EXTREMAL PROBLEMS FOR SCHLICHT FUNCTIONS IN THE EXTERIOR OF THE UNIT CIRCLE

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1. Introduction. Let Σ represent the class of functions

$$w = g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}, \qquad g(z) \neq 0, \tag{1}$$

which are schlicht and regular, except for the pole at infinity, in $|z| > 1$. Further let Σ^{-1} be the class of inverse functions of Σ which at $w = \infty$ have the expansion

$$z = \gamma(w) = w - \sum_{n=0}^{\infty} \beta_n w^{-n}. \tag{2}$$

In a recent paper W. C. Royster **(4)** considers the subclass of functions of Σ which map $|z| > 1$ onto a domain whose complement is starlike with respect to the origin. He found, by using the Julia variational method, the exact bounds for $|\beta_2|$ and $|\beta_3|$ and conjectured that for this subclass

$$|\beta_n| \leqslant \frac{2}{n+1}\binom{2n-1}{n-1}. \tag{3}$$

In this paper it is proved that the sharp inequality (3) holds for Σ^{-1}, and as the extremal function is essentially only the function inverse to

$$w = z + 2 + \frac{1}{z}, \tag{4}$$

the above-mentioned conjecture is answered in the affirmative.

2. The differential equation satisfied by the extremal functions. Let

$$z = \gamma(w) = w - \beta_0 - \frac{\beta_1}{w} - \frac{\beta_2}{w^2} - \ldots - \frac{\beta_n}{w^n} - \ldots \tag{5}$$

be an extremal function maximizing $|\beta_n|$ which is inverse to the schlicht function in $|z| > 1$

$$w = g(z) = z + b_0 + \frac{b_1}{z} + \ldots. \tag{6}$$

We assume that $g(z)$ satisfies the condition

$$g(z) \neq 0, \tag{7}$$

and normalize it so that $\beta_n > 0$.

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We sketch a derivation of the differential equation satisfied by $g(z)$ and refer the reader to (**1**, **5**, **6**, **7**) for the rigorous treatment.

We note first that the function $w = g(z)$ maps $|z| = 1$ onto a system of curves Γ. For suppose, on the contrary, that the complement of the image of $|z| > 1$ in the w-plane has an interior point w_0; then the function

$$w^* = w + \frac{e^{i\theta}\rho^2}{w - w_0} + \frac{e^{i\theta}\rho^2}{w_0}, \tag{8}$$

$0 \leqslant \theta < 2\pi$, $\rho > 0$ and small enough, will be schlicht and $\neq 0$ in $|z| > 1$. By taking the inverse of

$$w^* = g^*(z) = g(z) + \frac{e^{i\theta}\rho^2}{g(z) - w_0} + \frac{e^{i\theta}\rho^2}{w_0} \tag{9}$$

and denoting the coefficient of $1/w^n$ by $-\beta_n{}^*$, it can be proved that for an appropriate choice of θ we obtain

$$|\beta_n{}^*| > \beta_n \tag{10}$$

in contradiction to β_n being maximal.

Now let w_0 be a point of Γ_ρ, an arc of Γ of outer conformal radius ρ. Then there is a conformal mapping

$$\zeta(w) = w - w_0 + B_0(\rho)\rho + \frac{B_1(\rho)\rho^2}{w - w_0} + \frac{B_2(\rho)\rho^3}{(w - w_0)^2} + \ldots \tag{11}$$

which maps the exterior of Γ_ρ into the exterior of $|\zeta| > \rho$ in the ζ-plane, and it is known that for each n the coefficients $B_n(\rho)$ are bounded uniformly in ρ.

We introduce the functions

$$\eta(\zeta) = \zeta + \frac{C_1\rho^2}{\zeta} \tag{12}$$

with

$$|C_1| \leqslant 1 \tag{13}$$

which are schlicht in $|\zeta| > \rho$. Except for (13) C_1 can be chosen arbitrarily.

By composing the mappings (6), (11), and (12), we get for $|z| > 1$ the schlicht function

$$g_1(z) = \eta(\zeta(g(z))) = g(z) - w_0 + B_0\,\rho + \frac{(B_1 + C_1)\rho^2}{g(z) - w_0} + o(\rho^2). \tag{14}$$

Normalizing this function in accordance with (7), we obtain the function

$$w^* = g^*(z) = w + \frac{(B_1 + C_1)\rho^2}{w - w_0} + \frac{(B_1 + C_1)\rho^2}{w_0} + o(\rho^2) \tag{15}$$

schlicht and regular, except for the pole at infinity, in $|z| > 1$.

