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ON APPROXIMATION PROPERTIES OF THE PARABOLIC POTENTIALS

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In this paper the approximation properties of parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$ generated by the heat operators $\left(-\Delta_x + \dfrac{\partial}{\partial t}\right)$ and $\left(E - \Delta_x + \dfrac{\partial}{\partial t}\right)$, where

$$\Delta_x = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2},$$

are studied as $\alpha \to 0^+$.

1. Introduction and formulation of main results

The parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$ (of Riesz and Bessel type, respectively) are defined in the Fourier terms by

$$F[H^\alpha f](x,t) = \left(|x|^2 + it\right)^{-\alpha/2} F[f](x,t), \tag{1.1}$$

$$F[\mathcal{H}^\alpha f](x,t) = \left(1 + |x|^2 + it\right)^{-\alpha/2} F[f](x,t), \tag{1.2}$$

where $\alpha > 0$, $x \in R^n$, $t \in R^1$.

These potentials are interpreted as negative fractional powers of the heat operators $(-\Delta_x + \partial/\partial t)$ and $(E - \Delta_x + \partial/\partial t)$, that is formally,

$$H^\alpha f(x,t) = (-\Delta_x + \partial/\partial t)^{-\alpha/2} f(x,t),$$
$$\mathcal{H}^\alpha f(x,t) = (E - \Delta_x + \partial/\partial t)^{-\alpha/2} f(x,t)$$

(E is identity operator and $\Delta_x = \sum\limits_{k=1}^{n} \dfrac{\partial^2}{\partial x_k^2}$ is Laplacian).

The parabolic potentials were introduced by Jones [**7**] and Sampson [**11**] and studied by Bagby, Gopala Rao, Chanillo, Nogin, Rubin, Aliev and many other mathematicians (see: [**1, 3, 4, 6, 9, 10**]).

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In this paper we investigate the approximation properties of the families $H^\alpha f$ and $\mathcal{H}^\alpha f$ as $\alpha \to 0^+$. One should note that the classical Riesz and Bessel kernels as approximations of the identity have been studied by Kurokawa [**8**].

First, we shall give some necessary notations and auxiliary facts.

Let $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}^1 = \{(x,t) : x \in \mathbb{R}^n, t \in \mathbb{R}^1\}$. Define $L_p \equiv L_p(\mathbb{R}^{n+1}), 1 \leqslant p < \infty$ as the class of measurable functions f on $\mathbb{R}^{n+1}$ with the norm

$$\|f\|_p = \left(\int\limits_{\mathbb{R}^{n+1}} |f(x,t)|^p dxdt\right)^{1/p}, \ dx = dx_1 \dots dx_n.$$

$C_0 \equiv C_0(\mathbb{R}^{n+1})$ will denote the class of all continuous functions on $\mathbb{R}^{n+1}$ vanishing at infinity. $C \equiv C(\mathbb{R}^{n+1})$ is the class of all continuous functions on $\mathbb{R}^{n+1}$. We set, as usual, $\|f\|_\infty = ess \sup\limits_{\mathbb{R}^{n+1}} |f(x,t)|$ and denote by $W(x,t)$ the classical Gauss–Weierstrass kernel, defined in Fourier terms by

$$F[W(.,t)](\zeta) \equiv \int\limits_{\mathbb{R}^n} e^{-ix\cdot\zeta}\, W(x,t)\, dx = e^{-t|\zeta|},$$

where $t > 0$, $\zeta \in \mathbb{R}^n$ and $x \cdot \zeta = x_1\zeta_1 + \dots + x_n\zeta_n$.

It is well known that

$$W(x,t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t), \quad t > 0, \ x \in R^n, \tag{1.3}$$

and

$$\int\limits_{\mathbb{R}^n} W(x,t)\, dx = 1, \quad \forall t > 0. \tag{1.4}$$

The potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$, initially defined in terms of Fourier transform by (1.1) and (1.2), have the following convolution type integral representations (see: [**1**, p. 396]).

$$H^\alpha f(x,t) = \frac{1}{\Gamma(\alpha/2)} \int\limits_{\mathbb{R}^n\times(0,\infty)} \tau^{(\alpha/2)-1} W(y,\tau) f(x-y, t-\tau)\, dyd\tau; \tag{1.5}$$

$$\mathcal{H}^\alpha f(x,t) = \frac{1}{\Gamma(\alpha/2)} \int\limits_{\mathbb{R}^n\times(0,\infty)} \tau^{(\alpha/2)-1} e^{-\tau} W(y,\tau) f(x-y, t-\tau) dyd\tau; \tag{1.6}$$

The following theorem characterises the behaviour of the operators H^α and $\mathcal{H}^\alpha$ on L_p-spaces.

