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FUNCTIONS WITH UNBOUNDED $\bar{\partial}$-DERIVATIVE AND THEIR BOUNDARY FUNCTIONS

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Abstract

Let $F(z)$ be a continuous complex-valued function defined on the closed upper half plane $\bar{H}$ whose generalized derivative $\bar{\partial}F(z)$ is unbounded. In this paper, we discuss the relationship between the increasing order of $|\bar{\partial}F(x+iy)|$ when $y \to 0$ and that of $\lambda_F(x,t) = |(F(x+t) - 2F(x) + F(x-t))/t|$, $(x, t \in \mathbb{R})$, when $t \to 0$.

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1. Introduction

Let $F(z)$ be a continuous complex-valued function defined on $\bar{H} = \{z; \operatorname{Im} z \geq 0\}$. When $\|\bar{\partial}F(z)\|_\infty < +\infty$, it is called (in the terminology of Ahlfors [1]) a *quasiconformal deformation* on H. Denote by $Q_*(H)$ the class of quasiconformal deformations on H normalized by $\operatorname{Im} F(x) = 0$ when $x \in \mathbb{R}$ and $F(0) = F(1) = \lim_{z\to\infty} F(z)/z^2 = 0$. The importance of the class $Q_*(H)$ lies in the fact that it generates a family of quasiconformal mappings $w = f_t(z)$, $t \geq 0$, of H onto itself with $0, 1, \infty$ three fixed points, which is the solution of the differential equation

$$\frac{dw}{dt} = F(w), \qquad w \in H \tag{1.1}$$

with initial condition $w(0) = z$. In addition, the dilatations $K_t(z)$ of $f_t(z)$ are controlled by $K_t(z) \leq e^{2\|\bar{\partial}F\|_\infty t}$.

A continuous real-valued function $F(x)$ defined on $\mathbb{R}$ is said to belong to *Zygmund*

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class Λ_* [9] if

$$\lambda_F(x,t) = \left|\frac{F(x+t)-2F(x)+F(x-t)}{t}\right| \tag{1.2}$$

is bounded for all $x \in \mathbb{R}$ and $t > 0$. Define the Zygmund norm of $F(x)$ by $\|F\|_z = \sup_{x,t\in\mathbb{R}} \lambda_F(x,t)$. It was proved independently by Gardinar and Sullivan [5] and Reich and Chen [7] that the necessary and sufficient condition for a real-valued function $F(x)$ on $\mathbb{R}$ to have a quasiconformal deformation extension to $\bar H$ is that $F(x) \in \Lambda_*$. In [4] and [5], the relationship between $\|F\|_z$ and $\|\bar\partial F\|_\infty$ was discussed and estimations of them were obtained. In this paper, we will discuss the situation when $\lambda_F(x,t)$ and $\bar\partial F(z)$ are unbounded. It is based on the following consideration: when $\bar\partial F(z)$ is unbounded, equation (1.1) will not have a quasiconformal mapping solution, but it might have as solution an orientation-preserving homeomorphism of $\bar H$ onto itself, which is almost everywhere quasiconformal in the sense of Lehto [6] (see Section 4). So it is of interest to study the relationship between $\lambda_F(x,t)$, where $x, t \in \mathbb{R}$, and $\bar\partial F(z)$, where $z \in H$. In Section 2, under the assumption that $\bar\partial F(z) \in L^p(H)$ $(p > 2)$, we obtain an estimate of the increase of $\lambda_F(x,t)$ when $t \to 0$, which is sharp in the order. In Section 3, using the Beurling–Ahlfors extension, we obtain an estimate of the increase of $|\bar\partial F(x+iy)|$ when $y \to 0$, over that of $\lambda_F(x,t)$ when $t \to 0$.

2. The estimation of $\lambda_F(x,t)$

When $F(z) \in Q_*(H)$, we know from [5] that

$$F(z) = -\frac{z(z-1)}{\pi}\iint_H \left(\frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} + \frac{\overline{\mu(\zeta)}}{\bar\zeta(\bar\zeta-1)(\bar\zeta-z)}\right) d\sigma_\zeta, \tag{2.1}$$

where $\mu(z) = \bar\partial F(z) \in L^\infty(H)$. We firstly prove that when $\mu(z) \in L^p(H)$ $(p > 2)$, (2.1) still holds.

