ON THE MOTION OF PARTICLES IN GENERAL RELATIVITY THEORY

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1. Introduction. The gravitational field manifests itself in the motion of bodies. Therefore the problem of determining the motion of such bodies from the field equations alone is of fundamental importance. This problem was solved for the first time some ten years ago and the equations of motion for two particles were then deduced [1]. A more general and simplified version of this problem was given shortly thereafter [2].

Mr. Lewison pointed out to us, that from our approximation procedure, it does not follow that the field equations can be solved up to an arbitrarily high approximation. This is indeed true. We believe that the present work not only removes this difficulty, but that it gives a new and deeper insight into the problem of motion. From the logical point of view the present theory is considerably simpler and clearer than the old one. But as always, we must pay for these logical simplifications by prolonging the chain of technical argument.

The subject matter is presented here from the beginning and the knowledge of previous work is not assumed. To facilitate the reading for those who have studied the previous papers we use here essentially the same notation as before.

Let us start with some general remarks.

All attempts to represent matter by an energy-momentum tensor are unsatisfactory and we wish to free our theory from any particular choice of such a tensor. Therefore we shall deal here only with gravitational equations in empty space, and matter will be represented by singularities of the gravitational field.

In Newtonian mechanics, particles are represented as singularities of a scalar field φ, which satisfies Laplace's equation everywhere outside the singularities. Because the classical equation is linear, the field can be decomposed into partial fields, each part due to a single particle. Each particle *is* in a field due to all other particles. The theory is completed by the equation of motion, that is by putting the acceleration equal to the negative gradient of the field, the proportionality factor being a universal constant. Thus classical physics postulates the equations of motion independently of the field laws. The masses of the sources of the field are assumed to be independent of time. The laws of motion are supposed to be valid in an inertial system. Therefore space-time appears as an independent physical entity. The conceptual weakness of such a space-time background in the classical theory was already recognized by Newton.

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If we compare this state of affairs with that in general relativity theory, in its original formulation, we see striking similarities and differences. Laplace's equation

$$\Delta\varphi = 0$$

is replaced by the gravitational equation

$$R_{kl} = 0,$$

which, however, unlike the classical equation, satisfies the general relativity principle. The classical principle of inertia becomes in relativity theory the principle of the geodesic line valid for a particle with infinitely small mass. True enough, the difficulty with the inertial system disappears in relativity theory, as does the independent physical reality of space-time. Yet the equations of motion still appear independently of the field equations.

Our aim is to investigate to what extent the field equations *alone* contain the equations of motion of particles; also to develop a method that will allow us to find these equations of motion up to an arbitrary approximation.

Let us start with a simple remark: a *linear* law always means that the motion of singularities is arbitrary. If to a world-line of a singularity with mass m_1 there belongs a field $F_{(1)}$ and if to a world-line of a singularity with mass m_2 there belongs a field $F_{(2)}$, then the superposition of these two fields, that is $F_{(1)}+F_{(2)}$ is also a solution of the linear field equations. In such a solution the same two world-lines would appear together that before appeared singly. Therefore the field with its linear laws cannot imply any interaction between the singularities. Thus only non-linear field equations can provide us with equations of motion because only non-linearity can express the interaction between singularities.

But the argument cannot be reversed. Non-linearity is necessary but not sufficient for the equations of motion to follow from the field equations.

The reason why the gravitational field equations do provide us with equations of motion lies not in their non-linear character alone, but also in the fact that these equations are not independent from each other Indeed, among the ten components four are free, this being due to the freedom of choice in the coordinate system. The ten equations are valid, so to speak, only for six effective functions. They would be inconsistent were it not for the four (Bianchi) identities that they satisfy. This must be so for every relativistic system of equations derived from a variational principle. These identities are (besides the non-linearity) responsible for the *equations of motion being determined by the field equations.*

The ideas leading to the equations of motion are not easy and are mutually interwoven.

One of the essential ideas in this paper is the treatment of gravitational equations by a "new approximation method." In it we treat space and time differently. We regard the changes of the field in time as small compared with those in space. Only then do we arrive at a consistent, manageable set of

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equations that can be solved step by step. This idea is not new and was contained in the previous papers.

The other important idea is the deduction of the equations of motion, which are *ordinary* differential equations, from the field equations which are *partial* differential equations. This idea, treated here differently than in the previous papers, leads to the use of surface integrals taken around the singularities of the field. These surface integrals will depend only on the motion of the singularities and not on the shape of the surface.

These and other ideas will be treated in detail in this paper. To make them clear we have decided to delegate all the more tedious calculations to the Appendices. (If we refer, for example, to **A.4**, this means the Appendix belonging to Sec. **4**.) But even so, many straightforward but long calculations had to be omitted. This is especially true for the calculations that lead beyond Newtonian motion. We included here a short section on this subject, just for the sake of completeness. But, as in [1], so here we have to refer those who would like to see the full calculations to the manuscript which is deposited at the Institute for Advanced Study.

Finally we should like to thank Mr. Lewison for his critical study of our previous papers, and Mr. Schild for a careful and critical reading of this manuscript.

2. Notations: the gravitational equations. Since in the greater part of our work, we shall have to separate space and time, our notation will not be the usual four-dimensional one. We make the conventions: Latin indices take the values 1, 2, 3, and they refer to space co-ordinates only. Greek indices refer to both space and time, running over the values 0, 1, 2, 3. Repetition of indices implies summation.

The expression

(2.1) $$g_{\mu\nu|\sigma} \text{ etc. stands for } \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \text{ etc.}$$

At infinity the gravitational field takes the Galilean values $\eta_{\mu\nu}$, that is:

(2.2) $$\eta_{mn} = -\delta_{mn};\ \eta_{0m} = 0\ ;\ \eta_{00} = 1.$$

We write:

(2.3) $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu};\ g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu},$$

where $h_{\mu\nu}$ represents the deviation of space-time from flat space and it is not assumed to be small.

The $h^{\mu\nu}$ can be calculated as functions of $h_{\mu\nu}$ by means of the relation

(2.4) $$g_{\mu\sigma}g^{\mu\nu} = \delta_\sigma^{\ \nu}.$$

It turns out to be convenient to replace the h's by γ's which are their linear combinations:

(2.5) $$\gamma_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\,\eta_{\mu\nu}\eta^{\sigma\rho}h_{\sigma\rho}\,,$$

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or more explicitly:

(2.6) $$\gamma_{00} = \tfrac{1}{2}\, h_{00} + \tfrac{1}{2}\, h_{ss}$$

(2.7) $$\gamma_{0n} = h_{0n}$$

(2.8) $$\gamma_{mn} = h_{mn} - \tfrac{1}{2}\, \delta_{mn} h_{ss} + \tfrac{1}{2}\, \delta_{mn} h_{00}.$$

This replacement is, of course, not very important but it does simplify the calculations.

