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ON THE NUMBER OF PROLONGATIONS OF A FINITE RANK VALUATION

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A (non-archimedean) valuation v on a field K is said to be *henselian* if it has a unique prolongation to a valuation on K_a, the algebraic closure of K. A *henselization* (K^h, v^h) of a valuated field (K, v) is a smallest separable extension of K containing a henselian prolongation v^h of v. (K^h, v^h) is unique up to K-isomorphism, and $(K^h, v^h) = (K, v)$ if and only if v is henselian. In this paper we confine ourselves to valuations of finite rank.

If v is a non-henselian rank one valuation on K, and if $[K_s:K] = \infty$, K_s being the separable closure of K, then it is known that v has infinitely many prolongations to K_a [**1**, (27.11)]. We shall see that this is no longer true if the rank of v is greater than one. Endler has shown that if K is any field such that $1 < [K_s:K] < \infty$, then K is real closed, that is $K_a = K_s = K(\sqrt{-1})$ (see [**2**]). With the aid of this, he proves that if (K, v) is any rank one non-henselian valuated field with henselization (K^h, v^h) such that $[K^h:K] < \infty$, then again K is real closed and so $K_a = K^h = K(\sqrt{-1})$ (see [**2**]). Hence if on a field K, a rank one valuation has finitely many prolongations to K_a, it must have exactly one or two. The question naturally arises as to how many such prolongations a valuation v of rank $d > 1$ can have. We conclude that if v has a finite number it still has exactly one or two by showing that if (K^h, v^h) is a finite proper extension of (K, v), we again have $K^h = K(\sqrt{-1})$, even though it is not necessary that K be real closed.

Our notation will follow that of [**5**]. Let v be a valuation on a field K with residue class field K/v. If A is the valuation ring of v with maximal ideal M, and if P is a prime ideal of A, then P determines two valuations:

(i) v_P on K with valuation ring A_P and residue class field $A_P/PA_P = K/v_P$, and

(ii) v/P on K/v_P with valuation ring A/P and residue class field $A/M = K/v$.

On the other hand, given an arbitrary valuation w on a field K and a valuation u on the residue class field K/w, there is a unique valuation v on K with a valuation ring A and prime ideal $P \subset A$ such that $v_P = w$ and $v/P = u$. We say v is composed of v_P and v/P.

If $\bar{K}$ is an algebraic extension of K and P a prime ideal of the valuation ring A of v, we denote by g_P the number of distinct prolongations of v_P to $\bar{K}$. Letting $\bar{v}_1, \ldots, \bar{v}_k$ (where $k = g_P$) be these valuations, we let $(g/P)_{\bar{v}_i}$ be the number of prolongations of v/P on K/v_P to $\bar{K}/\bar{v}_i$.

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We recall the following facts:

(a) In the notation above, if g is the number of prolongations of v to $\bar{K}$, then

$$g = g_M = \sum_{i=1}^{g_p} (g/P)_{\bar{v}_i},$$

for any prime ideal P of A (see [**5**, p. 174]).

(b) If (K^h, v^h) is the henselization of (K, v), P^h and $P^h \cap A = P$ corresponding prime ideals of the valuation rings of v^h and v respectively, then K^h contains the henselization of K with respect to v_P, and $K^h/(v^h)_{P^h}$ contains the henselization of K/v_P with respect to v/P (see [**5**, p. 210]).

(c) Let (K, v) be a valuated field and P a prime ideal of the valuation ring of v. If (K, v_P) and $(K/v_P, v/P)$ are henselian, then (K, v) is henselian (see [**5**, p. 211]).

(d) If K is complete with respect to a rank one valuation v, then (K, v) is henselian (see [**5**, p. 198]).

(e) If K has two distinct rank one henselian valuations, it is separably closed (see [**3**]).