Theorem A. (See [**3, 6**].) I. *Let $f \in L_p$, $1 \leqslant p < \infty$, $0 < \alpha < (n+2)/p$ and $q = (n+2)p/(n+2-\alpha p)$.*

(a) *The integral $H^\alpha f(x,t)$ converges absolutely for almost all $(x,t) \in \mathbb{R}^{n+1}$.*

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(b) *For $p > 1$, the operator H^α is bounded from L_p into L_q.*

(c) *For $p = 1$, the operator H^α is weak $(1, q)$, that is,*

$$\operatorname{meas}\left\{(x,t) : \left|H^\alpha f(x,t)\right| > \lambda\right\} \leqslant \left(c\|f\|_1/\lambda\right)^q, \ \forall\lambda > 0,$$

where $q = (n+2)/(n+2-\alpha)$.

II *The operator $\mathcal{H}^\alpha$ is bounded in L_p for all $\alpha \geqslant 0$ and $1 \leqslant p \leqslant \infty$.*

We shall need the following classes of "anisotropic" Lipschitz functions on $\mathbb{R}^{n+1} \times \mathbb{R}^1$.

A. The Lipschitz class Λ_β.

$$\Lambda_\beta = \left\{f \in L_\infty(\mathbb{R}^{n+1}) : \left\|f(x-y, t-\tau) - f(x,t)\right\|_\infty \leqslant c_f\left(|y|^2 + |\tau|\right)^{\beta/2}\right\} \tag{1.7}$$

B. The local Lipschitz class $\Lambda_\beta(x_0, t_0)$.

$$\Lambda_\beta(x_0,t_0) = \left\{f : \left|f(x_0-y, t_0-\tau) - f(x_0,t_0)\right| \leqslant c_f\left(|y|^2 + |\tau|\right)^{\beta/2}\right\} \tag{1.8}$$

(Here $x, x_0, y \in \mathbb{R}^n$; $t, t_0, \tau \in \mathbb{R}^1$ and $0 < \beta \leqslant 1$.)

Throughout the paper the letters $c, c_1, c_2, \dots c_1(\delta), c_2(\delta), \dots$ are used for constants (the constants $c_i(\delta)$ depend on parameter $\delta > 0$). We shall write "$\varphi(\alpha) = O\big(\psi(\alpha)\big)$ as $\alpha \to 0^+$" if $\left|\varphi(\alpha)\right| \leqslant c\,\psi(\alpha)$ as $\alpha \to 0^+$.

The main theorems of the paper are as follows.

Theorem 1. *Let $f \in L_p(\mathbb{R}^{n+1})$, $1 \leqslant p < \infty$, and A^α is one of the operators H^α and $\mathcal{H}^\alpha$. Then:*

(a) *If at a point $(x,t) \in \mathbb{R}^{n+1}$ there exist limit*

$$\lim_{(z,s)\to(x,t)} f(z,s) = l, \ -\infty \leqslant l \leqslant \infty,$$

then $\lim\limits_{\alpha\to 0^+} A^\alpha f(x,t) = l$. In particular, if f is continuous at the point $(x,t) \in \mathbb{R}^{n+1}$, then $\lim\limits_{\alpha\to 0^+} A^\alpha f(x,t) = f(x,t)$.

(b) *If $f \in L_p \cap C_0$, the convergence $\lim\limits_{\alpha\to 0^+} A^\alpha f = f$ is uniform on $\mathbb{R}^{n+1}$. If $f \in L_p \cap C$, the convergence is uniform on any compact $K \subset \mathbb{R}^{n+1}$.*

Theorem 2. *If $f \in L_p(\mathbb{R}^{n+1})$, $1 \leqslant p < \infty$, then $\lim\limits_{\alpha\to 0^+} \mathcal{H}^\alpha f(x,t) = f(x,t)$, where the limit is understood in the L_p-norm, or pointwise for almost all $(x,t) \in \mathbb{R}^{n+1}$.*

The next theorem gives an estimation of the order of approximation of the "Lipchitz functions" by the families $H^\alpha f$ and $\mathcal{H}^\alpha f$.

