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NORM CLOSED INVARIANT SUBSPACES IN L^∞ AND H^∞

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Abstract. We characterize norm closed subspaces B of $L^\infty(\partial D)$ such that $C(\partial D)B \subset B$ and maximal ones in the family of proper closed subspaces B of $L^\infty(\partial D)$ such that $A(D)B \subset B$, where $A(D)$ is the disk algebra. Analogously, we characterize closed subspaces of H^∞ that are simultaneously invariant under S and S^*, the forward and the backward shift operators, and maximal invariant subspaces of H^∞.

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1. Introduction and preliminaries. Let L^∞ be the Banach space of essentially bounded functions on the unit circle ∂D, and H^∞ be the norm closed subspace of functions that admit an analytic extension to D. Let z be the identity function on ∂D. A norm closed subspace B of L^∞ is called *invariant* if $zB \subset B$ and doubly invariant if $zB \subset B$ and $\overline{z}B \subset B$. Weak-star closed invariant subspaces of L^∞ were characterized long ago in Beurling's theorem. See [**1**, pp. 131–133]. They have one of the following forms.

(a) $B = \chi_E L^\infty$, where $E \subset \partial D$ is a measurable set and χ_E denotes its characteristic function. This happens when B is doubly invariant.

(b) $B = uH^\infty$, where $|u(z)| = 1$ for almost every $z \in \partial D$.

It follows immediately that every weak-star closed invariant subspace of H^∞ has the form (b) with u an inner function. The structure of inner functions is known completely. See [**2**]. By Beurling's characterization, one can write down all weak-star closed invariant subspaces of H^∞ in an explicit way.

Despite these results, very little is known about closed invariant subspaces of L^∞ and H^∞ with respect to the norm topology. In this paper, we consider only the norm topology. In the family of proper invariant subspaces of L^∞ and H^∞, a maximal one is called a maximal invariant subspace of L^∞ and H^∞, respectively.

First, we give a complete characterization of doubly invariant subspaces of L^∞. From this, we are able to determine maximal invariant subspaces of L^∞. Let $Sf = zf, f \in H^\infty$ and S^* be the operator on H^∞ defined by $(S^*f)(z) = \overline{z}(f(z) - f(0))$. We characterize the closed subspaces of H^∞ that are simultaneously invariant under S and S^*. Also, we describe the maximal invariant subspaces of H^∞.

Let A be a uniform algebra. We denote by $M(A)$ the maximal ideal space of A. Now $M(A)$ consists of the linear functionals of A that are multiplicative and nonzero. Also $M(A)$ is a compact Hausdorff space with the weak-star topology induced by the dual space of A. The Gelfand transform, defined by $\hat{a}(\varphi) = \varphi(a)$, for $a \in A$ and $\varphi \in M(A)$, establishes an isometric isomorphism between A and a closed subalgebra of $C(M(A))$, the space of continuous functions on $M(A)$.

When A is also a C^* algebra, the Gelfand transform is a $*$-isomorphism from A onto $C(M(A))$. This allows us to identify L^∞ with $C(M(L^\infty))$, from which the dual space $(L^\infty)^*$ is identified with the space $\mathfrak{M}(M(L^\infty))$ of finite regular Borel measures on $M(L^\infty)$ with the total variation norm. Specifically, every element of $(L^\infty)^*$ has the form

$$L_\mu(f) = \int_{M(L^\infty)} \hat{f}\, d\mu \quad (f \in L^\infty),$$

where $\mu \in \mathfrak{M}(M(L^\infty))$. Also, for every such μ, the formula above defines a linear functional of L^∞ with $\|L_\mu\| = \|\mu\|$. Put $\ker L_\mu = \{f \in L^\infty : L_\mu(f) = 0\}$. When $\int_{M(L^\infty)} \hat{f}\, d\mu = 0$ holds, we write as $\hat{f} \perp \mu$. For a subspace B of L^∞, we write $B \perp \mu$ if $\hat{f} \perp \mu$ for every $f \in B$. We denote by $\operatorname{supp} \mu$ the closed support set of μ.

