EACH JOIN-COMPLETION OF A PARTIALLY ORDERED SET IS THE SOLUTION OF A UNIVERSAL PROBLEM

Dedicated to the memory of Hanna Neumann

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The main result of this paper is the theorem in the title. Only special cases of it seem to be known so far. As an application, we obtain a result on the unique extension of Galois connexions. As a matter of fact, it is only by the use of Galois connexions that we obtain the main result, in its present generality. We first introduce the notions of join-extensions and completions, ideal-completions in particular, and corresponding types of mappings.

1. Join-extensions

A subset P of a partially ordered set E is called join-dense in E if each element $x \in E$ is the join of a subset $P' \subset P$. One may take $P' = P \cap (x] = \{p \mid p \in P,\ p \leqq x\}$, so that

$$x = \sup_E P \cap (x], \tag{1}$$

for each $x \in E$. Then E is called a join-extension of P. By virtue of (1), *E then becomes order-isomorphic with its canonical image*

$$\dot{E} = \{P \cap (x] \mid x \in E\}. \tag{2}$$

For $\mathscr{C} = \dot{E}$, we have

$$\dot{P} \subset \mathscr{C} \subset \mathscr{L}(P), \tag{3}$$

where $\dot{P}$ — the canonical image of the least join-extension P — denotes the family of all principal lower ends (principal ideals) $(p]_P = \{p' \mid p' \in P,\ p' \leqq p\}$, $\mathscr{L}(P)$ — the canonical image of the largest join-extension — the family of all lower ends (unions of principal lower ends). Note that each family $\mathscr{C}$ with the property (3) is an $\dot{E}$, for a unique — up to natural isomorphism over P — join-extension E of P.

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We will call this the $\mathscr{C}$-(join-) extension of P. E is complete if $\mathscr{C} = \dot{E}$ is a closure family over P. In this case, we call E a join-completion, more specifically, the $\mathscr{C}$-(join-) completion of P.

Let now P and Q be partially ordered sets, $\mathscr{C}$ the canonical image of a join-extension of P. A mapping $\phi : P \to Q$ is called a $\mathscr{C}$-homomorphism if, for each $q \in Q$, the inverse image of the principal ideal $(q]$ of Q belongs to $\mathscr{C}$,

$$\{p \mid \phi(p) \leqq q\} = \phi^{-1}((q]) \in \mathscr{C}. \tag{4}$$

E.g., ϕ is an $\mathscr{L}(P)$-homomorphism if and only if ϕ is order-preserving (isotone). Consequently, each $\mathscr{C}$-homomorphism, for any $\mathscr{C}$, is order-preserving. Actually, we have established a huge scale of order-preserving mappings: the smaller $\mathscr{C}$, the better the $\mathscr{C}$-homomorphisms, the $\dot{P}$-homomorphisms being the best. As a matter of fact, $\phi : P \to Q$ is a $\dot{P}$-homomorphism if and only if it admits a right adjoint, i.e. a mapping $\psi : Q \to P$ such that

$$\phi(p) \leqq q \text{ if and only if } p \leqq \psi(q), \tag{5}$$

for each $p \in P$, $q \in Q$. This right adjoint is uniquely determined by

$$\psi(q) = \max\{p \mid \phi(p) \leqq q\} \tag{6}$$

whereas its left adjoint, the original mapping $\phi : P \to Q$, is uniquely determined by

$$\phi(p) = \min\{q \mid p \leqq \psi(q)\}. \tag{7}$$

We call (ϕ, ψ) an adjoint situation between P and Q. Both ϕ and ψ are then order-preserving (covariant), and $\psi \circ \phi : P \to P$ is a closure operator, $\phi \circ \psi : Q \to Q$ a kernel operator. We also call such a pair a Galois connexion of mixed type. For dualizing Q, we get what has been called a Galois connexion by Ore [7] (cf. also Everett [6]), characterized by the equivalence

$$q \leqq \phi(p) \text{ if and only if } p \leqq \psi(q). \tag{8}$$

Here, ϕ and ψ are order-reversing (contravariant), and both $\psi \circ \phi$ and $\phi \circ \psi$ are closure operators. It is a curious fact that Galois connexions of mixed type, in spite of their frequent occurrence, have not been paid much attention.

