-1} and γ_i either coincide or else have at most a finite number of points of intersection lying in any bounded subset of E_2. Hence from these results, and by arguing with the function $g(\lambda)=(\phi_1(b_1, \lambda, \nu_{i-1}(\lambda))-1)$ for the case where C_{i-1} and γ_i coincide, we arrive at the following alternative: either the singular points of $F=0$ lying on S_i (resp. S_{i-1}) are all isolated or S_i, S_{i-1}, C_{i-1}, and γ_i all coincide. It is clear that only the latter case can occur if $(A_1(x_1)/B_1(x_1))$ is constant in $a_1\leq x_1\leq b_1$, that is to say, if $\Theta_1=\Theta_2$. Now let $(\lambda^\dagger, \mu^\dagger)$ (where $\mu^\dagger=\mu_i(\lambda^\dagger)$) be a singular point of $F=0$; we note that this point is necessarily a double point since $F_{\mu\mu}(\lambda^\dagger, \mu^\dagger)<0$ (here F_μ denote the partial derivative of F with respect to μ). If $(\lambda^\dagger, \mu^\dagger)$ is non-isolated, then S_i and S_{i-1} coincide, and S_i (resp. S_{i-1}) has a continuously turning tangent whose slope at $(\lambda^\dagger, \mu^\dagger)$ is

$$(5) \qquad -\left(\int_{a_1}^{b_1} A_1(x_1)\psi_1^2(x_1, \lambda^\dagger, \mu^\dagger)\, dx_1\right)\left(\int_{a_1}^{b_1} B_1(x_1)\psi_1^2(x_1, \lambda^\dagger, \mu^\dagger)\, dx_1\right)^{-1}.$$

If $(\lambda^\dagger, \mu^\dagger)$ is an isolated singular point (and for this case we must have $\Theta_1 < \Theta_2$), then there is an open disc in E_2 with $(\lambda^\dagger, \mu^\dagger)$ as centre which contains no other singular point of $F=0$ and such that in this disc S_i meets S_{i-1} in precisely the one point $(\lambda^\dagger, \mu^\dagger)$. Thus it follows that $(\lambda^\dagger, \mu^\dagger)$ is either a node or a cusp and that through this point there passes precisely two branches of the curve $F=0$. For the case of a node the two branches have distinct tangents at $(\lambda^\dagger, \mu^\dagger)$ whose slopes assume values lying in the interval $[\tan \Theta_1, \tan \Theta_2]$; and since $\mu_i(\lambda)$ (resp. $\mu_{i-1}(\lambda)$) lies on that branch which, corresponding to λ, has largest (resp. smallest) ordinate, it therefore follows that S_i (resp. S_{i-1}) cannot possess a tangent at $(\lambda^\dagger, \mu^\dagger)$. If $(\lambda^\dagger, \mu^\dagger)$ is a cusp, then at this point the two branches of $F=0$, C_{i-1}, and γ_i all have the same tangent whose slope is given by (5).

Finally, we conclude from these results that for any non-negative integer i, $\mu_i(\lambda)$ has a continuous or sectionally continuous first derivative in any compact λ interval, and moreover, at each λ the right (resp. left) derivative of $\mu_i(\lambda)$ assumes a value lying in the interval $[\tan \Theta_1, \tan \Theta_2]$. Hence it follows that for $\lambda_1 \neq \lambda_2$,

$$(6) \qquad \tan \Theta_1 \le \{(\mu_i(\lambda_2) - \mu_i(\lambda_1))/(\lambda_2 - \lambda_1)\} \le \tan \Theta_2.$$

Analogous results hold for the curve S_j^* if we let ϕ_2 and ψ_2 be solutions of (1b) satisfying $\phi_2(a_2, \lambda, \mu) = p_2(a_2)\psi_2'(a_2, \lambda, \mu) = 1$, $\phi_2'(a_2, \lambda, \mu) = \psi_2(a_2, \lambda, \mu) = 0$, and put

$$F^*(\lambda, \mu) = (\phi_2(b_2, \lambda, \mu) + p_2(b_2)\psi_2'(b_2, \lambda, \mu) - 2).$$

The analogue of (6) then becomes

$$(7) \qquad \tan \Theta_1^* \le \{(\mu_j^*(\lambda_2) - \mu_j^*(\lambda_1))/(\lambda_2 - \lambda_1)\} \le \tan \Theta_2^*$$

for $\lambda_1 \neq \lambda_2$.

From (6) and (7) we conclude that S_i intersects S_j^* in precisely one point, and this completes the proof of our theorem. Moreover, if $(\lambda^\dagger, \mu^\dagger)$ denotes this point of intersection, then we have also shown that any branch of the curve $F=0$ passing through $(\lambda^\dagger, \mu^\dagger)$ cannot be tangent at this point to any branch of the curve $F^*=0$ passing through this point, and where of course in each case there is precisely one branch if $(\lambda^\dagger, \mu^\dagger)$ is either an ordinary point or a non-isolated singular point of the curve. In this sense we see that at the point of intersection, S_i and S_j^* cannot touch.

References

1. F. V. Atkinson, *Multiparameter Eigenvalue Problems*, Vol. I, Academic, New York, N.Y., 1972.

2. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, N.Y., 1955.

3. M. Faierman, *Asymptotic formulae for the eigenvalues of a two parameter system of ordinary differential equations of the second order*, Canad. Math. Bull. (to appear).

4. A. Howe, *Klein's oscillation theorem for period boundary conditions*, Canad. J. Math. **23** (1971), 699–703.

5. R. G. D. Richardson, *Theorems of oscillation for two linear differential equations of the second order*, Trans. Amer. Math. Soc. **13** (1912), 22–34.

6. C. A. Stewart, *Advanced Calculus*, Methuen, London, 1940.

LOYOLA COLLEGE,
MONTREAL, QUEBEC

The Author's current address is:
Department of Mathematics,
Ben Gurion University of the Negev,
Beer-Sheva, Israel.