Note on the Peano-Baker Method of solving Linear Differential Equations.

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In *Mathematische Annalen*, Vol. 32 (1888) Peano discusses the solution of a system of homogeneous linear differential equations

$$\frac{dx_1}{dt} = r_{11} x_1 + \ldots + r_{1n} x_n$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{dx_n}{dt} = r_{n1} x_1 + \ldots + r_{nn} x_n,$$

where r_{ij} denotes a real function of the variable t, and shows how, by a series of repeated substitutions, this system of equations may be replaced by the equivalent equation

$$\frac{dX}{dt} = RX,$$

where X denotes the complex $[x_1, x_2, \ldots x_n]$ and R the matrix

$$\left\{ \begin{matrix} r_{11} & \ldots & r_{1n} \\ \vdots & & \vdots \\ r_{n1} & \ldots & r_{nn} \end{matrix} \right\},$$

of which equation the solution X can be represented as a sum of integrals.

Later, Baker in *Proc. Lond. Math. Soc.*, Vol. 34 (1902) deals with further applications of matrix notation to integration problems, and shows in particular that the solution of the linear differential equation of the second order

$$\frac{d^2x}{dt^2} = \omega x,$$

where ω is a function of t, can be expressed in the form

$$x = \Delta_1 x_0 + \Delta_2 x_0'$$

where x_0 and x_0' are the values of x and $\frac{dx}{dt}$ when $t=0$, and

$$\Delta_1 = 1 + Q^2\omega + Q^2\omega\, Q^2\omega + Q^2\omega\, Q^2\omega\, Q^2\omega + \ldots$$

$$\Delta_2 = t + Q^2\omega t + Q^2\omega\, Q^2\omega t + Q^2\omega\, Q^2\omega\, Q^2\omega t + \ldots,$$

Q denoting the operation of integrating a matrix from 0 to t, so that

$$Q^2\omega = \int_0^t dt \int_0^t \omega\, dt.$$

The especial point of interest of the method from the theoretical view is that the solutions furnished by it are valid over much larger areas of the plane than are the solutions expressed in power-series. An obvious question that arises is— Is the method equally important when viewed from the practical standpoint, that is to say, are the series obtained well adapted for calculation, and do they readily furnish values of the unknown function to a high degree of accuracy? As I had already calculated various tables of values of the confluent hypergeometric function (Whittaker & Watson's *Modern Analysis*, p. 331), I tested the matter in connection with these.

If in the linear differential equation of these, viz.

$$z^2 \frac{d^2 W_{k,m}(z)}{dz^2} + \left[-\tfrac{1}{4} z^2 + kz + (\tfrac{1}{4} - m^2)\right] W_{k,m}(z) = 0$$

we put $z = e^{\theta}$ and $W_{k,m}(z) = v\, e^{\frac{1}{2}\theta}$, this equation reduces to

$$\frac{d^2 v}{d\theta^2} = (\tfrac{1}{4} e^{2\theta} - k e^{\theta} + m^2)\, v.$$

Hence in Baker's notation the solution of this equation may be written

$$v = \Delta_1 v_0 + \Delta_2 v_0',$$

where v_0 and v_0' are the values of v and $\frac{dv}{d\theta}$ respectively when $\theta = 0$, *i.e.*, $z=1$, when we revert to z as independent variable. Hence, since $v = z^{-\frac{1}{2}} W_{k,m}(z)$ we have

$$v_0 = \left[W_{k,m}(z) \right]_{z=1} \text{ and } v_0' = \left[\frac{d\, W_{k,m}(z)}{dz} - \tfrac{1}{2} W_{k,m}(z) \right]_{z=1}.$$

For purposes of calculation this value of v_0' may be more conveniently determined if we make use of the recurrence-formula

$$z\frac{dW_{k,m}(z)}{dz}-(\tfrac{1}{2}z-k)\,W_{k,m}(z)+W_{k+1,m}(z)=0,$$

so that we have $v_0'=\left[-W_{k+1,m}(z)-kW_{k,m}(z)\right]_{z=1}$.

Accordingly, the solution of our original equation can be written in the form

$$W_{k,m}(z)=z^{\frac{1}{2}}\left[\Delta_1 W_1-\Delta_2 W_2\right]$$

where $W_1=\left[W_{k,m}(z)\right]_{z=1}$ and $W_2=\left[W_{k+1,m}(z)+kW_{k,m}(z)\right]_{z=1}$.