Let now $Q = E$ be an extension of P, ϕ the inclusion mapping. P is called $\mathscr{C}$-faithful in E and E a $\mathscr{C}$-faithful extension of P if $\phi : P \to E$ is a $\mathscr{C}$-homomorphism, i.e., if

$$P \cap (x] \in \mathscr{C} \tag{9}$$

for each $x \in E$. If, in particular, E is a join-extension, we may simply write

$$\dot{E} \subset \mathscr{C}. \tag{10}$$

Hence the $\mathscr{C}$-extension ($\dot{E} = \mathscr{C}$) is the largest $\mathscr{C}$-faithful joint-extension. As a special case, the normal or Dedekind-MacNeille completion i.e. the least join-completion, with the canonical image

$$\mathscr{N}(P)$$

consisting of all intersections of principal ideals, is characterized as the only $\mathscr{N}(P)$-faithful join-completion of P ([11]). In particular, since P is $\mathscr{N}(P)$-faithful in each meet-extension ([11]) and since the normal completion is, in fact, also a meet-completion, one gets Banaschewski's characterization ([3]) of the normal completion as the only join- and meet-completion of P.

2. Ideal-completions

For another type of completions, the above maximal property has been observed by Doctor [5]. Let $\mathscr{F}$ be an arbitrary family of subsets of P, $\mathscr{F} \subset \mathscr{P}(P)$. A subset $J \subset P$ is $\mathscr{F}$-join-closed in P provided that the following holds:

(11) for each $F \in \mathscr{F}$, if $F \subset J$ and $x = \sup_P F$, then $x \in J$. An $\mathscr{F}$-ideal of P is an $\mathscr{F}$-join-closed lower end.

The family of all $\mathscr{F}$-ideals will be denoted

$$\mathscr{I}_{\mathscr{F}} \text{ or } \mathscr{I}_{\mathscr{F}}(P).$$

It is the canonical image of a join-completion E of P which will be called the $\mathscr{F}$-ideal-completion. E.g., $\mathscr{L}(P)$ represents such an ideal-completion: one may take $\mathscr{F} = \varnothing$. Or choosing as $\mathscr{F}$ the family of all k-small subsets i.e. all sets $F \subset P$ such that $|F| < k$, where k is an (infinite regular) cardinal number, we get the k-ideal-completion studied in [13] with the canonical image

$$\mathscr{I}_k(P).$$

If we choose $k > |P|$, i.e. $\mathscr{F} = \mathscr{P}(P)$, we get the least ideal-completion, whose canonical image we might denote

$$\mathscr{I}_\infty(P).$$

The elements of $\mathscr{I}_k(P)$ or $\mathscr{I}_\infty(P)$ are the k-join-closed lower ends (k-ideals) of the completely join-closed lower ends respectively. If P is not complete, the least ideal-completion may still be properly larger than the least join-completion. In this case ($\mathscr{I}_\infty(P) \neq \mathscr{N}(P)$), the latter will be no $\mathscr{F}$-ideal-completion at all. One can show, however, that for each join-completion $\mathscr{C}$, there is a least $\mathscr{F}$-ideal completion larger than $\mathscr{C}$; one may, for that matter, taker the largest family $\mathscr{F}$ such that $\mathscr{C} \subset \mathscr{I}_{\mathscr{F}}$.

Now, as the notion of $\mathscr{C}$-homomorphisms corresponds to general join-extensions, there is a natural property of mappings connected with ideal-completions, or rather with the generating families $\mathscr{F}$ themselves. We call the mapping $\phi : P \to Q$ $\mathscr{F}$-join-preserving provided that the following holds:

(12) for each $F \in \mathscr{F}$, if $x = \sup_P F$, then $\phi(x) = \sup_Q \phi(F)$.