Theorem 3. *Let A^α be either of the potentials H^α and $\mathcal{H}^\alpha$, $\alpha > 0$. Then:*

(a) *If $f \in L_p \cap \Lambda_\beta$, $1 \leqslant p < \infty$, $0 < \beta \leqslant 1$, where Λ_β is the Lipschitz class defined as in (1.7), then*

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(1.9) $$\|A^\alpha f - f\|_\infty = O(1)\alpha \text{ as } \alpha \to 0^+;$$

(b) *If* $f \in L_p \cap \Lambda_\beta(x_0, t_0)$, $1 \leqslant p < \infty$, $0 < \beta \leqslant 1$, *where* $\Lambda_\beta(x_0, t_0)$ *is the Lipschitz class defined as* (1.8), *then*

(1.10) $$A^\alpha f(x_0, t_0) - f(x_0, t_0) = O(1)\alpha \quad \text{as} \quad \alpha \to 0^+.$$

REMARK 1. It is interesting to observe that the order of approximation does not depend on the "Lipschitz degree" β of the function f.

2. Proofs of the main results

PROOF OF THE THEOREM 1. (a) By making use of the Fubini theorem, we can write the formulas (1.5) and (1.6) in the form of

(2.1) $$H^\alpha f(x,t) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \tau^{(\alpha/2)-1} \Big(\int_{\mathbb{R}^n} W(y,\tau) f(x-y, t-\tau) dy \Big) d\tau,$$

(2.2) $$\mathcal{H}^\alpha f(x,t) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \tau^{(\alpha/2)-1} e^{-\tau} \Big(\int_{\mathbb{R}^n} W(y,\tau) f(x-y, t-\tau) dy \Big) d\tau.$$

We shall prove the statements of theorem in the case of $A^\alpha = H^\alpha$. (See Remark 2 below about the $\mathcal{H}^\alpha$).

Suppose a function $f \in L_p$ has the limit $l \in (-\infty, \infty)$ at the point $(x,t) \in \mathbb{R}^{n+1}$. Using the identity (1.4) we get

$$H^\alpha f(x,t) - l = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \tau^{(\alpha/2)-1} \int_{\mathbb{R}^n} W(y,\tau) \big(f(x-y, t-\tau) - l\, e^{-\tau}\big) dy\, d\tau.$$

Given $\varepsilon > 0$ there exist $\delta > 0$ such that

(2.3) $$|f(x-y, t-\tau) - l| < \varepsilon \quad \text{and} \quad (1 - e^{-\tau}) < \varepsilon$$

for all $|y| < \sqrt{\delta}$ and $0 < \tau < \delta$. We have

$$\begin{aligned} \big|H^\alpha f(x,t) - l\big| \leqslant{}& \frac{1}{\Gamma(\alpha/2)} \Big| \int_0^\delta \tau^{(\alpha/2)-1} \int_{|y|<\sqrt{\delta}} W(y,\tau) \big(f(x-y, t-\tau) - l\, e^{-\tau}\big) dy d\tau \Big| \\ &+ \frac{1}{\Gamma(\alpha/2)} \Big| \int_0^\delta \tau^{(\alpha/2)-1} \int_{|y|\geqslant\sqrt{\delta}} W(y,\tau) \big(f(x-y, t-\tau) - l\, e^{-\tau}\big) dy d\tau \Big| \end{aligned}$$

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$$+\frac{1}{\Gamma(\alpha/2)}\left|\int_{\delta}^{\infty}\tau^{(\alpha/2)-1}\int_{R^n}W(y,\tau)\big(f(x-y,t-\tau)-l\,e^{-\tau}\big)dyd\tau\right|$$

$$\equiv i_1(\alpha)+i_2(\alpha)+i_3(\alpha) \tag{2.4}$$

The application of the estimates (2.3) leads to

$$\begin{aligned} i_1(\alpha) &\leqslant \frac{1}{\Gamma(\alpha/2)}\int_0^{\delta}\tau^{(\alpha/2)-1}\int_{|y|<\sqrt{\delta}}W(y,\tau)\big|f(x-y,t-\tau)-l\big|\,dy\,d\tau \\ &\quad +\frac{|l|}{\Gamma(\alpha/2)}\int_0^{\delta}\tau^{(\alpha/2)-1}(1-e^{-\tau})\int_{|y|<\sqrt{\delta}}W(y,\tau)dy\,d\tau \\ &\leqslant \frac{1+|l|}{\Gamma(\alpha/2)}\varepsilon\int_0^{\delta}\tau^{(\alpha/2)-1}\int_{\mathbb{R}^n}W(y,\tau)dy\,d\tau \\ &\overset{(1.4)}{=} \frac{1+|l|}{\Gamma(\alpha/2)}\varepsilon\int_0^{\delta}\tau^{(\alpha/2)-1}d\tau=\frac{(1+|l|)\delta^{\alpha/2}}{(\alpha/2)\Gamma(\alpha/2)}\varepsilon=\frac{(1+|l|)\delta^{\alpha/2}}{\Gamma(1+\alpha/2)}\varepsilon. \end{aligned} \tag{2.5}$$