When we evaluate Δ_1 and Δ_2 by performing the successive integrations and replace e^{θ} by z, we find that

$$\begin{aligned}\Delta_1=1&+\left\{\tfrac{1}{16}z^2-kz+\tfrac{1}{2}m^2L^2+L\left(k-\tfrac{1}{8}\right)+k-\tfrac{1}{16}\right\}\\&+\left\{\tfrac{1}{1024}z^4-\tfrac{5}{144}kz^3+z^2\left[\tfrac{m^2}{32}L^2-L\left(\tfrac{m^2}{16}-\tfrac{k}{16}+\tfrac{1}{128}\right)+\tfrac{m^2}{16}+\tfrac{k^2}{4}+\tfrac{1}{256}\right]\right.\\&\quad-z\left[\tfrac{km^2}{2}L^2-L\left(2km^2-k^2+\tfrac{k}{8}\right)+4km^2-k^2+\tfrac{3k}{16}\right]\\&\quad+\tfrac{m^4}{24}L^4+\tfrac{m^2}{6}\left(k-\tfrac{1}{8}\right)L^3+\tfrac{m^2}{2}\left(k-\tfrac{1}{16}\right)L^2-\left(\tfrac{1}{256}-\tfrac{5k}{48}+\tfrac{k^2}{2}+\tfrac{m^2}{16}-2km^2\right)L\\&\quad\left.-\left(\tfrac{5}{1024}-\tfrac{2k}{9}+\tfrac{5k^2}{4}+\tfrac{m^2}{16}-4km^2\right)\right\}+\ldots\ldots\end{aligned}$$

and

$$\begin{aligned}\Delta_2=L&+\left\{\tfrac{1}{16}z^2(L-1)-kz(L-2)+\tfrac{m^2}{6}L^3-\left(k-\tfrac{1}{16}\right)L-\left(2k-\tfrac{1}{16}\right)\right\}\\&+\left\{\tfrac{1}{64}z^4\left(\tfrac{1}{16}L-\tfrac{3}{32}\right)-\tfrac{k}{16}z^3\left(\tfrac{5}{9}L-\tfrac{37}{27}\right)\right.\\&\quad+z^2\left[\tfrac{m^2}{96}L^3-\tfrac{m^2}{32}L^2+\left(\tfrac{1}{256}-\tfrac{k}{16}+\tfrac{k^2}{4}+\tfrac{m^2}{16}\right)L-\left(\tfrac{k}{16}+\tfrac{3k^2}{4}+\tfrac{m^2}{16}\right)\right]\\&\quad-kz\left[\tfrac{m^2}{6}L^3-m^2L^2+\left(\tfrac{1}{16}-k+4m^2\right)L-\left(\tfrac{51}{16}+8m^2\right)\right]\\&\quad+\tfrac{m^4}{120}L^5-\tfrac{m^2}{6}\left(k-\tfrac{1}{16}\right)L^3-m^2\left(k-\tfrac{1}{32}\right)L^2\\&\quad+\left(\tfrac{1}{1204}-\tfrac{5}{144}k+\tfrac{k^2}{4}+\tfrac{m^2}{16}-4km^2\right)L\\&\quad\left.+\tfrac{3}{2048}-\tfrac{37}{432}k+\tfrac{3k^2}{4}+\tfrac{m^2}{16}-8km^2\right\}+\ldots\ldots\end{aligned}$$

where $L=\log_e z$.

The result was that when k and m were both small, *e.g.* $k=0{\cdot}1$, $m=0{\cdot}2$, the terms given above for Δ_1 and Δ_2 were

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sufficient to ensure accuracy to four significant figures for values of z ranging from 0·1 to 2 5. When $z=3$, the accuracy was to three figures, and for $z=5$ to one figure. If either of the parameters was increased the accuracy was less. Thus, when $k=1{\cdot}1$, $m=0{\cdot}2$ or $k=0\ 1$, $m=1{\cdot}2$, the accuracy obtainable was to two figures for values of z not exceeding 3. Beyond that, the number of terms used was insufficient to give values of the function. As one might naturally expect, when z was equal to unity, $\Delta_1=1$ and $\Delta_2=0$.