Thus we can, throughout, replace the h's by the γ's. The equations of the gravitational field for empty space,

(2.9) $$R_{\mu\nu} = 0,$$

can be written (see **A.2**) in the following way:

(2.10) $$\Phi_{00} + 2\Lambda_{00} = 0$$

(2.11) $$\Phi_{0n} + 2\Lambda_{0n} = 0$$

(2.12) $$\Phi_{mn} + 2\Lambda_{mn} = 0,$$

where:

(2.13) $$\Phi_{00} = -\gamma_{00|ss}$$

(2.14) $$\Phi_{0m} = -\gamma_{0m|ss} + \gamma_{0s|sm}$$

(2.15) $$\Phi_{mn} = -\gamma_{mn|ss} + \gamma_{ms|ns} + \gamma_{ns|ms} - \delta_{mn}\gamma_{rs|rs}$$

and:

(2.16) $$2\Lambda_{00} = \gamma_{sr|sr} + 2\Lambda'_{00}$$

(2.17) $$2\Lambda_{0m} = \gamma_{ms|s0} - \gamma_{00|m0} + 2\Lambda'_{0m}$$

(2.18) $$2\Lambda_{mn} = -\gamma_{0m|0n} - \gamma_{0n|0m} + 2\delta_{mn}\gamma_{0s|0s} + \gamma_{mn|00} - \delta_{mn}\gamma_{00|00} + 2\Lambda'_{mn}.$$

In these formulae, all the linear terms are written out explicitly, while $\Lambda'_{\mu\nu}$ stands for all the non-linear terms in the γ's. The division of the linear expressions into those belonging to $\Phi_{\mu\nu}$ and those belonging to $\Lambda_{\mu\nu}$ may seem artificial at this moment. In anticipation of further development, we shall remark here, that, in the actual approximation procedure, by which we shall solve the gravitational equations, these linear terms collected in $\Lambda_{\mu\nu}$ will behave like the non-linear terms.

3. Lemma. We mentioned in the introduction that the differential equations of motion will be derived by forming surface integrals. The technique of calculating such surface integrals will reappear many times in this paper and it is based on a lemma to which we shall refer as *the* lemma. Here we shall give its formulation and its proof.

We have a set of functions:

(3.1) $$F_{(\alpha a \cdots)kl}.$$

It is immaterial whether these functions of x^{μ} have tensorial character, or not. The bracketed indices are Greek, or Latin, and they will not play any role in our argument. But we do assume that these functions are skew-symmetric in the indices k, l:

(3.2) $$F_{(\ldots)kl} = -F_{(\ldots)lk}.$$

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We now form an integral

$$\int_{(S_2)} F_{(\ldots)kl|l}\, n_k\, dS \tag{3.3}$$

over an *arbitrary two-dimensional closed* surface that does not pass through the singularities of the field. In (3.3)

$$n_k = \cos\,(x^k, \vec{n}) \tag{3.4}$$

are the components of the "normal unit" vector to the surface. The words "normal," and "unit" are used in the conventional sense to designate the corresponding functions of the co-ordinates, which are implied by these terms in Euclidean geometry. They have nothing to do with any particular metric.

Our lemma is:

$$\int_{(S_2)} F_{(\ldots)kl|l}\, n_k\, dS = 0. \tag{3.5}$$

We see that the integral (3.3) is certainly independent of the shape of the surface, because

$$F_{(\ldots)kl|lk} = 0, \tag{3.6}$$

and because of Green's theorem. We can also write the integral (3.3) in the form

$$\int_{(S_2)} \mathrm{curl}_n \vec{A}\, dS, \tag{3.7}$$

where

$$F_{(\ldots)23} = A_1;\; F_{(\ldots)31} = A_2;\; F_{(\ldots)12} = A_3.$$

But (3.7) and therefore (3.3) can be changed, by Stokes' theorem, into a line integral over the rim of the surface. If the surface is closed, the rim is of zero length. Therefore, our lemma as expressed by (3.5) is proved.

4. Surface integrals. We treat particles of matter as singularities of the field. Let us assume p particles and the knowledge of their world lines. Thus we denote by

$$\overset{s}{\xi}{}^k(x^0);\; s = 1, 2, 3, \ldots, p, \tag{4.1}$$

the world-line of the sth singularity. Here and later, the index written on the top will always label the particular singularity.

The gravitational field, that is the γ's, will depend on the x^μ's but also on the ξ's and their time derivatives. The equations that the γ's fulfill are

$$\Phi_{\mu\nu} + 2\Lambda_{\mu\nu} = 0. \tag{4.2}$$

At an arbitrary moment x^0, let us surround the sth singularity, and it alone, by a closed surface. Then:

$$\int^{s} (\Phi_{\mu k} + 2\Lambda_{\mu k}) n_k\, dS = 0, \tag{4.3}$$

where the s over the integral indicates here, and later too, that the integral is to be taken on a two-dimensional surface surrounding the sth singularity and it alone.

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We shall show that

$$\int^{s} \Phi_{\mu k} n_k dS = 0. \tag{4.4}$$

Indeed it follows from the definition (2.14) and (2.15) of $\Phi_{\mu k}$ that it can be written in the following form:

$$\Phi_{\mu k} = F_{(\mu)kl|l} \tag{4.5}$$

$$F_{(\mu)kl} = \gamma_{\mu l|k} - \gamma_{\mu k|l} - \delta_{\mu k}\gamma_{lr|r} + \delta_{\mu l}\gamma_{kr|r}. \tag{4.6}$$

But $F_{(\mu)kl}$ is skew-symmetric in k and l. Therefore (4.4) is fulfilled. From it and from (4.3) we deduce:

$$\int^{s} 2\Lambda_{\mu k} n_k dS = 0. \tag{4.7}$$

Also, because of the structure of $\Phi_{\mu k}$ we easily verify:

$$\Phi_{\mu n|n} = 0, \tag{4.8}$$

therefore also:

$$\Lambda_{\mu n|n} = 0. \tag{4.9}$$

Equation (4.9) tells us that no surface integral of the form (4.7) can depend on the shape of the surface. But equation (4.7) tells us more; namely, that such an integral vanishes.

The $4p$ surface integrals in (4.7) can give us no relation between the space co-ordinates of the field, because the surface is entirely arbitrary. They can only give us relations between the co-ordinates of the singularities and their time derivatives. Thus we may have at most $4p$ differential equations. Anticipating the later development, we may remark here that these equations will determine $3p$ functions of time

$$\overset{s}{\xi}{}^k(x^0),$$

that is, the motion of singularities.

5. The method of approximation. The problem before us is to solve our field equations and to deduce the equations of motion. This we shall do by a new approximation procedure. Let us assume a function $\varphi(x^\mu, \lambda)$ developed into a power series in the parameter λ (for small λ):

$$\varphi(x^\mu, \lambda) = \lambda^0 \underset{0}{\varphi} + \lambda^1 \underset{1}{\varphi} + \lambda^2 \underset{2}{\varphi} + \cdots = \sum_{l=0}^{\infty} \lambda^l \underset{l}{\varphi}. \tag{5.1}$$

The indices below indicate the *order* (l in λ^l is always the exponent, not the index).