Define

$$\hat\mu(z) = \begin{cases} \mu(z), & z \in H; \\ \overline{\mu(\bar z)}, & z \in L, \end{cases} \tag{2.2}$$

where L represents the lower half plane. Then $\hat\mu(z) \in L^p(\mathbb{C})$ $(p > 2)$. By an integral operator P defined by Ahlfors [2],

$$(P\hat\mu)(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \hat\mu(z)\left(\frac{1}{\zeta-z} - \frac{1}{\zeta}\right) d\sigma_\zeta, \tag{2.3}$$

we have

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LEMMA 2.1. [2]. *For $\hat{\mu}(z) \in L^p(\mathbb{C})$ $(p > 2)$, the relation*

$$(P\hat{\mu})_{\bar{z}}(z) = \hat{\mu}(z) \tag{2.4}$$

holds in the distributional sense.

LEMMA 2.2. *Let $F(z)$ be a continuous complex-valued function on $\bar{H}$ normalized by* $\operatorname{Im} F(x) = 0$ *when $x \in \mathbb{R}$ and* $F(0) = F(1) = \lim_{z\to\infty} F(z)/z^2 = 0$. *If $\mu(z) = \bar{\partial} F(z) \in L^p(H)$ $(p > 2)$, then* (2.1) *still holds.*

PROOF. Set

$$Q(z) = (P\hat{\mu})(z) - z(P\hat{\mu})(1) = -\frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\hat{\mu}(\zeta)}{\zeta(\zeta-1)(\zeta-z)}\, d\sigma_\zeta. \tag{2.5}$$

Then $Q(0) = Q(1) = 0$, and by Lemma 2.1, $\bar{\partial} Q(z) = \hat{\mu}(z)$. Hence $F(z) = Q(z) + \phi(z)$ $(z \in H)$, where $\phi(z)$ is a holomorphic function in H. Since $\operatorname{Im} F(x) = 0$ and $\operatorname{Im} Q(x) = 0$ when $x \in \mathbb{R}$, $\phi(z)$ can be extended by the reflection principle to be a holomorphic function in $\mathbb{C}$ with normalization $\phi(0) = \phi(1) = 0$. When $z \to \infty$, we have

$$\begin{aligned}
|Q(z)| &\le \frac{|z(z-1)|}{\pi} \iint_{|\zeta|\le|z|/2} \frac{|\hat{\mu}(\zeta)|}{|\zeta(\zeta-1)(\zeta-z)|}\, d\sigma_\zeta \\
&\quad + \frac{|z(z-1)|}{\pi} \iint_{|\zeta|>|z|/2} \frac{|\hat{\mu}(\zeta)|}{|\zeta(\zeta-1)(\zeta-z)|}\, d\sigma_\zeta \\
&= I_1 + I_2.
\end{aligned}$$

Let

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad (p > 2,\ 1 < q < 2). \tag{2.6}$$

It follows from $|\zeta| \le |z|/2$ that $|\zeta - z| \ge |z|/2$. Hence

$$\begin{aligned}
I_1 &\le \frac{2|z-1|}{\pi} \iint_{|\zeta|\le|z|/2} \frac{|\hat{\mu}(\zeta)|}{|\zeta(\zeta-1)|}\, d\sigma_\zeta \\
&\le \frac{2|z-1|}{\pi} \left(\iint_{\mathbb{C}} |\hat{\mu}(\zeta)|^p\, d\sigma_\zeta \right)^{1/p} \left(\iint_{\mathbb{C}} \left| \frac{1}{\zeta(\zeta-1)} \right|^q d\sigma_\zeta \right)^{1/q} \\
&\le c_1 |z|.
\end{aligned}$$