Let us estimate $i_2(\alpha)$. We have

$$\begin{aligned} i_2(\alpha) &\leqslant \frac{1}{\Gamma(\alpha/2)}\int_0^{\delta}\tau^{(\alpha/2)-1}\int_{|y|>\sqrt{\delta}}W(y,\tau)\big|f(x-y,t-\tau)\big|dy\,d\tau \\ &\quad+\frac{|l|}{\Gamma(\alpha/2)}\int_0^{\delta}\tau^{(\alpha/2)-1}e^{-\tau}\int_{|y|>\sqrt{\delta}}W(y,\tau)dy\,d\tau\equiv i_2'(\alpha)+i_2''(\alpha). \end{aligned} \tag{2.6}$$

Taking into account (1.3) and the Hölder inequality we get for small $\alpha>0$

$$\begin{aligned} i_2'(\alpha) &\leqslant \frac{\|f\|_p}{\Gamma(\alpha/2)}\left(\int_0^{\delta}d\tau\int_{|y|>\sqrt{\delta}}\left(\tau^{(\alpha/2)-1}W(y,\tau)\right)^{p'}dy\right)^{1/p'} \\ &\overset{(1.3)}{=} c_1\frac{\|f\|_p}{\Gamma(\alpha/2)}\left(\int_0^{\delta}\tau^{((\alpha/2)-1-(n/2))p'}d\tau\int_{|y|>\sqrt{\delta}}e^{-|y|^2p'/(4\tau)}dy\right)^{1/p'} \\ &\qquad \left(\text{set } y=2\sqrt{\frac{\tau}{p'}}z\ ,\ dy=2^n(p')^{-n/2}\tau^{n/2}dz\right) \\ &= c_2\frac{\|f\|_p}{\Gamma(\alpha/2)}\left(\int_0^{\delta}\tau^{((\alpha/2)-1-(n/2))p'+\frac{n}{2}}d\tau\int_{|z|>\frac{1}{2}\sqrt{\delta p'/\tau}}e^{-|z|^2}dz\right)^{1/p'} \end{aligned}$$

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$$(\text{use } e^{-|z|^2} = e^{-(|z|^2/2)}e^{-(|z|^2/2)} \leqslant e^{-(|z|^2/2)}e^{-(\delta p'/8\tau)} \text{ for } |z| > \frac{1}{2}\sqrt{\delta p'/\tau})$$

$$\leqslant c_3 \frac{\|f\|_p}{\Gamma(\alpha/2)} \left(\int_0^\delta \tau^{(n/2)-(1+(n/2))^p} e^{-\delta p'/(8\tau)} d\tau \right)^{1/p'} \left(\int_{\mathbb{R}^n} e^{-|z|^2/2} dz \right)^{1/p}$$

$$\leqslant c_1(\delta)\|f\|_p \alpha. \tag{2.7}$$

By a similar way,

$$i_2''(\alpha) = \frac{|l|}{\Gamma(\alpha/2)} \int_0^\delta \tau^{(\alpha/2)-1} e^{-\tau} d\tau \int_{|y|>\sqrt{\delta}} (4\pi\tau)^{-n/2} e^{-|y|^2/(4\tau)} dy$$

$$(\text{set } y = 2\sqrt{\tau} z \text{ , } dy = 2^n \tau^{n/2} dz)$$

$$\leqslant \frac{c_4|l|}{\Gamma(\alpha/2)} \int_0^\delta \tau^{(\alpha/2)-1-\frac{n}{2}} d\tau \int_{|z|>\sqrt{\delta}/(2\sqrt{\tau})} e^{-|z|^2/2} e^{-\delta/(8\tau)} dz$$

$$\leqslant \frac{c_4|l|}{\Gamma(\alpha/2)} \int_0^\delta \tau^{(\alpha/2)-1-\frac{n}{2}} e^{-\delta/(8\tau)} d\tau \int_{\mathbb{R}^n} e^{-|z|^2/2} dz$$

$$= c_2(\delta)|l|\alpha. \tag{2.8}$$

From (2.6), (2.7) and (2.8) it follows that

$$i_2(\alpha) \leqslant \big(c_1(\delta)\|f\|_p + c_2(\delta)|l|\big)\alpha = c_3(\delta)\alpha \quad \text{as} \quad \alpha \to 0^+. \tag{2.9}$$