If the function φ varies quickly in space, but slowly with x^0, then we are justified in not treating all its derivatives in the same fashion. The derivatives with respect to x^0 will be of a higher order than space derivatives. We can formalize the procedure by introducing an *auxiliary time* τ,

$$\tau = x^0 \lambda, \tag{5.2}$$

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so that derivatives with respect to τ can be treated on the same footing as the space derivatives:

(5.3) $$\varphi_{|0} = \frac{\partial \varphi}{\partial x^0} = \frac{\partial \varphi}{\partial \tau} \lambda = \lambda \varphi_{,0}.$$

We conclude: the "stroke differentiation" of a quantity with respect to x^0, can be replaced by the "comma differentiation" with respect to τ if the power of λ with which this quantity is associated is simultaneously raised by one. To express this explicitly we use numbers under zeros, written after the comma, e.g.:

(5.4) $$\lambda^{2l} \underset{2l}{\gamma}_{mn|0} = \lambda^{2l+1} \underset{2l}{\gamma}_{mn,\underset{1}{0}} \text{ or: } \lambda^{2l} \underset{2l}{\gamma}_{mn|00} = \lambda^{2l+2} \underset{2l}{\gamma}_{mn,\underset{2}{00}}.$$

From now on, all differentiations will be with respect to (τ, x^1, x^2, x^3) and they will be denoted by commas:

(5.5) $$\gamma_{\ldots|s} = \gamma_{\ldots,s}\,;\quad \gamma_{\ldots|0} = \lambda \gamma_{\ldots,\underset{1}{0}}.$$

Thus we shall develop all functions that appear in the field equations in power series in λ. We start with the γ's in the following way:

(5.6) $$\begin{cases} \gamma_{00} = \lambda^2 \underset{2}{\gamma}_{00} + \lambda^4 \underset{4}{\gamma}_{00} + \lambda^6 \underset{6}{\gamma}_{00} + \cdots \\ \gamma_{0m} = \lambda^3 \underset{3}{\gamma}_{0m} + \lambda^5 \underset{5}{\gamma}_{0m} + \cdots \\ \gamma_{mn} = \lambda^4 \underset{4}{\gamma}_{mn} + \lambda^6 \underset{6}{\gamma}_{mn} + \cdots \end{cases}$$

Why do we start with different powers of λ? This is an assumption, but it can be justified heuristically. Assuming for a moment the usual energy momentum tensor for matter, we have, for a quasi-stationary field, approximately:

(5.7) $$\begin{cases} \Delta \gamma_{00} = -2\rho \\ \Delta \gamma_{0m} = -2\rho \dfrac{dx^m}{d\tau} \lambda \\ \Delta \gamma_{mn} = -2\rho \dfrac{dx^m}{d\tau} \dfrac{dx^n}{d\tau} \lambda^2, \end{cases}$$

therefore

(5.8) $$\gamma_{mn} \sim \lambda \gamma_{0m} \sim \lambda^2 \gamma_{00},$$

and it is pure convention that we start with λ^2 for γ_{00}.

The other question suggested by (5.6) is: why do we omit the odd powers of λ in the developments of γ_{00}, γ_{mn}, and the even powers in γ_{0m}? Indeed, we could have introduced all powers in (5.6). A more thorough investigation shows that our choice (5.6) means that what we are doing here is similar to the procedure in electro-magnetic theory when we take not the retarded, but the half-retarded plus half-advanced potentials [3].

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All the functions that will appear later are gained from the γ's by summation, multiplication, differentiation. Thus to every component, the following rule applies throughout: Any component having an $\left\{\begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right\}$ number of zero suffixes will have only $\left\{\begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right\}$ powers of λ in its expansion.

6. Field equations and the approximation method. We go back to the field equations

$$\Phi_{\mu\nu} + 2\Lambda_{\mu\nu} = 0 \tag{6.1}$$

into which we introduce the γ's in their power-series development. Thus the (00) equation in (6.1) can be written:

$$\sum_l \lambda^{2l}(\underset{2l}{\Phi_{00}} + 2\underset{2l}{\Lambda_{00}}) = 0. \tag{6.2}$$

Now we cut up (6.2), and the other field equations, into equations for each approximation step. We write them down in the following form:

$$\underset{2l-2}{\Phi_{00}} + 2\underset{2l-2}{\Lambda_{00}} = 0 \tag{6.3a}$$

$$\underset{2l-1}{\Phi_{0m}} + 2\underset{2l-1}{\Lambda_{0m}} = 0 \tag{6.3b}$$

$$\underset{2l}{\Phi_{mn}} + 2\underset{2l}{\Lambda_{mn}} = 0. \tag{6.3c}$$

Let us analyse more closely the structure of (6.3). Remembering (2.13) to (2.15) we can write more explicitly:

$$\underset{2l-2}{\Phi_{00}} = -\underset{2l-2}{\gamma_{00}}{}_{,rr} \tag{6.4a}$$

$$\underset{2l-1}{\Phi_{0m}} = -\underset{2l-1}{\gamma_{0m}}{}_{,rr} + \underset{2l-1}{\gamma_{0r}}{}_{,mr} \tag{6.4b}$$

$$\underset{2l}{\Phi_{mn}} = -\underset{2l}{\gamma_{mn}}{}_{,rr} + \underset{2l}{\gamma_{mr}}{}_{,nr} + \underset{2l}{\gamma_{nr}}{}_{,mr} - \delta_{mn}\underset{2l}{\gamma_{rs}}{}_{,rs} \tag{6.4c}$$

and:

$$2\underset{2l-2}{\Lambda_{00}} = \underset{2l-2}{\gamma_{rs}}{}_{,rs} + 2\underset{2l-2}{\Lambda'_{00}} \tag{6.5a}$$

$$2\underset{2l-1}{\Lambda_{0m}} = -\underset{2l-2}{\gamma_{00}}{}_{,\underset{1}{0}m} + \underset{2l-2}{\gamma_{mr}}{}_{,\underset{1}{0}r} + 2\underset{2l-1}{\Lambda'_{0m}} \tag{6.5b}$$

$$\left\{\begin{aligned} 2\underset{2l}{\Lambda_{mn}} = & -\underset{2l-1}{\gamma_{0m}}{}_{,\underset{1}{0}n} - \underset{2l-1}{\gamma_{0n}}{}_{,\underset{1}{0}m} + 2\delta_{mn}\underset{2l-1}{\gamma_{0r}}{}_{,\underset{1}{0}r} \\ & + \underset{2l-2}{\gamma_{mn}}{}_{,\underset{2}{00}} - \delta_{mn}\underset{2l-2}{\gamma_{00}}{}_{,\underset{2}{00}} + 2\underset{2l}{\Lambda'_{mn}}. \end{aligned}\right. \tag{6.5c}$$

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Let us now assume that:

(6.6a) $$\underset{2}{\gamma_{00}} \quad \cdots \quad \underset{2l-4}{\gamma_{00}}$$

(6.6b) $$\underset{3}{\gamma_{0m}} \quad \cdots \quad \underset{2l-3}{\gamma_{0m}}$$

(6.6c) $$\underset{4}{\gamma_{mn}} \quad \cdots \quad \underset{2l-2}{\gamma_{mn}}$$

are all known. Then $\underset{2l-2}{\gamma_{00}}$ can be found from (6.3a). Indeed $\underset{2l-2}{\Lambda_{00}}$ contains only terms already known, since $\underset{2l-2}{\gamma_{mn}}$ is known and $\underset{2l-2}{\Lambda'_{00}}$ is non-linear and can therefore depend only on the known γ's. The same is true for (6.3b) and (6.3c). The unknown functions are contained in Φ's; the known functions in the Λ's. The $\underset{2l-2}{\gamma_{00}}$, already found from (6.3a), appears as a known function in $\underset{2l-1}{\Lambda_{0m}}$. Similarly $\underset{2l-1}{\gamma_{0m}}$ found from (6.3b) appears as known in $\underset{2l}{\Lambda_{mn}}$. Indeed we see now the reasons for our division of linear terms.