We begin with a simple example. Let $\mathbf{R}$ be the field of real algebraic numbers and $\mathbf{Z}$ the additive group of integers. Consider the field $K = \mathbf{R}((x))$ of all formal power series with coefficients in $\mathbf{R}$ and exponents in $\mathbf{Z}$. K consists of all expressions of the form $s = \sum_{i=n}^{\infty} a_i x^i$, where $n \in \mathbf{Z}$, $a_n \neq 0$, and $a_i \in \mathbf{R}$. If $s \neq 0$, n is called the order of s and denoted $\varphi(s)$. This φ (with $\varphi(0) = \infty$) is called the order valuation or natural valuation on K with valuation ring $\mathbf{R}[[x]]$, and residue class field $\mathbf{R}$. Moreover, K is complete and hence henselian with respect to φ (see [**5**, p. 103]). Now let v be a rank one valuation on $\mathbf{R}$. Then v is non-henselian (see [**4**]). Let w be the rank two valuation on K composed of φ and v. Thus if P is the minimal prime ideal of the valuation ring of w, $w_P = \varphi$ and $w/P = v$. By (a) then, since w_P is henselian and w/P has two prolongations to the algebraic closure $\mathbf{R}(\sqrt{-1})$ of $\mathbf{R}$, w has exactly two prolongations to K_a, while $[K_s:K] = \infty$. In a similar manner, using the real closure $\mathbf{R}((x))_{rc}$ of $\mathbf{R}((x))$ and the unique (as we shall see in Theorem 2) prolongation of w to $\mathbf{R}((x))_{rc}$, we can construct a field K and a rank three valuation on K with two prolongations to K_a. In fact, given any $d > 1$, we can find a rank d valuated field (K, v) such that v has exactly two prolongations to K_a while $[K_s:K] = \infty$. Moreover, (K^h, v^h) is a finite extension of (K, v). For, let v_1 and v_2 be the two prolongations of v to K_a, and $b \in K_a$ be such that $v_1(b) \neq v_2(b)$. Then $K(b)$ is henselian with respect to both $v_1|_{K(b)}$ and $v_2|_{K(b)}$, while $[K(b):K] < \infty$.

THEOREM 1. *Let (K, v) be a non-henselian valuated field and (K^h, v^h) its henselization. If $[K^h:K] < \infty$, then $K^h = K(\sqrt{-1})$, and v has exactly two prolongations to K_a.*

Proof. Since this is known for rank one valuations, we use induction on the rank of v. Let the rank of v be $d > 1$ and assume the theorem holds for any valuated field (F, w) where w is a non-henselian valuation of rank $(d - 1)$. If $[K_s:K] < \infty$, then $K_a = K(\sqrt{-1})$, so $K^h = K(\sqrt{-1})$, and we are through. Assume, then, that $[K_s:K] = \infty$. Let P be the minimal prime ideal of the valuation ring of v. Then, since K^h contains the henselization of K with respect to the rank one valuation v_P by (b), and since $[K^h:K] < \infty$, v_P must be henselian. Hence by (c), v/P is a non-henselian rank $(d - 1)$ valuation on K/v_P. Since $[K^h/(v^h)_{P^h}:K/v_P] \leqq [K^h:K] < \infty$, and since $K^h/(v^h)_{P^h}$ contains the henselization of K/v_P with respect to v/P (again by (b)), by induction, v/P has exactly two prolongations to $(K/v_P)_a$, and $(K/v_P)^h = K(\sqrt{-1})/\bar{v}_P$, $\bar{v}_P$ being the unique prolongation of v_P to $K(\sqrt{-1})$. By (a) then, v has exactly two prolongations to K_a. Similarly, since v/P has two prolongations to

$$K(\sqrt{-1})/\bar{v}_P = (K/v_P)^h,$$

so does v to $K(\sqrt{-1})$. Thus $K(\sqrt{-1})$ is henselian with respect to a prolongation of v (in fact, two), so that $K(\sqrt{-1}) = K^h$.