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$$\begin{aligned} I_2 &\le \frac{2|z-1|}{\pi} \iint_{|\zeta|>|z|/2} \frac{|\hat{\mu}(\zeta)|}{|(\zeta-1)(\zeta-z)|}\, d\sigma_\zeta \\ &\le \frac{4|z-1|}{\pi} \iint_{|\zeta|>|z|/2} \frac{|\hat{\mu}(\zeta)|}{|\zeta(\zeta-z)|}\, d\sigma_\zeta \\ &\le \frac{4|z-1|}{\pi} \left(\iint_{\mathbb{C}} |\hat{\mu}(\zeta)|^p\, d\sigma_\zeta \right)^{1/p} \left(\iint_{|\zeta|>|z|/2} \left| \frac{d\sigma_\zeta}{\zeta(\zeta-z)} \right|^q \right)^{1/q}. \end{aligned}$$

Substituting $z\tau$ for ζ in the last integral, we have

$$\begin{aligned} I_2 &\le \frac{4|z-1|}{\pi} \left(\iint_{\mathbb{C}} |\hat{\mu}(\zeta)|^p\, d\sigma_\zeta \right)^{1/p} \left(\iint_{|\tau|>\frac{1}{2}} \frac{d\sigma_\tau}{|z|^{2q-2}|\tau(\tau-1)|^q} \right)^{1/q} \\ &\le \frac{4|z-1|}{\pi|z|^{2/p}} \left(\iint_{\mathbb{C}} |\hat{\mu}(\zeta)|^p\, d\sigma_\zeta \right)^{1/p} \left(\iint_{\mathbb{C}} \frac{d\sigma_\tau}{|\tau(\tau-1)|^q} \right)^{1/q} \\ &\le c_2|z|. \end{aligned}$$

Therefore we have $Q(z) = O(|z|)$ when $z \to \infty$. It follows from $\lim_{z\to\infty} F(z)/z^2 = 0$ that $\lim_{z\to\infty} \phi(z)/z^2 = 0$, which implies $\phi \equiv 0$. Hence

$$\begin{aligned} F(z) = Q(z) &= -\frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\hat{\mu}(\zeta)}{\zeta(\zeta-1)(\zeta-z)}\, d\sigma_\zeta \\ &= -\frac{z(z-1)}{\pi} \iint_H \left(\frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} + \frac{\overline{\mu(\zeta)}}{\bar{\zeta}(\bar{\zeta}-1)(\bar{\zeta}-z)} \right) d\sigma_\zeta. \end{aligned}$$

THEOREM 2.3. *Suppose $F(z)$ satisfies the condition in Lemma 2.2; then for the boundary function $F(x)$, we have*

$$\lambda_F(x,t) \le ct^{-2/p}, \tag{2.7}$$

where

$$\begin{aligned} c &= \frac{2}{\pi} \left(\iint_H |\mu(\zeta)|^p d\sigma_\zeta \right)^{1/p} \left(\iint_H \frac{d\sigma_\zeta}{|\zeta(\zeta-1)(\zeta+1)|^q} \right)^{1/q} \\ &= \frac{2}{\pi} \|\mu\|_p \left\| \frac{1}{\zeta(\zeta-1)(\zeta+1)} \right\|_q. \end{aligned}$$

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PROOF. By [5], we have

$$\begin{aligned}\lambda_F(x,t) &= \left|\frac{F(x+t)-2F(x)+F(x-t)}{t}\right| \\ &= \frac{2}{\pi}\left|\iint_H \frac{\mu(x+t\zeta)}{\zeta(\zeta-1)(\zeta+1)}\,d\sigma_\zeta\right| \\ &\le \frac{2}{\pi}\left\|\frac{1}{\zeta(\zeta-1)(\zeta+1)}\right\|_q \left(\iint_H |\mu(x+t\zeta)|^p\,d\sigma_\zeta\right)^{1/p}.\end{aligned}$$