Thus our equations (6.3), if solved, will give us

(6.7) $$\underset{2l-2}{\gamma_{00}}, \ \underset{2l-1}{\gamma_{0m}}, \ \underset{2l}{\gamma_{mn}},$$

and if such a procedure converges, we can determine the field to any approximation we wish.

The important question to consider is: are the equations (6.3) always solvable?

7. The divergence condition. We go back to our equations (6.3). The first of them, that is

(7.1) $$\underset{2l-2}{\Phi_{00}} + 2\underset{2l-2}{\Lambda_{00}} = 0$$

is, because of (6.4a) and (6.5a), a Poisson equation, where $\underset{2l-2}{\Lambda_{00}}$ is known. There is no difficulty in integrating this equation and finding $\underset{2l-2}{\gamma_{00}}$. Next we have (6.3b), and because of (6.4b), we see:

(7.2) $$\underset{2l-1}{\Phi_{0m,m}} = 0.$$

Thus the next three equations can be integrated only if

(7.3) $$\underset{2l-1}{\Lambda_{0m,m}} = 0.$$

But $\underset{2l-1}{\Lambda_{0m}}$ is already known. Therefore we must be sure that our procedure leads us to $\underset{2l-1}{\Lambda_{0m}}$ satisfying (7.3). Similarly the last six equations (6.3c) lead us because of

(7.4) $$\underset{2l}{\Phi_{mn,n}} = 0$$

to the integrability condition:

(7.5) $$\underset{2l}{\Lambda_{mn,n}} = 0.$$

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We shall prove that (7.3) and (7.5) are satisfied, if the field equations are satisfied in *all the previous* approximations.

The tensor

(7.6) $$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R$$

satisfies the Bianchi identity

(7.7) $$G^{\mu}_{\nu|\mu} + \left\{ {\alpha \atop \alpha\beta} \right\} G_{\nu}{}^{\beta} - \left\{ {\beta \atop \nu\alpha} \right\} G_{\beta}{}^{\alpha} = 0.$$

We assume that all field equations up to the order $(2l - 2)$ are satisfied, that is including

$$\underset{2l-2}{\Phi_{00}} + 2\underset{2l-2}{\Lambda_{00}} = 0.$$

We know, that putting $\Phi_{\mu\nu} + 2\Lambda_{\mu\nu} = 0$ is equivalent to putting $R_{\mu\nu} = 0$. From **A.2** follows:

(7.8) $$\Phi_{\mu\nu} + 2\Lambda_{\mu\nu} = -2\,(R_{\mu\nu} - \tfrac{1}{2}\,\eta_{\mu\nu}\eta^{\alpha\beta}R_{\alpha\beta}),$$

which means, that our $\Phi_{\mu\nu} + 2\Lambda_{\mu\nu}$ are a linear combination of the $R_{\mu\nu}$. Thus, if our field equations are satisfied, then we have:

(7.9) $$\begin{cases} \underset{2}{G_{00}} = \underset{4}{G_{00}} = \cdots = \underset{2l-2}{G_{00}} = 0 \\ \underset{3}{G_{0m}} = \underset{5}{G_{0m}} = \cdots = \underset{2l-3}{G_{0m}} = 0 \\ \underset{2}{G_{mn}} = \underset{4}{G_{mn}} = \cdots = \underset{2l-2}{G_{mn}} = 0. \end{cases}$$

Let us write down the *zero* Bianchi identity of the order $(2l - 1)$. From the left-hand side of (7.7) we have, putting $\nu = 0$, the following linear terms:

(7.10) $$-\underset{2l-1}{G_{0m}}{}_{,m} + \underset{2l-2}{G_{00}}{}_{,\underset{1}{0}}.$$

The non-linear part contains the products of the G's and the γ's. But because of (7.9), both the non-linear part of the Bianchi identity and the second expression in (7.10) vanish. Thus the *zero* Bianchi identity, together with the field equations give:

(7.11) $$\underset{2l-1}{G_{0m}}{}_{,m} = 0.$$

Because of (7.8), (7.6) and (7.2) this means:

(7.12) $$\underset{2l-1}{\Lambda_{0m}}{}_{,m} = 0.$$

Going on to the next approximation step, let us now assume that besides (7.9), we have also:

(7.13) $$\underset{2l-1}{G_{0m}} = 0.$$

Putting into Bianchi identity (7.7) $\nu = m$, we have in the $2l$ order, because of (7.9) and (7.13):

(7.14) $$\underset{2l}{G_{mn}}{}_{,n} = 0$$

and therefore because of (7.4), (7.8):

(7.15) $$\underset{2l}{\Lambda_{mn}}{}_{,n} = 0.$$

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Thus the divergence conditions are satisfied in each approximation step, though not identically. They are satisfied because of the Bianchi identities and because of the previous field equations.

8. The surface condition and the equations of motion. We now approach the most essential part of our argument. We are faced with the task of solving the following system of equations:

(8.1a) $$\underset{2l-2}{\Phi_{00}} + 2\underset{2l-2}{\Lambda_{00}} = 0$$

(8.1b) $$\underset{2l-1}{\Phi_{0m}} + 2\underset{2l-1}{\Lambda_{0m}} = 0$$

(8.1c) $$\underset{2l}{\Phi_{mn}} + 2\underset{2l}{\Lambda_{mn}} = 0.$$

We know that because of the Bianchi identities and because (as we assumed) similar equations had been solved in the previous approximations, we have

(8.2) $$\underset{2l-1}{\Lambda_{0m}}{}_{,m} = 0; \quad \underset{2l}{\Lambda_{mn}}{}_{,n} = 0.$$

Let us also remember, that there is no difficulty in solving (8.1a) which is a Poisson equation. But what about (8.1b) and (8.1c)?

Before we return to this fundamental question, we wish to discuss the *start* of our approximation procedure which determines the character of our calculations.

In (8.1) we put $l = 2$ and write the first two equations explicitly:

(8.3a) $$\underset{2}{\gamma_{00}}{}_{,ss} = 0$$

(8.3b) $$-\underset{3}{\gamma_{0m}}{}_{,ss} + \underset{3}{\gamma_{0s}}{}_{,ms} = \underset{2}{\gamma_{00}}{}_{,\underset{1}{0m}}.$$

The character of the entire solution will depend on the choice of the harmonic function we take as the solution of (8.3a). As we are interested in solutions representing particles, we shall write:

(8.4) $$\begin{cases} \underset{2}{\gamma_{00}} = 2\varphi \; ; \; \varphi = \sum\limits_{s=1}^{p} \left\{ -2\underset{2}{\overset{s}{m}}\overset{s}{\psi} \right\} \\ \overset{s}{\psi} = [(x^k - \overset{s}{\xi^k})(x^k - \overset{s}{\xi^k})]^{-\frac{1}{2}} = (\overset{s}{r})^{-1}. \end{cases}$$

Here $\overset{s}{r}$ is the "distance" in space of a point from the sth singularity.

We leave it undecided, for the moment, whether $\underset{2}{\overset{s}{m}}$ is a function of time, or a constant. Now we introduce this $\underset{2}{\gamma_{00}}$ into (8.3b) and again obtain three equations for the three functions $\underset{3}{\gamma_{0m}}$. But is (8.3b) always solvable? True, the divergence of both sides vanishes. But this is not sufficient. The surface integral of the left-hand side of (8.3b) vanishes, as follows from the lemma. But then the surface integral of the right-hand side of (8.3b) must vanish too.