Let us now estimate $i_3(\alpha)$. We have

$$i_3(\alpha) \leqslant \frac{1}{\Gamma(\alpha/2)} \int_\delta^\infty \tau^{(\alpha/2)-1} \left(\int_{\mathbb{R}^n} W(y,\tau) \big| f(x-y,t,\tau) \big| \, dy \right) d\tau$$

$$+ \frac{|l|}{\Gamma(\alpha/2)} \int_\delta^\infty \tau^{(\alpha/2)-1} e^{-\tau} \left(\int_{\mathbb{R}^n} W(y,\tau) dy \right) d\tau$$

$$\equiv i_3'(\alpha) + i_3''(\alpha). \tag{2.10}$$

By (1.4) it follows that

$$i_3''(\alpha) = \frac{|l|}{\Gamma(\alpha/2)} \int_\delta^\infty \tau^{(\alpha/2)-1} e^{-\tau} d\tau \leqslant \frac{c_4(\delta)|l|}{\Gamma(\alpha/2)} \leqslant c_5(\delta)|l|\alpha \quad \text{as} \quad \alpha \to 0^+. \tag{2.11}$$

Further, using Hölder's inequality, we have for $\alpha < \frac{n+2}{p}$

$$i_3'(\alpha) \leqslant \frac{\|f\|_p}{\Gamma(\alpha/2)} \left(\int_\delta^\infty \tau^{((\alpha/2)-1)p'} d\tau \int_{\mathbb{R}^n} \big(W(y,\tau)\big)^{p'} dy \right)^{1/p'}$$

$$\leqslant \frac{\|f\|_p}{\Gamma(\alpha/2)}\Big(\int_\delta^\infty \tau^{((\alpha/2)-1-(n/2))p'}d\tau \int_{\mathbb{R}^n} e^{-(|y|^2p'/4\tau)}\,dy\Big)^{1/p'}$$

$$= c\frac{\|f\|_p}{\Gamma(\alpha/2)}\Big(\int_\delta^\infty \tau^{((\alpha/2)-1-(n/2))p'+\frac{n}{2}}d\tau\Big)^{1/p'}\Big(\int_{\mathbb{R}^n} e^{-|z|^2}dz\Big)^{1/p'}$$

$$\leqslant c_6(\delta)\|f\|_p\alpha. \tag{2.12}$$

Therefore, from (2.10), (2.11) and (2.12) we have

$$i_3(\alpha) \leqslant (c_5(\delta)|l| + c_6(\delta)\|f\|_p)\alpha = c_7(\delta)\alpha \quad \text{as} \quad \alpha \to 0^+. \tag{2.13}$$

Finally, from (2.4), (2.5), (2.9) and (2.13) it follows that

$$\big|H^\alpha f(x,t) - l\big| \leqslant \frac{(1+|l|)\delta^{\alpha/2}}{\Gamma(1+\alpha/2)}\varepsilon + c_3(\delta)\alpha + c_7(\delta)\alpha.$$

The last estimate yields

$$\limsup_{\alpha\to 0^+} |H^\alpha f(x,t) - l| \leqslant (1+|l|)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we have

$$\lim_{\alpha\to 0^+} |H^\alpha f(x,t) - l| = 0.$$

Let now $l = +\infty$, that is $\lim_{(y,\tau)\to(x,t)} f(y,\tau) = +\infty$ (the case of $l = -\infty$ is examined analogously).

For a given $M > 0$ there exists $\delta > 0$ such that $f(x-y,t-\tau) > M$ for any $|y| < \sqrt{\delta}$, $0 < \tau < \delta$. Using this observation we have

$$H^\alpha f(x,t) = \frac{1}{\Gamma(\alpha/2)}\int_0^\delta \tau^{(\alpha/2)-1}\int_{|y|<\sqrt{\delta}} W(y,\tau)f(x-y,t-\tau)dyd\tau$$

$$+\frac{1}{\Gamma(\alpha/2)}\int_0^\delta \tau^{(\alpha/2)-1}\int_{|y|\geqslant\sqrt{\delta}} W(y,\tau)f(x-y,t-\tau)dyd\tau$$

$$+\frac{1}{\Gamma(\alpha/2)}\int_\delta^\infty \tau^{(\alpha/2)-1}\int_{\mathbb{R}^n} W(y,\tau)f(x-y,t-\tau)dyd\tau$$

$$\equiv j_1(\alpha) + j_2(\alpha) + j_3(\alpha). \tag{2.14}$$

It is clear that

$$j_1(\alpha) \geqslant \frac{M}{\Gamma(\alpha/2)}\int_0^\delta \tau^{(\alpha/2)-1}\int_{|y|<\sqrt{\delta}} W(y,\tau)dyd\tau$$