The fiber over $\lambda \in \partial D$ in $M(L^\infty)$ is defined by $M_\lambda = \{\varphi \in M(L^\infty) : \hat{z}(\varphi) = \lambda\}$. Since $|\hat{z}| \equiv 1$, $M(L^\infty) = \bigcup_{\lambda \in \partial D} M_\lambda$. Measures that are supported on a single fiber will be of particular interest in our discussion. We define

$$\mathfrak{F} = \{\mu \in \mathfrak{M}(M(L^\infty)) : \operatorname{supp} \mu \subset M_\lambda \text{ for some } \lambda \in \partial D\}.$$

2. Doubly, and maximal invariant subspaces in L^∞. Recall that a norm closed subspace $B \subset L^\infty$ is called *invariant* if $zB \subset B$ (i.e.: $A(D)B \subset B$), and is called *doubly invariant* if $zB \subset B$ and $\bar{z}B \subset B$ (i.e.: $C(\partial D)B \subset B$). If $f \in C(\partial D)$ and $\lambda \in \partial D$ then $\hat{f}|_{M_\lambda} = f(\lambda)$. Hence, if $\mu \in \mathfrak{F}$ is supported in M_λ for some $\lambda \in \partial D$, then $\hat{f} = f(\lambda)$ on $\operatorname{supp} \mu$, and consequently

$$\hat{f} \ker L_\mu \subset \ker L_\mu.$$

That is, $\ker L_\mu$ is a doubly invariant subspace of L^∞ for every $\mu \in \mathfrak{F}$. It follows immediately that if $\mathfrak{G} \subset \mathfrak{F}$, then $\bigcap \{\ker L_\mu : \mu \in \mathfrak{G}\}$ is doubly invariant. The following theorem shows that the converse also holds.

THEOREM 1. *Every doubly invariant subspace B of L^∞ has the form*

$$B = \bigcap_{\mu \in \mathfrak{G}} \ker L_\mu, \tag{1}$$

for some family $\mathfrak{G} \subset \mathfrak{F}$.

To prove our theorem, we need the following lemma due to Glicksberg; see [**1**, p. 61].

LEMMA 2. *Let B be a doubly invariant subspace of L^∞ and $f \in L^\infty$. Then $f \in B$ if and only if $\hat{f}|_{M_\lambda} \in \hat{B}|_{M_\lambda}$, for every $\lambda \in \partial D$. Also, if $\mu \perp B$, then $\mu|_{M_\lambda} \perp B|_{M_\lambda}$.*

Proof of Theorem 1. Put $\mathfrak{G} = \{\mu \in \mathfrak{F} : \mu \perp B\}$. For $\lambda \in \partial D$, let $\mathfrak{G}_\lambda$ denote the set of measures μ in $\mathfrak{G}$ that are concentrated on M_λ. Then $\mathfrak{G} = \bigcup \{\mathfrak{G}_\lambda : \lambda \in \partial D\}$. By Lemma 2 we also have $\mu|_{M_\lambda} \perp B|_{M_\lambda}$, for all $\mu \perp B$. Then, by [**1**, p. 57], $\hat{B}|_{M_\lambda}$ is closed in $C(M_\lambda)$.

Hence we have

$$\begin{aligned} B &= \bigcap_{\lambda\in\partial D} \{f \in L^\infty : \hat{f}|_{M_\lambda} \in \hat{B}|_{M_\lambda}\} \qquad \text{(by Lemma 2)} \\ &= \bigcap_{\lambda\in\partial D} \{f \in L^\infty : \hat{f} \perp \mu \text{ for every } \mu \in \mathfrak{G}_\lambda\} \qquad (\text{because } \hat{B}|_{M_\lambda} \text{ is closed}) \\ &= \{f \in L^\infty : \hat{f} \perp \mu \text{ for every } \mu \in \mathfrak{G}\} \\ &= \bigcap_{\mu\in\mathfrak{G}} \ker L_\mu. \end{aligned}$$

Let B be an invariant subspace of L^∞. We can define maximal invariant subspaces of B similarly.