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If we calculate the surface integral around each singularity, we find (see **A.4**) that it vanishes only if

$$\frac{d}{d\tau}\big(\overset{s}{\underset{2}{m}}\big) = \overset{s}{\underset{2}{m}}{}_{,\underset{1}{0}} = \overset{s}{\underset{3}{\dot{m}}} = 0, \tag{8.5}$$

that is if the $\underset{2}{m}$'s do not depend on time. This is so, because

$$\overset{s}{\psi}_{,0} = -\overset{s}{\psi}_{,k}\,\overset{s}{\dot{\xi}}{}^{k}\,; \quad \left(\dot{\xi}^{k} = \frac{d\xi^{k}}{d\tau}\right) \tag{8.6}$$

and because only expressions proportional to r^{-2} can give a contribution to the surface integral. Thus, going back to (8.4), we have to assume that

$$\overset{1}{\underset{2}{m}},\ \overset{2}{\underset{2}{m}},\ \overset{3}{\underset{2}{m}},\ \ldots,\ \overset{p}{\underset{2}{m}} \tag{8.7}$$

are constant.

These constants (8.7) can be positive or negative. We shall assume that $\overset{s}{\underset{2}{m}}$ are positive. Indeed, by taking the first particle and removing all others, we see that $\overset{1}{\underset{2}{m}}$ is its *gravitational mass*, since for large r the field is that of a particle with gravitational mass $\overset{1}{\underset{2}{m}}$. This is the same constant of integration that appears in the Schwarzschild solution, since our field for one particle is that of a Schwarzschild singularity when r is large. *Thus we shall have to exclude from our solution negative gravitational masses. But then we must also exclude dipoles and poles of higher order.*

Yet if we try to solve (8.1) we see (the details will be presented later) that we cannot do so without adding certain poles and dipoles to $\underset{2l-2}{\gamma_{00}}$. This we shall have to do, in order to insure the integrability of (8.1) in each approximation. But then the solution of the total field will contain dipoles which are not allowed, since they represent physically meaningless solutions. We shall have to remove them *after* the total field has been calculated. This can be done by *restricting the motion of particles.* That is, the condition that the dipole field vanishes will give us $3p$ ordinary differential equations for the motion of p particles. Thus the motion is undetermined in the approximation procedure. It becomes determined after the approximation procedure is finished and the dipole fields are removed.

In practice, we find solutions both for the field and for the equations of motion only to a certain approximation, say $2n$. We obtain the equations of motion to the $2n$ approximation, by removing all the dipole fields to such an approximation.

Although we have developed our field equations with respect to an arbitrary parameter λ, this λ can be absorbed by the actual equations of motion through the change of scale in $\overset{s}{\underset{2}{m}}$ and τ, so that λ is absent from the final form of the equations.

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We have given a general outline of our treatment. Turning to the details, let us see why (8.1) will not, generally, be integrable. We know, from the contents of Sec. **4**, particularly from (4.4) that the surface integrals of the Φ functions vanish. Although this was proved for the total field it is equally true in each approximation step, since the proof made use only of the structure of the Φ's, which is the same for the total field, as for the field in each approximation. Thus we have:

$$(8.8) \qquad \int^s \underset{2l-1}{\Phi_{0r}} n_r \, dS = 0 \; ; \; \int^s \underset{2l}{\Phi_{mr}} n_r \, dS = 0.$$

But then our equations (8.1) can be self-consistent, only if we have:

$$(8.9) \qquad \int^s 2\underset{2l-1}{\Lambda_{0r}} n_r \, dS = 0 \; ; \; \int^s 2\underset{2l}{\Lambda_{mr}} n_r \, dS = 0.$$

But the Λ's in (8.1) are already known; they are functions of the *known* field calculated in the previous approximation steps. Therefore we can calculate the integrals (8.9) and find whether they vanish or not.

At this point it is convenient to introduce a new notation. Because of (8.2), the surface integrals (8.9) will not depend on the shape of the surface, but only on the singularities and their motion. Thus the surface integrals, even if they do not vanish, can be functions of τ only.

We write:

$$(8.10) \qquad \frac{1}{4\pi} \int^s 2\underset{2l-1}{\Lambda_{0r}} n_r \, dS = \underset{2l-1}{\overset{s}{C_0}}(\tau) = \underset{2l-1}{\overset{s}{C_0}}$$

$$(8.11) \qquad \frac{1}{4\pi} \int^s 2\underset{2l}{\Lambda_{mr}} n_r \, dS = \underset{2l}{\overset{s}{C_m}}(\tau) = \underset{2l-1}{\overset{s}{C_m}}$$

and assume that we have calculated the C's. If they vanish identically, and if they vanish always as we proceed with our approximation, then our equations are self-consistent.

Let us assume, however, that the C's in (8.10) and (8.11) are *not* zero. Then (8.1b, c) cannot be solved. There is no difficulty in solving (8.1a). This equation is of the form

$$(8.12) \qquad \underset{2l-2}{\gamma_{00}}{}_{,rr} = 2\underset{2l-2}{\Lambda_{00}},$$

where the right-hand side is known. We see that the solution of this equation is determined only up to an additive harmonic function. Thus we can add to any solution either single "poles" or "poles" and "dipoles."

By adding single poles we can insure the integrability of (8.1b). Then by adding dipoles we can insure the integrability of (8.1c). We could have done all that in *one* step, adding poles and dipoles, but the division into two steps makes for a simpler presentation.

After finding $\underset{2l-2}{\gamma_{00}}$ from (8.12), we calculate $\underset{2l-1}{\overset{s}{C_0}}$ and, in general, find $\underset{2l-1}{\overset{s}{C_0}} \neq 0$.

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We then replace in (8.1b):

$$\text{(8.13)} \qquad \underset{2l-2}{\gamma_{00}} \text{ by } \underset{2l-2}{\gamma_{00}} - \sum_s 4 \underset{2l-2}{\overset{s}{m}} \overset{s}{\psi}$$

where $\underset{2l-2}{\overset{s}{m}}$ are certain functions of time to be determined soon, and ψ's are the functions defined in (8.4). Of course this change in $\underset{2l-2}{\gamma_{00}}$ induces a change in $\underset{2l-1}{\overset{s}{C_0}}$. Indeed,

$$\text{(8.14)} \qquad \begin{cases} 2\underset{2l-1}{\Lambda_{0m}} \quad \text{changes now to} \\ 2\underset{2l-1}{\Lambda_{0m}} + \sum_s (4 \underset{2l-2}{\overset{s}{m}} \overset{s}{\psi})_{,\underset{1}{0m}}, \end{cases}$$

as follows from (6.5b) because $\underset{2l-2}{\gamma_{00}}$ appears in $\underset{2l-1}{\Lambda_{0m}}$ only as $-\underset{2l-2}{\gamma_{00}}{}_{,\underset{1}{m0}}$. Now obviously the old surface integral

$$\text{(8.15)} \qquad \frac{1}{4\pi} \int^s 2\underset{2l-1}{\Lambda_{0r}} n_r \, dS = \underset{2l-1}{\overset{s}{C_0}}$$

changes into **A.4**

$$\text{(8.16)} \qquad \underset{2l-1}{\overset{s}{C_0}} - 4 \underset{2l-1}{\overset{s}{\dot{m}}},$$

therefore it can be made zero by choosing

$$\text{(8.17)} \qquad 4 \underset{2l-1}{\overset{s}{\dot{m}}} = \underset{2l-1}{\overset{s}{C_0}}.$$

Thus by adding a pole we can insure the integrability of (8.1b). The next step is to insure the integrability of (8.1c). Thus we assume that $\underset{2l-2}{\gamma_{00}}$, $\underset{2l-1}{\gamma_{0m}}$ are known, that (8.1b) is integrable and we have once more to return to $\underset{2l-2}{\gamma_{00}}$ looking for a different solution of (8.1a) so as to insure the integrability of (8.1c) without destroying the integrability of (8.1b).