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$$= c_1 \frac{M}{\Gamma(\alpha/2)} \int_0^{\delta} \tau^{(\alpha/2)-1-(n/2)} \int_{|y|<\sqrt{\delta}} e^{-|y|^2/4\tau} dy d\tau$$

$$= c_1 \frac{M}{\Gamma(\alpha/2)} \int_0^{\delta} \tau^{(\alpha/2)-1} \int_{|x|<\sqrt{\delta/\tau}} e^{-|x|^2} dx$$

$$= c_2 \frac{M}{\Gamma(\alpha/2)} \int_0^{\delta} \tau^{(\alpha/2)-1} \int_0^{\sqrt{\delta/\tau}} e^{-r^2} r^{n-1} dr d\tau$$

$$\geqslant c_2 \frac{M}{\Gamma(\alpha/2)} \int_0^{\delta} \tau^{(\alpha/2-1)} d\tau \left(\int_0^1 e^{-r^2} r^{n-1}\, dr \right)$$

$$= c_3 \frac{M}{\Gamma(\alpha/2)} \frac{2}{\alpha} \delta^{(\alpha/2)} = c_3 \frac{M}{\Gamma(1+\alpha/2)} \delta^{\alpha/2}, \quad (c_3 > 0). \tag{2.15}$$

Further, by making use of the estimates for $i_2'(\alpha)$ and $i_3'(\alpha)$ (see (2.7) and (2.12), respectively), we have

$$|j_2(\alpha)| \leqslant c_1(\delta)\|f\|_p \alpha \quad \text{and} \quad |j_3(\alpha)| \leqslant c_2(\delta)\|f\|_p \alpha. \tag{2.16}$$

Thus, it follows from (2.14), (2.15) and (2.16) that

$$H^\alpha f(x,t) \geqslant c_3 \frac{M}{\Gamma(1+\alpha/2)} \delta^{\alpha/2} - c_1(\delta)\|f\|_p \alpha - c_2(\delta)\|f\|_p \alpha,$$

and therefore,

$$\liminf_{\alpha \to 0^+} H^\alpha f(x,t) \geqslant c_3 M, \quad (c_3 > 0).$$

Since $M > 0$ is arbitrary, the last estimate yields that $\lim\limits_{\alpha \to 0^+} H^\alpha f(x,t) = \infty$.

(b) Let now $f \in L_p \cap C_0$. The condition $f \in C_0$ yields that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{(x,t)\in\mathbb{R}^{n+1}} |f(x-y,t-\tau) - f(x,t)| < \varepsilon \quad \text{and} \quad (1 - e^{-\tau}) < \varepsilon$$

for all $|y| < \sqrt{\delta}$ and $0 < \tau < \delta$.

Setting $l = f(x,t)$ in (2.4) and using (2.17), we have as in proof of part (a) (see, (2.4), (2.5), (2.9) and (2.13))

$$\|H^\alpha f - f\|_\infty \leqslant \frac{(1+\|f\|_\infty)\delta^{\alpha/2}}{\Gamma(1+\alpha/2)} \varepsilon + \big(c_1(\delta)\, \|f\|_p + c_2(\delta)\, \|f\|_\infty\big)\alpha$$
$$+ \big(c_5(\delta)\, \|f\|_\infty + c_6(\delta)\, \|f\|_p\big)\alpha, \quad \alpha \to 0^+.$$

The latter estimate yields that $\limsup_{\alpha\to 0+}\|H^\alpha f - f\|_\infty \leqslant (1+\|f\|_\infty)\varepsilon$, and therefore, $\lim_{\alpha\to 0+}\|H^\alpha f - f\|_\infty = 0$. $\square$

REMARK 2. The proof of the statements of Theorem 1 for $A^\alpha = \mathcal{H}^\alpha$ follows the same lines and is based on the equality

$$\begin{aligned}\mathcal{H}^\alpha f(x,t) - l &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty \tau^{(\alpha/2)-1}e^{-\tau}\int_{\mathbb{R}^n} W(y,\tau)\big(f(x-y,t-\tau)-l\big)\,dyd\tau\\ &= \frac{1}{\Gamma(\alpha/2)}\int_0^\delta \tau^{(\alpha/2)-1}e^{-\tau}\int_{|y|\leqslant\sqrt{\delta}} W(y,\tau)\big(f(x-y,t-\tau)-l\big)dyd\tau\\ &\quad+\frac{1}{\Gamma(\alpha/2)}\int_0^\delta \tau^{(\alpha/2)-1}e^{-\tau}\int_{|y|>\sqrt{\delta}} W(y,\tau)\big(f(x-y,t-\tau)-l\big)dyd\tau\\ &\quad+\frac{1}{\Gamma(\alpha/2)}\int_\delta^\infty \tau^{(\alpha/2)-1}e^{-\tau}\int_{\mathbb{R}^n} W(y,\tau)\big(f(x-y,t-\tau)-l\big)dyd\tau. \end{aligned} \tag{2.17}$$

Slight additional technicalities related to the factor $e^{-\tau}$ are left to the reader.