NOTE: if $\mathscr{F}$ contains all sets $\{p', p\}$ such that $p' \leqq p$, then each $\mathscr{F}$-preserving mapping is order-preserving and if $\mathscr{F}$ is just that family, the converse is also true. If, in particular, $\mathscr{F}$ is the family of all k-small subsets, we call ϕ k-join-preserving, completely join-preserving if k is large enough, i.e. $\mathscr{F} = \mathscr{P}(P)$. And P is $\mathscr{F}$-join-faithful, k-join-faithful completely, join-faithful in E or E such a faithful extension of P if the inclusion mapping has the corresponding preservation property. The connection with the $\mathscr{C}$-homomorphisms defined above is established by

THEOREM 1. *Let $\phi: P \to Q$ be a mapping between partially ordered sets, $\mathscr{F} \subset \mathscr{P}(P)$. Then the following statements are equivalent:*

(i) *ϕ is order-preserving and $\mathscr{F}$-join-preserving;*

(ii) *ϕ is an $\mathscr{I}_{\mathscr{F}}$-homomorphisms, i.e. the inverse image of any principal ideal is an $\mathscr{F}$-ideal.*

The proof is left to the reader. Note that in most practical applications, the first condition of (i) can be disregarded.

Theorem 1 is an obvious analogue of the characterization of continuous (limit-preserving) mappings by inverse images. One would certainly dislike missing this tool in topology. It is a curious fact, however, that analogues like the very general one above have not been paid much attention to in partially ordered sets and lattices, maybe because one insisted on the algebraic rather than analytic character of lattice theory.

As its analogue in topology, Theorem 1 has many applications. For example the $\mathscr{F}$-ideal-completion is now the largest $\mathscr{F}$-join-faithful join-extension of P. This is the aforementioned result due to Doctor [5, Chapter I, Theorem 5]. In particular, one may replace $\mathscr{F}$ by k. In particular, the least ideal-completion is the largest completely join-faithful join-extension.

As a second application, Theorem 1 makes quite clear that the usual description of the normal completion as a completely join-and meet-faithful completion (complete extension) fails to characterize it. For as stated above, P is $\mathscr{N}(P)$-faithful, hence — Theorem 1 — completely join-faithful in each meet-extension. Dually, P becomes completely meet-faithful in each join-extension, in particular both completely join-and meet-faithful in the least ideal-completion — which may differ from the normal completion.

For another application, observe that the partially ordered set P is complete iff $\dot{P} = \mathscr{N}(P)$; in this case, one even has $\dot{P} = \mathscr{I}_{\infty}(P)$. So Theorem 1 immediately yields the old result, due to Pickert [8] and others, that $\phi : P \to Q$, where at least P is complete, has a right adjoint iff ϕ is completely join-preserving. Without the completeness of P, the necessary and sufficient condition for ϕ to have a right adjoint (the "adjoint functor theorem") becomes less attractive.

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3. The universal property of join-completions

We are now ready to prove

THEOREM 2. *Let $\mathscr{C}$ be the canonical image of a join-completion of P. Let E be an extension of P. Then the following statements are equivalent:*

(i) *E is the $\mathscr{C}$-completion of P.*

(ii) *E is a $\mathscr{C}$-faithful completion of P. Moreover, for each complete lattice F and each $\mathscr{C}$-homomorphism $\phi_0 : P \to F$ there is exactly one completely join-preserving extension $\phi : E \to F$.*

According to the first condition of (ii), P is just a completion of P. That P is join-dense in it, is not stated: it follows.