We replace now our $\underset{2l-2}{\gamma_{00}}$ (containing the additional poles) by

$$\text{(8.18)} \qquad \underset{2l-2}{\gamma_{00}} - \sum_{s=1}^{p} \underset{2l-2}{\overset{s}{S_r}} \overset{s}{\psi}_{,r}.$$

These are additional *dipole* solutions, and we assume that no other dipole expressions are contained in $\underset{2l-2}{\gamma_{00}}$. Again the S_r are functions of τ only, to be determined later. The $\underset{2l-2}{\gamma_{00}}$ now contain the single pole solutions so as to enforce the integrability of (8.1b). We can easily see what change in $\underset{2l-1}{\gamma_{0m}}$ is induced by (8.18). The answer is, that $\underset{2l-1}{\gamma_{0m}}$ changes into

$$\text{(8.19)} \qquad \underset{2l-1}{\gamma_{0m}} - \sum_s (\underset{2l-2}{\overset{s}{S_m}} \overset{s}{\psi})_{,\underset{1}{0}}.$$

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Indeed, if the old $\underset{2l-1}{\gamma_{0m}}$ satisfies the original equation (8.1b):

$$\underset{2l-1}{\gamma_{0m,ss}} - \underset{2l-1}{\gamma_{0s,ms}} = \underset{2l-2}{\gamma_{ms}}\underset{1}{{}_{,0s}} - \underset{2l-2}{\gamma_{00}}\underset{1}{{}_{,m0}} + 2\underset{2l-1}{\Lambda'_{0m}}, \tag{8.20}$$

then $\underset{2l-2}{\gamma_{00}}, \underset{2l-1}{\gamma_{0m}}$ *with* the additional expressions written out in (8.18) and (8.19) satisfy the equation too. This is so, because $2\underset{2l-1}{\Lambda'_{0m}}$ being non-linear can contain neither $\underset{2l-2}{\gamma_{00}}$ nor $\underset{2l-1}{\gamma_{0m}}$. Therefore the addition of dipoles does not affect the integrability of (8.1b).

Now the last and decisive step: we replace in (8.1c) $\underset{2l-2}{\gamma_{00}}, \underset{2l-1}{\gamma_{0m}}$, by the new expressions according to (8.18) and (8.19) and adjust the S's so that the surface integrals will vanish identically. This requires a somewhat more lengthy calculation.

Written out explicitly, equation (8.1c) is:

$$\begin{aligned} &\underset{2l}{\gamma_{mn,ss}} - \underset{2l}{\gamma_{ms,ns}} - \underset{2l}{\gamma_{ns,ms}} + \delta_{mn}\underset{2l}{\gamma_{rs,rs}} \\ &= -\underset{2l-1}{\gamma_{0m}}\underset{1}{{}_{,0n}} - \underset{2l-1}{\gamma_{0n}}\underset{1}{{}_{,0m}} + 2\delta_{mn}\underset{2l-1}{\gamma_{0r}}\underset{1}{{}_{,0r}} + \underset{2l-2}{\gamma_{mn}}\underset{2}{{}_{,00}} - \delta_{mn}\underset{2l-2}{\gamma_{00}}\underset{2}{{}_{,00}} + 2\underset{2l}{\Lambda'_{mn}} \\ &= 2\underset{2l}{\Lambda_{mn}}. \end{aligned} \tag{8.21}$$

We introduce into (8.21)

$$\underset{2l-2}{\gamma_{00}} - \sum_s \underset{2l-2}{\overset{s}{S_r}\,\overset{s}{\psi}}{}_{,r} \tag{8.22}$$

$$\underset{2l-1}{\gamma_{0m}} - \sum_s \Big(\underset{2l-2}{\overset{s}{S_m}\,\overset{s}{\psi}}\Big)\underset{1}{{}_{,0}} \tag{8.23}$$

for the old $\underset{2l-2}{\gamma_{00}}, \underset{2l-1}{\gamma_{0m}}$. We now obtain new expressions added to the old $\underset{2l}{\Lambda_{mn}}$. The difficulty is, that now the contributions come not only from the linear expressions, but also from Λ'_{mn} which will contain terms of the type $\underset{2l-2}{\gamma_{00}} \cdot \underset{2}{\gamma_{00}}$. The result of the calculations is given in **A.8**, and contains many expressions of which we shall here write only the first three which arise from the linear terms (the others, as we shall see, are unimportant). Instead of the old $2\underset{2l}{\Lambda_{mn}}$ we have:

$$\begin{aligned} &2\underset{2l}{\Lambda_{mn}} \\ &+ \sum_s \Big(\underset{2l}{\overset{s}{\ddot{S}}_m\overset{s}{\psi}}{}_{,n} + \underset{2l}{\overset{s}{\ddot{S}}_n\overset{s}{\psi}}{}_{,m} - \delta_{mn}\underset{2l}{\overset{s}{\ddot{S}}_r\overset{s}{\psi}}{}_{,r}\Big) \\ &+ \cdots \end{aligned} \tag{8.24}$$

where the dots at the end indicate the omitted expressions. As we are here discussing the problem of surface integrals, we are justified in omitting them because they do not give any contribution to the surface integrals. We see

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too, that the expressions written out here have a vanishing divergence, and this is true for the omitted terms also. Calculating the surface integrals (**A.4**), we find that the old surface integral

(8.25) $$\frac{1}{4\pi}\int^{s} 2\underset{2l}{\Lambda}_{mn} n_n \, dS = \overset{s}{\underset{2l}{C}}_m$$

changes into

(8.26) $$\overset{s}{\underset{2l}{C}}_m - \overset{s}{\underset{2l}{\ddot{S}}}_m \, .$$

Therefore it can be made zero, by choosing

(8.27) $$\overset{s}{\underset{2l}{\ddot{S}}}_m = \overset{s}{\underset{2l}{C}}_m \, .$$

Thus we can always, by adding dipole solutions in $\underset{2l-2}{\gamma_{00}}$, force the surface integrals to vanish identically.

By proceeding in this way, we accumulate single poles and dipoles, and the additional expressions in γ_{00} are:

(8.28) $$-\sum_{l} \lambda^{2l-2} \sum_{s=1}^{p} \left(4\overset{s}{\underset{2l-2}{m}}\, \overset{s}{\psi} + \overset{s}{\underset{2l-2}{S}}_r \, \overset{s}{\psi}_{,r} \right).$$

We violated our rule of not introducing dipoles. However, this was done for γ_{00} only. We can, at the end of the approximation procedure, annihilate all these additional dipole expressions by taking

(8.29) $$\sum_{l} \lambda^{2l-2} \overset{s}{\underset{2l-2}{S}}_r = 0.$$

Differentiating this twice, we obtain, because of (8.27):

(8.30) $$\sum_{l} \lambda^{2l} \overset{s}{\underset{2l}{\ddot{S}}}_m = \sum_{l} \lambda^{2l} \overset{s}{\underset{2l}{C}}_m = 0.$$

These are the 3p equations of motion. Thus the motion is determined, if dipole solutions are rejected.