REMARK 3. In the estimation of $i_2'(\alpha)$ and $i_3'(\alpha)$ we use the Hölder inequality when $p>1$. An attentive examination shows that the estimates for $i_2'(\alpha)$ and $i_3'(\alpha)$ are true also for $p=1$. This follows from the facts that the quantities

$$A_1(\delta) = \sup_{0<\alpha<1}\ \sup_{\substack{0<\tau<\delta\\ |y|>\sqrt{\delta}}}\big(\tau^{(\alpha/2)-1}\,W(y,\tau)\big)$$

and

$$A_2(\delta) = \sup_{0<\alpha<1}\ \sup_{\substack{\tau>\delta\\ |y|\varepsilon\mathbb{R}^n}}\big(\tau^{(\alpha/2)-1}\,W(y,\tau)\big)$$

are finite.

PROOF OF THEOREM 2: The L_p-continuity of the translation operator yields that for $\forall\ \varepsilon>0$ there exist $\delta>0$ such that $\big\|f(x-y,t-\tau)-f(x,t)\big\|_p<\varepsilon$ for all $|y|<\sqrt{\delta}$ and $0<\tau<\delta$. Using this and relation (2.18) for $l=f(x,t)$, we have

$$\begin{aligned}&\big\|\mathcal{H}^\alpha f(x,t)-f(x,t)\big\|_p\\ &\quad\leqslant \frac{1}{\Gamma(\alpha/2)}\int_0^\delta \tau^{(\alpha/2)-1}e^{-\tau}\int_{|y|<\sqrt{\delta}} W(y,\tau)\big\|f(x-y,t-\tau)-f(x,t)\big\|_p\,dy\,d\tau\end{aligned}$$

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$$+\frac{1}{\Gamma(\alpha/2)}\int_0^{\delta}\tau^{(\alpha/2)-1}e^{-\tau}\int_{|y|\geqslant\sqrt{\delta}}W(y,\tau)\big\|f(x-y,t-\tau)-f(x,t)\big\|_p\,dy\,d\tau$$

$$+\frac{1}{\Gamma(\alpha/2)}\int_{\delta}^{\infty}\tau^{(\alpha/2)-1}e^{-\tau}\int_{\mathbb{R}^n}W(y,\tau)\big\|W(x-y,t-\tau)-f(x,t)\big\|_p\,dy\,d\tau$$

(2.18) $$\equiv k_1(\alpha)+k_2(\alpha)+k_3(\alpha).$$

Further,

(2.19) $$k_1(\alpha)\leqslant\frac{\varepsilon}{\Gamma(\alpha/2)}\int_0^{\infty}\tau^{(\alpha/2)-1}e^{-\tau}d\tau\int_{\mathbb{R}^n}W(y,\tau)dy=\varepsilon;$$

$$k_2(\alpha)\leqslant\frac{2\|f\|_p}{\Gamma(\alpha/2)}\int_0^{\delta}\tau^{(\alpha/2)-1}e^{-\tau}\int_{|y|\geqslant\sqrt{\delta}}W(y,\tau)\,dy\,d\tau$$

$$\leqslant\frac{2\|f\|_p}{\Gamma(\alpha/2)}\int_0^{\delta}\tau^{(\alpha/2)-1}\int_{|y|\geqslant\sqrt{\delta}}W(y,\tau)\,dy\,d\tau$$

$$\leqslant\ldots(\text{see}\quad(2.8))\ldots$$

(2.20) $$\leqslant\frac{2\|f\|_p}{\Gamma(\alpha/2)}c_1(\delta)=c_2(\delta)\ \|f\|_p\ \alpha;$$

$$k_3(\alpha)\leqslant\frac{2\|f\|_p}{\Gamma(\alpha/2)}\int_{\delta}^{\infty}\tau^{(\alpha/2)-1}e^{-\tau}\,d\tau\int_{\mathbb{R}^n}W(y,\tau)\,dy$$

(2.21) $$=\frac{2\|f\|_p}{\Gamma(\alpha/2)}\int_{\delta}^{\infty}\tau^{(\alpha/2)-1}e^{-\tau}\,d\tau\leqslant\frac{2\|f\|_p}{\Gamma(\alpha/2)}c_3(\delta)=c_4(\delta)\ \|f\|_p\ \alpha.$$

It follows from (2.19)-(2.22) that $\lim\limits_{\alpha\to 0^+}\big\|\mathcal{H}^\alpha f(x,t)-f(x,t)\big\|_p=0$. By similar reason, for $f\in L_p\cap C_0$ $\mathcal{H}^\alpha f\to f$, uniformly as $\alpha\to 0^+$.