Substituting τ for $t\zeta$ in the last integral, we obtain

$$\lambda_F(x,t) \le ct^{-2/p}.$$

The increasing order $t^{-2/p}$ when $t \to \infty$ is sharp; because we can choose

$$\mu(\zeta) = \begin{cases} \zeta^{-\alpha/p}, & \zeta \in \{|\zeta| \le 1\} \cap \{\operatorname{Im}\zeta > 0\}, \\ 0, & \zeta \in \{|\zeta| > 1\} \cap \{\operatorname{Im}\zeta > 0\}, \end{cases} \tag{2.8}$$

where $p > 2$, $\alpha < 2$. Then $\mu(\zeta) \in L^p(H)$. It is not difficult to show that

$$\lambda(0,t) = \frac{2}{\pi}\left|\iint_{|\zeta|<1,\operatorname{Im}\zeta>0} \frac{d\sigma_\zeta}{\zeta^{1+\alpha/p}(\zeta-1)(\zeta+1)}\right| t^{-\alpha/p}.$$

Since α can approach 2 from below as close as we choose, the constant $-2/p$ cannot be improved.

Directly from this theorem we can easily obtain

COROLLARY 2.4. *Let $F(z)$ be a continuous complex-valued function on $\bar{H}$ normalized by* $\operatorname{Im} F(x) = 0$ *when* $x \in \mathbb{R}$ *and* $F(0) = F(1) = \lim_{z\to\infty} F(z)/z^2 = 0$. *If $\mu(z) = \bar{\partial}F(z) \in L^\infty(H) \oplus L^p(H)$ $(p > 2)$, that is, $\mu(z) = \mu_1(z) + \mu_2(z)$, where $\mu_1 \in L^\infty(H)$, $\mu_2 \in L^p(H)$ $(p > 2)$, then*

$$F(z) = -\frac{z(z-1)}{\pi}\iint_H \left(\frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} + \frac{\overline{\mu(\zeta)}}{\bar{\zeta}(\bar{\zeta}-1)(\bar{\zeta}-z)}\right) d\sigma_\zeta \tag{2.9}$$

and

$$\lambda_F(x,t) \le c_1 + ct^{-2/p} \tag{2.10}$$

where

$$c_1 = \frac{2}{\pi}\|\mu_1\|_\infty \left\|\frac{1}{\zeta(\zeta-1)(\zeta+1)}\right\|_1, \qquad c = \frac{2}{\pi}\|\mu_2\|_p \left\|\frac{1}{\zeta(\zeta-1)(\zeta+1)}\right\|_q.$$

3. The estimation of $\bar\partial F(x+iy)$

Let $F(x)$ be a continuous real-valued function on $\mathbb{R}$ with $\lim_{x\to\infty} F(x)/x^2 = 0$. Suppose $\lambda_F(x,t)$ is unbounded; then for any extension of $F(x)$ to H, its $\bar\partial$-derivative must be unbounded. Let $F(z) = u(x,y) + iv(x,y)$, where $z \in H$, be the Beurling–Ahlfors extension of $F(x)$:

$$\tag{3.1} \begin{aligned} u(x,y) &= \frac{1}{2y}\int_{x-y}^{x+y} h(t)\,dt, \\ v(x,y) &= \frac{1}{y}\left(\int_x^{x+y} h(t)dt - \int_{x-y}^{x} h(t)\,dt\right). \end{aligned}$$

It is obvious that $F(z) \in C^1(H)$, and

(3.2)

$$\begin{aligned} u_x &= \frac{1}{2y}(F(x+y) - F(x-y)), \\ u_y &= -\frac{1}{2y^2}\int_{x-y}^{x+y} F(t)dt + \frac{1}{2y}(F(x+y) + F(x-y)), \\ v_x &= \frac{1}{y}(F(x-y) - 2F(x) + F(x+y)), \\ v_y &= -\frac{1}{y^2}\left(\int_x^{x+y} F(t)\,dt - \int_{x-y}^{x} F(t)\,dt\right) + \frac{1}{y}(F(x+y) - F(x-y)). \end{aligned}$$

Now we have

THEOREM 3.1. *Suppose there exists $\delta > 0$, such that*

$$\tag{3.3} \lambda_F(x,t) \le \lambda(t)$$

holds for all $x \in \mathbb{R}$ and $0 < t < \delta$, where $t\lambda(t) \in L^1(0,\delta)$. Then for the $\bar\partial$-derivative of the Beurling–Ahlfors extension of $F(x)$,

$$\tag{3.4} |\bar\partial F(x_0 + iy_0)| \le \frac{1}{2}\lambda(y_0) + \sigma(y_0)$$

holds for all $x_0 \in \mathbb{R}$ and $0 < y_0 < \delta$, where

$$\tag{3.5} \sigma(y_0) = 2\int_0^{1/2} t\lambda(2y_0 t)\,dt = \frac{1}{2y_0^2}\int_0^{y_0} t\lambda(t)\,dt.$$