On the other hand, the m's can be calculated from the C_0's according to (8.17). Denoting the total coefficient at $\overset{s}{\psi}$ by $-4\overset{s}{M}$, we have:

(8.31) $$\overset{s}{M} = \lambda^2 \overset{s}{\underset{2}{m}} + \lambda^4 \overset{s}{\underset{4}{m}} + \lambda^6 \overset{s}{\underset{6}{m}} + \cdots$$

where $\overset{s}{\underset{4}{m}}, \overset{s}{\underset{6}{m}}, \ldots$ are functions of the original constants $\overset{s}{\underset{2}{m}}$ and of known functions of the time.

The equations (8.30) and (8.31) will contain only a finite number of terms depending on the order to which we wish to carry out the actual calculations.

9. On the choice of a co-ordinate system. We shall now see that it is possible to simplify our equations through the proper choice of a co-ordinate system. Let us assume that

(9.1) $$\underset{2l-2}{\gamma^{*}}_{00}\, , \; \underset{2l-1}{\gamma^{*}}_{0m}\, , \; \underset{2l}{\gamma^{*}}_{mn}$$

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are solutions of our system (6.3), where the Φ's and Λ's are defined by (6.4) and (6.5). Then we can show that any

$$\underset{2l-2}{\gamma_{00}} = \underset{2l-2}{\gamma^*_{00}}$$

(9.2)
$$\underset{2l-1}{\gamma_{0m}} = \underset{2l-1}{\gamma^*_{0m}} + \underset{2l-1}{a_{0,m}}$$

$$\underset{2l}{\gamma_{mn}} = \underset{2l}{\gamma^*_{mn}} + \underset{2l}{a_{m,n}} + \underset{2l}{a_{n,m}} - \delta_{mn} \underset{2l}{a_{r,r}} + \delta_{mn} \underset{2l-1}{a_0}{}_{,\underset{1}{0}}$$

with $\underset{2l-1}{a_0}$, $\underset{2l}{a_m}$ *arbitrary* are also solutions of our equations. This can be shown just by straightforward substitution in (6.4). A simple calculation shows that all the a's vanish from these equations. Thus we can, at each approximation step, impose four conditions upon the field. Let us choose, as is usually done, the following four co-ordinate conditions:

(9.3a)
$$\underset{2l-2}{\gamma_{00}}{}_{,\underset{1}{0}} - \underset{2l-1}{\gamma_{0r}}{}_{,r} = 0$$

(9.3b)
$$\underset{2l-1}{\gamma_{0m}}{}_{,\underset{1}{0}} - \underset{2l}{\gamma_{mr}}{}_{,r} = 0.$$

Indeed, if γ^* do not satisfy such a condition, then a's can be found that ensure it. The equations for the a's are:

(9.4a)
$$\underset{2l-1}{a_0}{}_{,rr} = \underset{2l-2}{\gamma^*_{00}}{}_{,\underset{1}{0}} - \underset{2l-1}{\gamma^*_{0r}}{}_{,r}$$

(9.4b)
$$\underset{2l}{a_m}{}_{,rr} = \underset{2l-1}{\gamma^*_{0m}}{}_{,\underset{1}{0}} - \underset{2l}{\gamma^*_{mr}}{}_{,r}.$$

With the co-ordinate condition (9.3) our system of equations is considerably simplified. Equations (6.3) now become:

(9.5a)
$$\underset{2l-2}{\gamma_{00}}{}_{,rr} = \underset{2l-4}{\gamma_{00}}{}_{,\underset{2}{00}} + 2\underset{2l-2}{\Lambda'_{00}}$$

(9.5b)
$$\underset{2l-1}{\gamma_{0m}}{}_{,rr} = \underset{2l-3}{\gamma_{0m}}{}_{,\underset{2}{00}} + 2\underset{2l-1}{\Lambda'_{0m}}$$

(9.5c)
$$\underset{2l}{\gamma_{mn}}{}_{,rr} = \underset{2l-2}{\gamma_{mn}}{}_{,\underset{2}{00}} + 2\underset{2l}{\Lambda'_{mn}},$$

which together with the co-ordinate conditions

(9.6a)
$$\underset{2l-2}{\gamma_{00}}{}_{,\underset{1}{0}} - \underset{2l-1}{\gamma_{0r}}{}_{,r} = 0$$

(9.6b)
$$\underset{2l-1}{\gamma_{0m}}{}_{,\underset{1}{0}} - \underset{2l}{\gamma_{mr}}{}_{,r} = 0,$$

now form a symmetrical system of equations, where in (9.5) all the known functions on the right-hand side are at least two orders lower than those on the left.

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The surface integrals that must vanish and which give the equations of motion are:

(9.7a) $$\int^{s}\left(\underset{2l-3}{\gamma}_{0m,\underset{2}{00}} - \underset{2l-2}{\gamma}_{00,\underset{1}{0m}} + 2\underset{2l-1}{\Lambda'}_{0m}\right) n_m\, dS = 0$$

(9.7b) $$\int^{s}\left(\underset{2l-2}{\gamma}_{nm,\underset{2}{00}} - \underset{2l-1}{\gamma}_{n0,\underset{1}{0m}} + 2\underset{2l}{\Lambda'}_{nm}\right) n_m\, dS = 0.$$

We can deduce them from our old formulae, using the lemma, or directly, differentiating (9.6), adding to (9.5) and using the lemma.

If, as in Sec. **8**, we now introduce dipoles in order to satisfy (9.7b), we do not violate (9.6a).

Sometimes it is more convenient to use other co-ordinate conditions. For example, the one used in the actual calculations is:

(9.8a) $$\underset{2l-2}{\gamma}_{00,\underset{1}{0}} - \underset{2l-1}{\gamma}_{0s,s} = 0$$

(9.8b) $$\underset{2l}{\gamma}_{mn,n} = 0.$$

The equations then are:

(9.9a) $$\underset{2l-2}{\gamma}_{00,rr} = 2\underset{2l-2}{\Lambda'}_{00}$$

(9.9b) $$\underset{2l-1}{\gamma}_{0m,rr} = 2\underset{2l-1}{\Lambda'}_{0m}$$

(9.9c) $$\underset{2l}{\gamma}_{mn,rr} = -\underset{2l-1}{\gamma}_{0m,\underset{1}{0n}} - \underset{2l-1}{\gamma}_{0n,\underset{1}{0m}} + \delta_{mn}\underset{2l-2}{\gamma}_{00,\underset{2}{00}} + \underset{2l-2}{\gamma}_{mn,\underset{2}{00}} + 2\underset{2l}{\Lambda'}_{mn}$$
$$= 2\underset{2l}{\Lambda}_{mn}$$

and the surface conditions are:

(9.10a) $$\int^{s}\left(2\underset{2l-1}{\Lambda'}_{0m} - \underset{2l-2}{\gamma}_{00,\underset{1}{0m}}\right) n_m dS = 0$$

(9.10b) $$\int^{s} 2\underset{2l}{\Lambda}_{nm} n_m dS = 0.$$

The question arises: to what extent does the co-ordinate condition influence the equations of motion? We shall return to this problem in the last section and we shall show that the equations of motion to the sixth order do not depend on the choice of the co-ordinate system.