COROLLARY. *Let v be a valuation on K. The number of prolongations of v to K_a is either one, two (when $K^h = K(\sqrt{-1})$), or is infinite (when $[K^h:K] = \infty$).*

Proof. We need only show that $[K^h:K] = \infty$ implies v has an infinite number of prolongations to K^h. Although this is rather well known, we prove it. Suppose that $v^h, v_1, \ldots, v_n$ are all the prolongations of v to K^h. For each $i = 1, \ldots, n$, choose $a_i \in K^h$ such that $v_i(a_i) \neq v^h(a_i)$, then let $F = K(a_1, \ldots, a_n)$. Thus $v^h|_F$ has a unique prolongation to K^h and so to K_a. Thus F is henselian with respect to $v^h|_F$, a prolongation of v, and $[F:K] < \infty$, which is impossible since we must have $F = K^h$.

The example preceding Theorem 1 shows that if v is a rank $d > 1$ valuation on K, it is possible for K^h to be a finite extension of K while K is not real closed, something that cannot happen for rank one valuations. However, in our example K is still a real field (i.e., -1 is not a sum of squares in K). Indeed, this is always true.

THEOREM 2. *If (K, v) is a valuated field such that $K^h = K(\sqrt{-1})$, then K is a real field and v has a unique prolongation to a valuation on K_{rc}.*

Proof. If the rank of v is one, or if $[K_s:K] < \infty$, we are through, for then K is real closed. Assume, then, that v is of rank $d > 1$, $[K_s:K] = \infty$, and that the theorem holds for valuations of rank $(d - 1)$. Then using the notation exactly as in Theorem 1, v/P is a non-henselian rank $(d - 1)$ valuation on K/v_P, and $(K/v_P)^h = K(\sqrt{-1})/\bar{v}_P$. So by induction, K/v_P is a real field. If $-1 = \sum_{i=1}^n (a_i)^2$, where $a_i \in K$ and $v_P(a_1)$ is minimal among the $v_P(a_i)$, then by multiplying through by $(a_1)^{-2}$ if necessary (that is, if $v_P(a_1) < 0$), $-1 = \sum_{i=1}^n (c_i)^2$, where $c_i \in A_P$, the valuation ring of v_P. Therefore,

$-1 + PA_P$ can be written as the sum of squares in $K/v_P = A_P/PA_P$, which is impossible since K/v_P is real. Thus K is a real field. Since $K^h = K(\sqrt{-1}) \not\subseteq K_{rc}$, v has a unique prolongation to a valuation on K_{rc}.

If v is a rank one valuation on K with exactly two prolongations to K_a, it is known that K/v is algebraically closed (see [**6**]). Actually this holds for a valuation of any rank d.

THEOREM 3. *If* $K^h = K(\sqrt{-1})$, K/v *is algebraically closed.*

Proof. Since half of our work is done, we again use induction on the rank of v. Suppose that v is of rank $d > 1$. If $[K_s:K] < \infty$, then $K_a = K^h = K(\sqrt{-1})$. Hence $(K/v)_a = K^h/v^h = K/v$. Thus suppose that $[K_s:K] = \infty$. Let P be the minimal prime ideal of the valuation ring of v. Then, as before, v_P is henselian and v/P is not. By induction, since v/P has two prolongations to $(K/v_P)_a$, and since the rank of v/P is $d - 1$, the residue class field K/v is algebraically closed.

Finally, while it is not possible that a field K possess three rank one valuations having one, two, and an infinite number of prolongations respectively to K_a, this can happen, for example, for three rank two valuations. Let K be a real closed field with a henselian rank one valuation v_1 (e.g. if K is the real closure of $\mathbf{R}((x))$ and v_1 the unique prolongation of the order valuation on $\mathbf{R}((x))$ to K). Let v_2 be a non-henselian rank one valuation on K and let w_1, w_2 be the rank two valuations on $F = K((y))$ composed of the order valuation φ on F and v_1, v_2. Now let w be a rank two prolongation of a suitable rank two valuation on $K(y)$ (namely, a valuation of $K(y)$ whose valuation ring is not contained in the valuation ring of the y-adic valuation). Then if P is the minimal prime ideal of the valuation ring of w, w_P is not henselian by (e), since $w_P \neq \varphi$ and $F_s \neq F$. Since $[F_s:F] = \infty$, w_P has an infinite number of prolongations to F_a. Thus w has an infinite number of prolongations, while w_1 and w_2 have one and two respectively.

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