COROLLARY 3. *Let B be a doubly invariant subspace of L^∞ and N an invariant subspace of B.*

(i) *N is a maximal invariant subspace of B if and only if $N = \ker L_\mu \cap B$, for some measure $\mu \in \mathfrak{F}$ with $\mu \not\perp B$.*

(ii) *N is contained in a maximal invariant subspace of B if and only if $\bigcup_{n\geq 0} \bar{z}^n N$ is not dense in B.*

Proof. Suppose that N is maximal in B. Then N is a proper subspace of B. Since $zN \subset N$, $N \subset \bar{z}N$ holds. Then either $\bar{z}N = N$ or $\bar{z}N = B$ holds. Suppose that $\bar{z}N = B$. Then for every $f \in B$, we have $\bar{z}f \in B$ and there is $h \in N$ such that $\bar{z}h = \bar{z}f$. This implies that $N = B$. This contradicts the properness of N in B. Thus, $\bar{z}N = N$ holds and N is double invariant. By Theorem 1, there exists $\mathfrak{G} \subset \mathfrak{F}$ such that $N = \bigcap\{\ker L_\mu : \mu \in \mathfrak{G}\}$. Since $N \neq B$, there must be some $\mu_1 \in \mathfrak{G}$ such that $\mu_1 \not\perp B$. Hence

$$N \subset B \cap \ker L_{\mu_1} \subset B,$$

where the last inclusion is proper. Since N is maximal in B, we have $N = B \cap \ker L_{\mu_1}$.

Conversely, let $\mu \in \mathfrak{F}$ be such that $\mu \not\perp B$. Then $B \cap \ker L_\mu$ is doubly invariant and $\dim B/(\ker L_\mu \cap B) = 1$, from which the maximality is clear. This proves (i).

Suppose that N is contained in a maximal invariant subspace M of B. In the first paragraph of the proof, we showed that M is doubly invariant. Thus, the closure of $\bigcup_{n\geq 0} \bar{z}^n N$ in L^∞ is contained in M. Since M is proper in B, $\bigcup_{n\geq 0} \bar{z}^n N$ is not dense in B. Conversely, suppose that $\bigcup_{n\geq 0} \bar{z}^n N$ is not dense in B. Let M be the closure of $\bigcup_{n\geq 0} \bar{z}^n N$ in L^∞. Then M is doubly invariant and $M \neq B$. By Theorem 1, there is some measure $\mu \in \mathfrak{F}$ such that $M \subset \ker L_\mu$ and $\mu \not\perp B$. Hence, by (i), $\ker L_\mu \cap B$ is a maximal invariant subspace of B containing N.

3. Invariant subspaces in H^∞. We recall that $Sf = zf$ and $S^*f = \bar{z}(f - f(0))$ for $f \in H^\infty$. Let $B \subset H^\infty$ be a closed subspace. Then B is an invariant subspace if and only if B is invariant under S. Put $\mathfrak{F}_0 = \{\mu \in \mathfrak{F} : \mu \perp \mathbb{C}\}$.

THEOREM 4. *Let $B \subset H^\infty$ be a closed subspace such that $B \neq \{0\}$. Then B is invariant under S and S^* if and only if there is $\mathfrak{G} \subset \mathfrak{F}_0$ such that*

$$B = \bigcap_{\mu\in\mathfrak{G}} \ker L_\mu \cap H^\infty.$$

Proof. For the sufficiency of the proof, observe that if $\mu \in \mathfrak{F}$ is supported on $M_\lambda(\lambda \in \partial D)$, then for every $f \in H^\infty$ we have

$$Sf - \lambda f \in \ker L_\mu \quad \text{and} \quad S^*f - \overline{\lambda}(f - f(0)) \in \ker L_\mu.$$

On the other hand, if $\mu \perp \mathbb{C}$ and $f \in \ker L_\mu$, then

$$\lambda f \in \ker L_\mu \quad \text{and} \quad \overline{\lambda}(f - f(0)) \in \ker L_\mu.$$

Consequently, if $\mu \in \mathfrak{F}_0$, then we have $Sf, S^*f \in \ker L_\mu$ for every $f \in \ker L_\mu$. That is, $\ker L_\mu \cap H^\infty$ is invariant under S and S^* for every $\mu \in \mathfrak{F}_0$.