Since the class $L_p\cap C_0$ is dense in L_p, $(1\leqslant p<\infty)$ and $\|\mathcal{H}^\alpha f\|_p\leqslant\|f\|_p$, $\forall\ \alpha>0$, it follows from [**12**, p. 60, Theorem 3.12] that $\lim\limits_{\alpha\to 0^+}\mathcal{H}^\alpha f(x,t)=f(x,t)$ for almost all $(x,t)\in\mathbb{R}^{n+1}$. ∎

Proof of Theorem 3: (a) We shall prove only the case when $A^\alpha=H^\alpha$. In the case of $A^\alpha=\mathcal{H}^\alpha$ the statements are proved in a similar way (see Remark 2).

Let $f\in L_p\cap\Lambda_\beta$ and $A^\alpha=H^\alpha$. Setting $l=f(x,t)$ in (2.4) and using (2.9) and (2.13) we have

$$\big|H^\alpha f(x,t)-f(x,t)\big|\leqslant i_1(\alpha)+i_2(\alpha)+i_3(\alpha),$$

where

(2.22) $$i_2(\alpha)\leqslant\big(c_1(\delta)\ \|f\|_p+c_2(\delta)\ \|f\|_\infty\big)\alpha\equiv c_3(\delta)\alpha,$$

and

$$i_3(\alpha) \leqslant \big(c_5(\delta)\ \|f\|_\infty + c_6(\delta)\ \|f\|_p\big)\alpha \equiv c_7(\delta)\alpha, \quad (\alpha \to 0^+). \tag{2.23}$$

Let us estimate $i_1(\alpha)$. We have

$$i_1(\alpha) \leqslant \frac{1}{\Gamma(\alpha/2)} \int_0^\delta \tau^{(\alpha/2)-1} \int_{|y|<\sqrt{\delta}} W(y,\tau)\ \big\|f(x-y,t-\tau) - f(x,t)\big\|_\infty\, dy\, d\tau$$

$$+ \frac{\|f\|_\infty}{\Gamma(\alpha/2)} \int_0^\delta \tau^{(\alpha/2)-1}(1-e^{-\tau}) \int_{|y|<\sqrt{\delta}} W(y,\tau)\, dy\, d\tau.$$

Taking into account that $1 - e^{-\tau} = \tau + O(1)\tau^2$ as $\tau \to 0^+$ and

$$\big\|f(x-y,t-\tau) - f(x,t)\big\|_\infty \leqslant c_f\ (|y|^2 + \tau)^{\beta/2},$$

we get

$$i_1(\alpha) \leqslant \frac{c}{\Gamma(\alpha/2)} \int_0^\delta \tau^{(\alpha/2)-1} \int_{R^n} W(y,\tau)\ (|y|^2+\tau)^{\beta/2} dy\, d\tau$$

$$+\frac{c}{\Gamma(\alpha/2)} \int_0^\delta \tau^{(\alpha/2)-1+1}\, d\tau \int_{\mathbf{R}^n} W(y,\tau)\, dy$$

$$\equiv i_1'(\alpha) + i_2'(\alpha).$$

After changing the variable y with $\sqrt{\tau}\ z$ a simple calculation leads to $i_1'(\alpha) = O(1)\alpha$ as $\alpha \to 0^+$. Further, using (1.4) we have

$$i_2'(\alpha) = \frac{c}{\Gamma(\alpha/2)} \int_0^\delta \tau^{(\alpha/2)}\, d\tau = O(1)\alpha \quad \text{as} \quad \alpha \to 0^+,$$

and therefore, $i_1(\alpha) = O(1)\alpha$ as $\alpha \to 0^+$.

Now from (2.23) it follows that $\|H^\alpha f - f\|_\infty = O(1)\alpha$ as $\alpha \to 0^+$.

Part (b) of the theorem is proved analogously, just replacing the expression $\|\cdots\|_\infty$ with $|\cdots|$. ∎

REMARK 4. The analogues of Theorems 1-3 can be formulated and proved for Parabolic potentials associated with the singular Laplace–Bessel differential operator

$$\Delta_\nu = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + (2\nu/x_n)\frac{\partial}{\partial x_n}, \ (\nu > 0).$$

For detailed information about these potentials the reader is referred to [**2**, **5**].

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