PROOF. (i) $\Rightarrow$ (ii): Since $\dot{E} = \mathscr{C}$, P is $\mathscr{C}$-faithful in E and E is complete. Consider now a complete lattice F and a $\mathscr{C}$-homomorphism $\phi_0 : P \to F$. Using the completeness of E and F, we define mappings $\phi : E \to F$ and $\psi : F \to E$ as follows:

$$\phi(x) = \sup_F \phi_0(P \cap (x]) = \sup_F \{\phi_0(P) \mid p \in P, p \leqq x\}, \tag{13}$$

$$\psi(y) = \sup_E \phi_0{}^{-1}((y]) = \sup_E \{p \mid p \in p, \phi_0(P) \leqq y\}, \tag{14}$$

for each $x \in E$, $y \in F$. ϕ extends ϕ_0. For ϕ_0, being order-preserving, preserves the maximum, and $x \in P$ is the maximum of $P \cap (x]$. We are now going to show that (ϕ, ψ) is an adjoint situation between E and F:

$$\phi(x) \leqq y \text{ if and only if } x \leqq \psi(y) \tag{15}$$

for each $x \in E$, $y \in F$. Suppose $\phi(x) \leqq y$ and $p \in P$, $p \leqq x$, by (13), $\phi_0(p) \leqq \phi(x) \leqq y$ whence $p \leqq \psi(y)$ by (14). This being true for each $p \in P \cap (x]$, $x \leqq \psi(y)$ since P is join-dence in E. For the proof of the other half of (15), note that $\phi_0{}^{-1}((y]) \in \mathscr{C}$ since ϕ_0 is a $\mathscr{C}$-homomorphism. But $\mathscr{C} = \dot{E}$, so using the natural isomorphism between E and $\dot{E}$ and (14), we get

$$\phi_0{}^{-1}((y)] = P \cap (\psi(y)]. \tag{16}$$

Suppose now that $x \leqq \psi(y)$ and $p \in P$, $p \leqq x$. Then $p \leqq \psi(y)$, so by (16), $\phi(p) \leqq y$. This being true for each $p \leqq P$, $p \leqq x$, $\phi(x) \leqq y$ by (13). This proves the existence of the completely join-preserving extension ϕ. The uniqueness is trivial since P is join-dense in E.

(ii) $\Rightarrow$ (i): Consider the $\mathscr{C}$-completion F of P. As just shown, F has the universality property (ii) as has, by hypothesis, E. So there is a (unique) isomorphism $\phi : E \to F$ extending the inclusion $\phi_0 : P \to F$, whence E is also a (model of the) $\mathscr{C}$-completion of P.

The special case $\mathscr{C} = \mathscr{L}(P)$ is also contained in [12, Theorem 1.2]; (ii) here simply says that E is complete (the inclusion of P into E being order-preserving anyway), and each order-preserving mapping into a complete lattice can be uniquely extended. For this simple special case, one does not need Galois connexions. All one has to know is the join in $\mathscr{L}(P)$, which is set-theoretical union. Likewise, we obtained our result for $\mathscr{C} = \mathscr{I}_k(P)$ for those two special types of partially ordered sets P which provided us with sufficient information about the joins in $\mathscr{I}_k(P)$ ([13, Theorems 1.6, 2.4]); again we did not need Galois connexions. For the k-ideal-completion of a k-join-semilattice P, there is a universality property somewhat stronger than (ii): each order-preserving mapping $\phi_0 : P \to F$ can be extended to a $\phi : E \to F$ preserving joins of k-directed sets ([13, Theorem 1.4]), the latter becoming completely join-preserving once ϕ_0 is k-join-preserving. For $\mathscr{C} = \mathscr{I}_\infty(P)$, (ii) says that P is completely join-faithful in E, and each completely join-preserving mapping $\phi_0 : P \to F$ can be extended to a mapping $\phi : E \to F$ with the same quality, — the point is that P might not be complete, but E is.