Let the inverse of (15) be

$$(16)\qquad z = \gamma^*(w^*) = \gamma^*\left(w + \frac{(B_1 + C_1)\rho^2}{w - w_0} + \frac{(B_1 + C_1)\rho^2}{w_0} + o(\rho^2)\right)$$
$$= \gamma^*(w) + \left[\frac{(B_1 + C_1)\rho^2}{w - w_0} + \frac{(B_1 + C_1)\rho^2}{w_0}\right]\gamma^{*\prime}(w) + o(\rho^2).$$

Replacing $\gamma^{*\prime}(w)$ by $\gamma'(w)$ in (16), the error will be $o(\rho^2)$ and so

$$(17)\quad z = \gamma^*(w^*) = \gamma^*(w) + \left[\frac{(B_1 + C_1)\rho^2}{w - w_0} + \frac{(B_1 + C_1)\rho_2}{w_0}\right]\gamma'(w) + o(\rho_2),$$

or

$$(18)\qquad \gamma^*(w) = \gamma(w) - \left[\frac{(B_1 + C_1)\rho^2}{w - w_0} + \frac{(B_1 + C_1)\rho^2}{w_0}\right]\gamma'(w) + o(\rho^2).$$

From the extremal property of β_n it follows that

$$(19)\qquad \left|\beta_n + (B_1 + C_1)\rho^2\left[\frac{(n-1)\beta_{n-1}}{w_0} + (n-2)\beta_{n-2} + (n-3)\beta_{n-3}\, w_0 + \ldots + w_0^{\,n-1}\right] + o(\rho^2)\right| \leqslant \beta_n.$$

We let $\rho \to 0$ and note that $B_1 \to -e^{2i\phi}$, where ϕ is the angle of inclination of the tangent to Γ at w_0; **(6)**. Since $\beta_n > 0$, we get from (19) in the limit, as $\rho \to 0$, the inequality

$$(20)\quad \operatorname{Re}\left\{(C_1 - e^{2i\phi})\left[\frac{(n-1)\beta_{n-1}}{w_0} + (n-2)\beta_{n-2} + (n-3)\beta_{n-3}\, w_0 + \ldots + w_0^{\,n-1}\right]\right\} \leqslant 0,$$

where C_1 is any complex number satisfying (13). Because of this property of C_1 we get from (20) the relation

$$(21)\quad \left[\frac{(n-1)\beta_{n-1}}{w_0} + (n-2)\beta_{n-2} + (n-3)\beta_{n-3}\, w_0 + \ldots + w_0^{\,n-1}\right] dw_0^{\,2} \geqslant 0,$$

where dw_0 is the differential element of Γ at the point w_0.

From (21) we can derive a differential equation satisfied by the function $g(z)$ whose inverse is the extremal function $\gamma(w)$. The expression

$$(22)\quad z^2 g'(z)^2\left[\frac{(n-1)\beta_{n-1}}{g(z)} + (n-2)\beta_{n-2} + (n-3)\beta_{n-3}\, g(z) + \ldots + g(z)^{n-1}\right] = z^{n+1} + \ldots$$

is an analytic function for $|z| > 1$, except for a pole at infinity, of order $n + 1$. According to (21), this function is real for $|z| = 1$, and hence it can be continued analytically into $|z| < 1$ by the Schwarz reflection principle. The function has a pole at the origin of order $n + 1$, and hence we get

$$(23)\quad z^2 g'(z)^2\left[\frac{(n-1)\beta_{n-1}}{g(z)} + (n-2)\beta_{n-2} + (n-3)\beta_{n-3}\, g(z) + \ldots + g(z)^{n-1}\right]$$
$$= z^{n+1} + A_1 z^n + \ldots + A_n z + A_{n+1} + \frac{\bar{A}_n}{z} + \ldots + \frac{\bar{A}_1}{z^n} + \frac{1}{z^{n+1}},$$

where $A_1, \ldots, A_{n+1}$ can be calculated explicity in terms of the coefficients $\beta_0, \beta_1, \ldots, \beta_n$.

3. The sharp bounds for β_n. From (5) we get

$$(24)\qquad 1 = g'(z)\left[1 + \frac{\beta_1}{g(z)^2} + \ldots + \frac{(n-1)\beta_{n-1}}{g(z)^n} + \ldots\right].$$

Hence

$$(25)\quad z^2 g(z)^{n-1} g'(z)^2\left[1 + \frac{\beta_1}{g(z)^2} + \ldots + \frac{(n-1)\beta_{n-1}}{g(z)^n}\right]$$
$$= z^2 g(z)^{n-1} g'(z) - \frac{n\beta_n z^2 g'(z)^2}{g(z)^2} - \frac{(n+1)\beta_{n+1} z^2 g'(z)^2}{g(z)^3} - \ldots .$$

As non-negative powers of z can occur only in the first two expressions of the right-hand side of (25), the coefficients $A_1, \ldots, A_{n+1}$ can be calculated from the expression

$$(26)\qquad z^2 g(z)^{n-1} g'(z) - \frac{n\beta_n z^2 g'(z)^2}{g(z)^2}.$$

By the residue theorem we obtain

$$(27)\qquad A_k = \frac{1}{2\pi i}\oint \frac{g(z)^{n-1} g'(z)}{z^{n-k}}\, dz = \frac{1}{2\pi i}\oint \frac{w^{n-1}}{\gamma(w)^{n-k}}\, dw, \qquad k = 1, \ldots, n,$$

where the integration is carried along a closed curve in $|z| > 1$ (and the corresponding image in the w-plane), and

$$(28)\qquad A_{n+1} = -(n+1)\beta_n.$$

From the second integral in (27) it is seen that A_k is a polynomial $P_k(\beta_0, \beta_1, \ldots, \beta_{k-1})$ in $\beta_0, \beta_1, \ldots, \beta_{k-1}$ with positive coefficients.