Now we prove the necessity. Suppose that B is invariant under S and S^*. Since $B \neq \{0\}$, there exist $f \in B$ and a nonnegative integer n such that $f = z^n g$, with $g \in H^\infty$ and $g(0) \neq 0$. Then $((S^*)^n - S(S^*)^{n+1})f = g(0) \in B$, so that B contains a nonzero constant. Consequently B contains the disk algebra $A(D)$.

Let $g \in H^\infty$ and $c \in C(\partial D)$ be such that $g + c$ is in the closure of $B + C(\partial D)$ in $H^\infty + C(\partial D)$. Then there are $f_n \in B$ and $c_n \in C(\partial D)$ such that $\|f_n + c_n - g - c\|_\infty \to 0$. It is well known (see [**2**, p. 137]) that $\mathrm{dist}(c_n - c, H^\infty) = \mathrm{dist}(c_n - c, A(D))$. Hence there exists $a_n \in A(D)$ such that $\|a_n - (c_n - c)\|_\infty \to 0$. Thus,

$$\|f_n + a_n - g\|_\infty \leq \|f_n + c_n - g - c\|_\infty + \|a_n - c_n + c\|_\infty \to 0.$$

Since $f_n + a_n \in B$ and B is closed, we have $g = \lim(f_n + a_n) \in B$. Hence, we have $g + c \in B + C(\partial D)$. Thus $B + C(\partial D)$ is closed in $H^\infty + C(\partial D)$. It follows that

$$B = (B + C(\partial D)) \cap H^\infty, \tag{2}$$

because $A(D) \subset B$.

Since $\overline{z}^n B \subset (S^*)^n B + C(\partial D) \subset B + C(\partial D)$ for every nonnegative integer n, we have that $B_\infty \stackrel{\text{def}}{=}$ the closure of $\bigcup_{n \geq 0} \overline{z}^n B$ in $H^\infty + C(\partial D)$ is contained in $B + C(\partial D)$. Therefore by (2)

$$B \subset B_\infty \cap H^\infty \subset (B + C(\partial D)) \cap H^\infty = B.$$

Thus $B = B_\infty \cap H^\infty$. Since B_∞ is a doubly invariant subspace of L^∞, by Theorem 1, there is a family $\mathfrak{G} \subset \mathfrak{F}$ such that $B_\infty = \bigcap\{\ker L_\mu : \mu \in \mathfrak{G}\}$. Since $\mathbb{C} \subset B \subset B_\infty$, we get $\mathfrak{G} \subset \mathfrak{F}_0$.

Corollary 5. *Let $B \subset H^\infty$ be a maximal invariant subspace. If there exists $f \in B$ that is invertible in H^∞, then $B = \ker L_\nu \cap H^\infty$ for some $\nu \in \mathfrak{F}$ with $\nu \not\perp H^\infty$.*