PROOF. Let $F^*(x) = F(2y_0x + x_0 - y_0)/2y_0 + cx + d$, and denote the Beurling–Ahlfors extension of $F^*(x)$ by $F^*(z)$. Then

(3.6) $$\lambda_{F^*(x,t)} = \lambda(2y_0x + x_0 - y_0, 2y_0t) \le \lambda(2y_0t), \qquad 0 < t < \delta/2y_0,$$

and

(3.7) $$F^*(z) = \frac{1}{2y_0}F(2y_0z + x_0 - y_0) + cz + d.$$

From the fact that $\bar{\partial}F^*(z) = \bar{\partial}F(2y_0z + x_0 - y_0)$, we have

(3.8) $$\bar{\partial}F^*\left(\frac{1+i}{2}\right) = \bar{\partial}F(x_0 + iy_0).$$

So to estimate $\bar{\partial}F(x_0 + iy_0)$, it suffices to estimate $|\bar{\partial}F(z)|$ at only one point $z = (1+i)/2$ with the condition that

(3.9) $$\lambda_F(x,t) \le \lambda(2y_0t)$$

holds for all $x \in \mathbb{R}$ and $0 < t < \delta/2y_0$.

Since the constants c and d can be chosen arbitrarily, we can also assume $F(0) = F(1) = \lim_{x\to\infty} F(x)/x^2 = 0$. Then it follows from (3.2) that

(3.10) $$|\bar{\partial}F((1+i)/2)|^2 = H(X,Y,Z) = 4(X-Y)^2 + (X+Y+2Z)^2,$$

where

(3.11) $$\begin{cases} X = \displaystyle\int_0^{1/2} F(t)\,dt, \\ Y = \displaystyle\int_{1/2}^{1} F(t)\,dt, \\ Z = F(1/2). \end{cases}$$

We now need the following lemma.

LEMMA 3.2. *For the expressions in* (3.11), *we have*

(3.12) $$-\sigma(y_0) \le Y - 3X \le \sigma(y_0),$$

(3.13) $$-\sigma(y_0) \le X - 3Y \le \sigma(y_0),$$

(3.14) $$-\frac{1}{4}\lambda(y_0) \le Z \le \frac{1}{4}\lambda(y_0).$$

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PROOF. Let $x \in (0, 1/2)$, then by (3.9),

$$-x\lambda(2y_0x) \le F(2x) - 2F(x) + F(0) \le x\lambda(2y_0x).$$

Integrating the above inequality with respect to x from 0 to $1/2$, we obtain (3.12).

Let $x \in (1/2, 1)$, then by (3.9)

$$-(1-x)\lambda(2y_0(1-x)) \le F(1) - 2F(x) + F(2x-1) \le (1-x)\lambda(2y_0(1-x)).$$

Integrating the above inequality with respect to x from $1/2$ to 1, we obtain (3.13).

The inequality (3.14) follows directly from the inequality $\lambda_F(1/2, 1/2) \le \lambda(y_0)$.

Now we continue the proof of Theorem 3.1. By Lemma 3.2, we know that the point (X, Y, Z), where X, Y, Z are defined by (3.11), lies in the closed parallelepiped bounded by planes $X - 3Y = \pm\sigma(y_0)$, $Y - 3X = \pm\sigma(y_0)$ and $Z = \pm\lambda(y_0)/4$. It is easy to see that $H(X, Y, Z)$ is convex, and hence reaches its maximum at one of the eight vertexes of the parallelepiped.

After some computation, we obtain

$$H(X, Y, Z) \le H(\sigma(y_0)/2, \sigma(y_0)/2, \lambda(y_0)/4) = (\sigma(y_0) + \lambda(y_0)/2)^2.$$

Hence

$$|\bar{\partial}F(x_0 + iy_0)| \le \lambda(y_0)/2 + \sigma(y_0),$$

which completes the proof of Theorem 3.1.