In view of the distinction between $\mathscr{I}_\infty(P)$ and $\mathscr{N}(P)$, we may look at the latter. Note that an $\mathscr{N}(P)$-homomorphism is in general still better than a completely join-preserving mapping (on the other hand worse than a $\dot{P}$-homomorphism). For $\mathscr{C} = \mathscr{N}(P)$, (ii) may now be replaced by

(ii$_{\mathscr{N}}$) *E is complete. Moreover for each complete lattice F, $\mathscr{N}$ each order-embedding $\phi_0\, P \to F$ can be extended to an order-embedding $\phi: E \to F$.*

An orded-embedding is, of course, an order-isomorphism onto the exact image, as is, e.g., the inclusion of P into E. Note that no uniqueness of ϕ is claimed anymore. (i) $\Rightarrow$ (ii$_{\mathscr{N}}$) has been proven by Aumann [1, Theorem 2.2.3]. The proof of the converse, similar to (ii) $\Rightarrow$ (i) above inasmuch as it uses Aumann's result, is left to the reader.

4. Extensions of Galois connexions

The main application of Theorem 2 is

THEOREM 3. *Let E be a join-completion of P. Let F be an arbitrary completion of Q, and let (ϕ_0, ψ_0) be an adjoint situation between P and Q. Suppose $\phi_0 : P \to F$ is an $\dot{E}$-homomorphism. Then there is exactly one adjoint situation between E and F, (ϕ, ψ), extending (ϕ_0, ψ_0).*

PROOF. Again the uniqueness follows from the fact that P is join-dense in E: there is at most one completely join-preserving mapping $\phi : E \to F$ extending ϕ_0. The existence of this ϕ has been shown above, also the right adjoint ψ of ϕ has been determined. All that is left to show is that ψ extends ψ_0. So let $q \in Q$. Then $\phi(\psi_0(q)) = \phi_0(\psi_0(q)) \leqq q$, whence $\psi_0(q) \leqq \psi(q)$ by (15). Let now $p \in P$, $p \leqq \psi(q)$. Then $\phi_0(p) = \phi(p) \leqq q$ by (15) so $p \leqq \psi_0(q)$. This being true for each $p \in P$, $p \leqq \psi(q)$, $\psi(q) \leqq \psi_0(q)$ since P is join-dense in E. So $\psi(q) = \psi_0(q)$, completing the proof.

As a sample for the application of this rather general theorem, let us consider the special case $\dot{E} = \mathscr{I}_k(P)$.

COROLLARY 1. *Let E be the k-ideal-completion of P, let F be a k-join-faithful completion of Q. Then each adjoint situatin* (ϕ_0, ψ_0) *between P and Q can be uniquely extended to an adjoint situation* (ϕ, ψ) *between E and F.*

For $\phi_0 : P \to Q$ is completely join-preserving anyway, and since the inclusion of Q into F is k-join-preserving, $\phi_0 : P - F$ is k-join-preserving, i.e. an $\mathscr{I}_k(P)$-homomorphism.

For a more symmetric special case, one may assume that F is also the k-ideal-completion of Q.

As another interesting special case of Theorem 3, we mention

COROLLARY 2. *Let E be a join-completion of P, F a meet-completion of Q. Then each adjoint situation* (ϕ_0, ψ_0) *between P and Q can be uniquely extended to an adjoint situation* (ϕ, ψ) *between E and F.*

For since Q is meet-dense in F, Q is $\mathscr{N}(Q)$-faithful in F. But $\phi_0 : P \to Q$ is a $\dot{P}$-homomorphism, hence an $\mathscr{N}(P)$-homomorphism. It follows that $\phi_0 : P \to F$ is an $\mathscr{N}(P)$-homomorphism, hence an $\dot{E}$-homomorphism since $\mathscr{N}(P) \subset \dot{E}$.

Dualizing F and Q, one obtains a nice symmetric form of Corollary 2, for Galois connexions in the sense of Ore. For an independent proof of the latter, one may adjust the definition of ψ to that of ϕ:

$$\phi(x) = \inf_F\{\phi_0(p) \mid p \in P, p \leqq x\}, \tag{17}$$

$$\psi(y) = \inf_E\{\psi_0(q) \mid q \in Q, q \leqq y\}. \tag{18}$$

There are results extending Galois connexions of Ore type between partially ordered sets P, Q to their power sets $\mathscr{P}(P)$, $\mathscr{P}(Q)$ (cf. Everett [6], Aumann [2], for related results also Raney [7]).

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