Proof. Let us assume first that $f = 1$. Then $A(D) \subset B$. Since $zB \subset B$, $B \subset S^*B$ holds. Thus, for $g \in B$ we have that $SS^*g = g - g(0) \in B \subset S^*B$. It is easy to see that S^*B is closed. Hence S^*B is an invariant subspace of H^∞. Since B is maximal in H^∞, either $S^*B = B$ or $S^*B = H^\infty$ holds. If $S^*B = H^\infty$, then for every $h \in H^\infty$ there is $g \in B$ such that $\overline{z}(g - g(0)) = h$, and consequently $zh \in B$. Thus $zH^\infty \subset B$ and, since zH^∞ is a maximal invariant subspace of H^∞ and B is a proper subspace of H^∞, then $B = zH^\infty$ holds. This contradicts the hypothesis that $1 \in B$. Hence, $S^*B = B$ holds and B turns out to be S^*-invariant. Then, by Theorem 4, there is a collection $\mathfrak{G} \subset \mathfrak{F}_0$ such that $B = \bigcap\{\ker L_\mu : \mu \in \mathfrak{G}\} \cap H^\infty$. Since B is a proper subspace of H^∞, there exists some $\nu \in \mathfrak{G}$ such that $\nu \not\perp H^\infty$. Since $\ker L_\nu \cap H^\infty$ is a maximal invariant subspace of H^∞ that contains B, we get $B = \ker L_\nu \cap H^\infty$.

For the case in which $f \in B$ is a general invertible function in H^∞, consider the space $f^{-1}B$. It is obvious that this space is also a maximal invariant subspace of H^∞, and $1 \in f^{-1}B$. By our previous case, there is some $\nu_0 \in \mathfrak{F}_0$ such that $\nu_0 \not\perp H^\infty$ and $f^{-1}B = \ker L_{\nu_0} \cap H^\infty$. Hence $B = \ker L_\nu \cap H^\infty$, where $\nu = \hat{f}^{-1}\nu_0$ is not orthogonal to $fH^\infty = H^\infty$.

For $w \in D$, we write $\varphi_\omega(z) = (w - z)(1 - \overline{w}z)$ for the special automorphism of the disk that interchanges w and 0.

Lemma 6. *Let $B \subset H^\infty$ be a maximal invariant subspace and b a finite Blaschke product. If $B \neq \varphi_w H^\infty$, for all $w \in D$, then $B \cap bH^\infty = bB$.*

Proof. First, we prove the following result.

Claim 1. If $B \neq zH^\infty$, then $B \cap z^n H^\infty = z^n B$ for every positive integer n.

Since $z^n B \subset B$, $B \subset \overline{z}^n B \cap H^\infty$ holds. By the maximality of B in H^∞, either

$$B = \overline{z}^n B \cap H^\infty \quad \text{or} \quad H^\infty = \overline{z}^n B \cap H^\infty. \tag{3}$$

The first equality is our claim. Suppose that $H^\infty = \overline{z}^n B \cap H^\infty$ holds for some n. We may assume that n is the smallest positive integer satisfying $H^\infty = \overline{z}^n B \cap H^\infty$. We have $z^n H^\infty = B \cap z^n H^\infty$. Hence

$$z^n H^\infty \subset B. \tag{4}$$

Here we have that $n \neq 1$. For, suppose that $zH^\infty \subset B$ holds. Since zH^∞ is a maximal invariant subspace of H^∞ and $B \subset H^\infty$ is proper, $B = zH^\infty$ holds. This contradicts our assumption of Claim 1. Hence $n \geq 2$. By (3), we have $B = \overline{z}B \cap H^\infty$. Hence by (4), we get

$$z^n H^\infty = z^n H^\infty \cap zH^\infty \subset B \cap zH^\infty = zB.$$

Thus we obtain $z^{n-1}H^\infty \subset B$. Hence $H^\infty = \overline{z}^{n-1} B \cap H^\infty$ holds. This contradicts the fact that n is the smallest positive integer such that $H^\infty = \overline{z}^n B \cap H^\infty$.

Next, we prove the following claim.

Claim 2. $B \cap \varphi_w^n H^\infty = \varphi_w^n B$ for every $w \in D$ and every positive integer n.

Consider the closed subspace of H^∞ given by $B \circ \varphi_w \stackrel{\text{def}}{=} \{f \circ \varphi_w : f \in B\}$. Since $(\varphi_w \circ \varphi_w)(z) = z$, it is clear that $B \circ \varphi_w$ is a maximal invariant subspace of H^∞. By our assumption, $B \neq \varphi_w H^\infty$ holds. Hence $B \circ \varphi_w \neq zH^\infty$. Therefore, by Claim 1 we have $(B \circ \varphi_w) \cap z^n H^\infty = z^n(B \circ \varphi_w)$ for every positive integer n. Composing this equality with φ_w we obtain the desired result.