The following corollaries follow directly from the above theorem.

COROLLARY 3.3. *Let $F(x)$ be a continuous real-valued function on $\mathbb{R}$ with $\lim_{x\to\infty} F(x)/x^2 = 0$. If*

$$\lambda_F(x, t) \le M|\log t| \tag{3.15}$$

holds for all $x \in \mathbb{R}$ and $0 < t < \delta$, then for the $\bar{\partial}$-derivative of the Beurling–Ahlfors extension of $F(x)$,

$$|\bar{\partial}F(x + iy)| \le \tfrac{3}{4}M|\log y| + c \tag{3.16}$$

holds for all $x \in \mathbb{R}$ and $0 < t < \delta$, where $c = M(1 + 4\log 2)/8$.

COROLLARY 3.4. *Let $F(x)$ be a continuous real-valued function on $\mathbb{R}$ with $\lim_{x\to\infty} F(x)/x^2 = 0$. If*

$$\lambda_F(x,t) \le M/t^\alpha \qquad (\alpha < 2) \tag{3.17}$$

holds for all $x \in \mathbb{R}$ and $0 < t < \delta$, then for the $\bar\partial$-derivative of the Beurling–Ahlfors extension of $F(x)$,

$$|\bar\partial F(x+iy)| \le c/y^\alpha \tag{3.18}$$

holds for all $x \in \mathbb{R}$ and $0 < y < \delta$, where $c = (3-\alpha)M/(2(2-\alpha))$.

4. An example

In Section 1, we stated that if $\bar\partial F(z)$ is unbounded in H, where $F(z)$ is continuous on $\bar H$ and is normalized by $\operatorname{Im} F(x) = 0$ $(x \in \mathbb{R})$ and $\lim_{z\to\infty} F(z)/z^2 = 0$, equation (1.1) might have as solutions a family of almost everywhere quasiconformal homeomorphisms of H onto itself. The following is an example:

$$F(z) = z(\log|z|)^{2/3}, \qquad z \in \bar H. \tag{4.1}$$

Then

$$\bar\partial F(z) = \frac{1}{3}\frac{z}{\bar z}(\log|z|)^{-1/3} \in L^\infty(H) \oplus L^p(H), \tag{4.2}$$

where $2 < p < 3$.

From $dw/dt = F(w)$, $w(0) = z$, we have

$$\frac{d\log|w| + i\,d\arg w}{(\log|w|)^{2/3}} = dt, \qquad w(0) = z, \tag{4.3}$$

which is equivalent to the system of equations

$$\begin{cases} \arg w & = \arg z \\ \displaystyle\int_0^t \frac{d\log|w|}{(\log|w|)^{2/3}} & = \displaystyle\int_0^t dt. \end{cases} \tag{4.4}$$

The solution is

$$w = f_t(z) = \frac{z}{|z|}e^{\phi(z,t)}, \qquad t \ge 0 \tag{4.5}$$

where $\phi(z,t) = [(\log|z|)^{1/3} + t/3]^3$.

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It is obvious that for any $t > 0$, $w = f_t(z)$ is a homeomorphism of H onto itself. After some computation, we have

$$\left|\frac{\bar{\partial} f_t(z)}{\partial f_t(z)}\right| = \left|\frac{[(\log|z|)^{1/3} + \frac{1}{3}t]^2/(\log|z|)^{2/3} - 1}{[(\log|z|)^{1/3} + \frac{1}{3}t]^2/(\log|z|)^{2/3} + 1}\right| < 1 \tag{4.6}$$

and $|\bar{\partial} f_t(z)/\partial f_t(z)| \to 1$ only when $|z| \to 1$. Hence $\{w = f_t(z),\ t > 0\}$ is a family of almost everywhere quasiconformal homeomorphisms of H onto itself. But there remains an open problem: under what general conditions on $\bar{\partial} F \in L^\infty(H) \oplus L^p(H)$ $(p > 2)$, does equation (1.1) have solutions which are almost everywhere quasiconformal homeomorphisms.

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