It is well known that under the normalization (7)

$$(29)\qquad |\beta_0| \leqslant 2.$$

Assuming that

$$(30)\qquad |\beta_\nu| \leqslant \frac{2}{\nu+1}\binom{2\nu-1}{\nu-1}, \qquad \nu = 1, \ldots, n-1,$$

we are going to prove that

$$(31)\qquad |\beta_n| \leqslant \frac{2}{n+1}\binom{2n-1}{n-1}.$$

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As the inverse of

$$w = z + 2 + \frac{1}{z} \tag{32}$$

is

$$z = w - 2 - \frac{1}{w} - \ldots - \frac{2}{n+1}\binom{2n-1}{n-1}\frac{1}{w^n} - \ldots, \tag{33}$$

we get, on the assumption (30), an upper bound for

$$A_k = P_k(\beta_0, \beta_1, \ldots, \beta_{k-1}), \qquad k = 1, 2, \ldots, n,$$

by inserting

$$g(z) = z + 2 + \frac{1}{z}$$

in (25). We obtain

$$|A_k| \leqslant \binom{2n-1}{k} - \binom{2n-1}{k-1}, \qquad k = 1, 2, \ldots, n. \tag{34}$$

Combining (28), (34), and the fact that the right-hand side of (23) must vanish for some value $z_0 = e^{i\theta_0}$ we obtain the relation

$$|\beta_n| \leqslant \frac{2}{n+1}\binom{2n-1}{n-1}. \tag{35}$$

As

$$\frac{2}{n+1}\binom{2n-1}{n-1}$$

is the absolute value of the coefficient of $1/w^n$ in (33), equality must hold in (35).

Now, equality in (35) holds only if

$$|\beta_\nu| = \frac{2}{\nu+1}\binom{2\nu-1}{\nu-1}, \qquad \nu = 0, 1, \ldots, n.$$

Hence $|\beta_0| = 2$, $|\beta_1| = 1$, but then, except for rotations,

$$w = z + 2 + \frac{1}{z}.$$

Hence, we have proved the following

THEOREM. *Let*

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots \tag{36}$$

be schlicht, different from zero, and regular, except for the pole at infinity, in $|z| > 1$. *Let*

$$z = \gamma(w) = w - \beta_0 - \frac{\beta_1}{w} - \frac{\beta_2}{w^2} - \ldots \tag{37}$$

be the inverse function of (36). *Then*

$$|\beta_n| \leqslant \frac{2}{n+1}\binom{2n-1}{n-1}, \tag{38}$$

where equality holds essentially only for the function which is inverse to

$$w = z + 2 + \frac{1}{z}. \tag{39}$$

The bound (38) is also the bound given by Löwner **(2)** for the coefficients of the inverses of schlicht functions in the unit circle.

We close by considering functions of the form

$$w = z + \frac{C_{n-1}}{z^{n-1}} + \frac{C_{2n-1}}{z^{2n-1}} + \ldots, \qquad n \geqslant 2, \tag{40}$$

schlicht and regular, except for the pole at infinity, in $|z| > 1$ (evidently $w \neq 0$ there).

The inverse functions of (40) are

$$z = w\sqrt[n]{1 - \frac{\beta_0}{w^n} - \frac{\beta_1}{w^{2n}} - \ldots} = w - \frac{q_1}{w^{n-1}} - \frac{q_2}{w^{2n-1}} - \ldots, \tag{41}$$

where

$$z = \gamma(w) = w - \frac{\beta_0}{w} - \frac{\beta_1}{w^2} - \ldots$$

is an inverse of a function of the form (5). Evidently q_ν ($\nu = 1, 2, \ldots$) are polynomials in $\beta_0, \beta_1, \ldots, \beta_{\nu-1}$ with positive coefficients. Hence we have the

COROLLARY. *The coefficients of functions inverse to functions of the form* (40) *attain their greatest absolute value for* (*essentially*) *the function inverse to*

$$w = z\left(1 + \frac{1}{z^n}\right)^{2/n}.$$

This throws some light also on the problem of finding bounds for the absolute value of the coefficients of functions of the form

$$g(z) = z + \frac{b_1}{z} + \ldots$$

dealt with in **(8)**.

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