Now let b be a finite Blaschke product. Obviously $bB \subset B \cap bH^\infty$. For the reverse inclusion, let $f \in H^\infty$ be such that $bf \in B$. Writing $b = \varphi_{w_1}^{n_1} \dots \varphi_{w_k}^{n_k}$, where $w_j \in D$ and $n_j \geq 1$ for $1 \leq j \leq k$, we have that

$$\varphi_{w_1}^{n_1} \dots \varphi_{w_k}^{n_k} f \in B.$$

Then Claim 2 asserts that $\varphi_{w_2}^{n_2} \dots \varphi_{w_k}^{n_k} f \in B$. We can repeat this argument $k - 1$ more times to obtain $f \in B$.

THEOREM 7. *Let $B \subset H^\infty$ be a maximal invariant subspace. Then either $B = \varphi_w H^\infty$, for some $w \in D$, or $B = \ker L_\nu \cap H^\infty$, for some $\nu \in \mathfrak{F}$ with $\nu \not\perp H^\infty$.*

Proof. Let B_∞ be the closure of $\bigcup_{n\geq 0} \bar{z}^n B$ in $H^\infty + C(\partial D)$. Assume first that $1 \in B_\infty$. Then there are $g \in B$ and a nonnegative integer n such that $\|\bar{z}^n g - 1\|_\infty < 1/2$. Hence, $\|g - z^n\|_\infty < 1/2$. Since $|\widehat{z^n}| \equiv 1$ on $M(H^\infty) \setminus D$, we have $|\hat{g}| \geq 1/2$ on $M(H^\infty) \setminus D$. It is well known that a function in H^∞ that never vanishes on $M(H^\infty) \setminus D$ can be factorised as $g = bf$, where $f \in (H^\infty)^{-1}$ and b is a finite Blaschke product.

If there is some $w \in D$ such that $B = \varphi_w H^\infty$, we are done. If not, Lemma 6 says that $f \in B$. Hence, Corollary 5 says that $B = \ker L_\mu \cap H^\infty$ for $\mu \in \mathfrak{F}$ with $\mu \not\perp H^\infty$. Thus our theorem holds when $1 \in B_\infty$.

Now suppose that $1 \notin B_\infty$. Since B_∞ is a doubly invariant subspace of L^∞, Theorem 1 states that there exists a family $\mathfrak{G} \subset \mathfrak{F}$ such that $B_\infty = \bigcap \{\ker L_\mu : \mu \in \mathfrak{G}\}$. Since $1 \notin B_\infty$, there must be some $\nu \in \mathfrak{G}$ such that $\nu \not\perp 1$. Thus

$$B \subset B_\infty \cap H^\infty \subset \ker L_\nu \cap H^\infty.$$

Since $1 \notin \ker L_\nu \cap H^\infty$, this space is a proper invariant subspace of H^∞. Also B is maximal in H^∞, so that $B = \ker L_\nu \cap H^\infty$ holds, as claimed.

4. Open problems. The most important open problem is to obtain a complete characterization of invariant subspaces of L^∞ and H^∞. If $B \subset H^\infty$ is invariant, the weak-star closure of B has the form uH^∞, where u is an inner function. Thus, $\bar{u}B$ is an invariant subspace of H^∞ that is weak-star dense in H^∞. Therefore, the problem for H^∞ reduces to characterize invariant subspaces that are weak-star dense in H^∞. A similar analysis can be carried out for L^∞, except that in this case we also have to characterize invariant subspaces whose weak-star closure is $\chi_E L^\infty$, where $E \subset \partial D$ is some measurable set.

We have other questions. Is every invariant subspace in H^∞ contained in a maximal one? What about L^∞? Obviously, these questions are less ambitious than the ones in the previous paragraphs.

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