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10. The Newtonian approximation. We shall discuss now the first three equations for $l = 2$. The equations are:

$$\underset{2}{\gamma}_{00,rr} = 0 \tag{10.1}$$

$$\underset{3}{\gamma}_{0m,rr} = 0 \tag{10.2}$$

$$\underset{4}{\gamma}_{nm,rr} = 2\underset{4}{\Lambda}_{nm}. \tag{10.3}$$

The co-ordinate conditions that we accept are:

$$\underset{3}{\gamma}_{0r,r} - \underset{2}{\gamma}_{00,\underset{1}{0}} = 0. \tag{10.4}$$

$$\underset{4}{\gamma}_{mr,r} = 0. \tag{10.5}$$

The explicit form of $\underset{4}{\Lambda}_{mn}$ is given in **A.10**.

The character of our entire solution will depend essentially upon the choice of the harmonic function we take as the solution of (10.1). As we are interested in solutions representing particles, we shall write:

$$\begin{aligned} &\underset{2}{\gamma}_{00} = 2\varphi; \quad \varphi = \sum_{s=1}^{p}\left\{-2\underset{2}{\overset{s}{m}}\overset{s}{\psi}\right\} \\ &\overset{s}{\psi} = \left[\left(x^r - \overset{s}{\xi}{}^r\right)\left(x^r - \overset{s}{\xi}{}^r\right)\right]^{-\frac{1}{2}} = \left(\overset{s}{r}\right)^{-1} \end{aligned} \tag{10.6}$$

From (10.2) we see that $\underset{3}{\gamma}_{0m}$ is a harmonic function too, which must, however, satisfy the co-ordinate condition also. From (10.4) we have:

$$\underset{3}{\gamma}_{0r,r} = \underset{2}{\gamma}_{00,\underset{1}{0}} = -\sum_{s}\left\{4\underset{2}{\overset{s}{m}}\,\overset{s}{\psi}_{,r}\,\underset{1}{\dot{\overset{s}{\xi}}}{}^r\right\}. \tag{10.7}$$

The constant $\underset{2}{\overset{s}{m}}$, which we identify with the gravitational mass of the particles is assumed to be positive. Therefore the exclusion of dipoles, together with the field equations and the co-ordinate condition determine uniquely $\underset{3}{\gamma}_{0n}$:

$$\underset{3}{\gamma}_{0n} = \sum_{s} 4\underset{2}{\overset{s}{m}}\overset{s}{\psi}\underset{1}{\dot{\overset{s}{\xi}}}{}^n \tag{10.8}$$

To this $\underset{3}{\gamma}_{0n}$ we could add, according to (9.2) the gradient of any function and in this way obtain a general solution. But as our entire procedure consists in employing only rational functions of $\left(x^r - \overset{s}{\xi}{}^r\right)$, any such addition would introduce new singularities (not of the character of a single pole), or a non-Galilean field at infinity. Thus we should regard $\underset{3}{\gamma}_{0n}$ in (10.8) as characterizing the problem of particles, regardless of whether we introduce the co-ordinate condition (10.4) or not.

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Just for the sake of simplicity, let us now restrict our consideration to *two* particles and write (omitting the indices below $\overset{s}{m}$, φ, f, g):

$$(10.9)\qquad \begin{cases} \varphi = f + g \\ f = -\,2\overset{1}{m}\overset{1}{\psi}\ ;\ g = -\,2\overset{2}{m}\overset{2}{\psi} \\ \overset{1}{\xi}{}^r = \eta^r;\ \overset{2}{\xi}{}^r = \zeta^r. \end{cases}$$

The next step then, since the surface integral (9.10a) vanishes for $l = 3$, because

$$(10.10)\qquad \int^s \left(2\underset{3}{\Lambda'}_{0m} - \underset{2}{\overset{}{\gamma}}_{\underset{1}{00,0m}}\right) n_m dS = -\int^s \underset{2}{\gamma}_{\underset{1}{00,0m}} n_m dS = 0,$$

is to determine

$$(10.11)\qquad \begin{aligned} \underset{4}{\overset{1}{C}}_m &= \frac{1}{4\pi}\int^1 2\underset{4}{\Lambda}_{mr}\, n_r\, dS \\ \underset{4}{\overset{2}{C}}_m &= \frac{1}{4\pi}\int^2 2\underset{4}{\Lambda}_{mr}\, n_r\, dS. \end{aligned}$$

If we wish to finish our approximation procedure here, the equations of motion up to the fourth, or as we shall call it, the Newtonian approximation, are:

$$(10.12)\qquad \underset{4}{\overset{1}{C}}_m = 0\ ;\ \underset{4}{\overset{2}{C}}_m = 0.$$

All we have to do now is to calculate the surface integrals, according to the method outlined in **A.4**. The result of this particular calculation is given in **A.10**. It is:

$$(10.13)\qquad \begin{cases} \underset{4}{\overset{1}{C}}_m(\tau) = 4\overset{1}{m}\left\{\ddot{\eta}^m + \dfrac{1}{2}\,\tilde{g}_{,m}\right\} = 0 \\ \underset{4}{\overset{2}{C}}_m(\tau) = 4\overset{2}{m}\left\{\ddot{\zeta}^m + \dfrac{1}{2}\,\tilde{f}_{,m}\right\} = 0 \\ \tilde{g}_{,m} = g_{,m} \text{ for } x^s = \eta^s \\ \tilde{f}_{,m} = f_{,m} \text{ for } x^s = \zeta^s. \end{cases}$$

The form (10.13) is actually independent of the variables x^s. In the last equations we see that $\tilde{g}_{,m}$, say, is obtained by differentiating g with respect to x^s and then by replacing x^s by η^s. But the result will be the same if we *first* replace x^s by η^s and *then* differentiate with respect to η^s or ζ^s. Thus:

$$\tilde{g}_{,s} = \frac{\partial g(r)}{\partial \eta^s} = -\frac{\partial g(r)}{\partial \zeta^s}$$

$$(10.14)\qquad g(r) = -\frac{2\overset{2}{m}}{r}\ ;\ r^2 = (\eta^s - \zeta^s)(\eta^s - \zeta^s).$$

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We can, therefore, think of our equations of motion as involving the differentiation of functions depending only on the position of singularities, as is characteristic of the theories based on the concept of action at a distance. Indeed, we see that our equations are precisely the Newtonian equations of motion, deduced here as the first approximation from the field equations. The treatment of p particles (instead of two) does not add any new difficulties if we deal with the Newtonian approximation only.

11. Transition to the next approximation. We wish to go now beyond the Newtonian approximation. But then we must calculate $\underset{4}{\gamma}_{mn}$, since $\underset{6}{\Lambda}_{mn}$ depends on $\underset{4}{\gamma}_{mn}$. The characteristic feature of this method is that generally, if we wish to find the equations of motion to the $2l$ approximation (inclusive) then we do not need to calculate $\underset{2l}{\gamma}_{mn}$, because $\underset{2l}{C}_m$ does not contain it. But now, if we wish to go one step further we must find $\underset{4}{\gamma}_{mn}$ for which the equations are:

$$\underset{4}{\gamma}_{mn,rr} = 2\underset{4}{\Lambda}_{mn}. \tag{11.1}$$

This is "the transition step" that we have to take before proceeding to the next approximation. These equations are integrable only if we *do* assume Newtonian motion. Otherwise we would have to add dipoles. Yet if we wish to proceed *only* to the next approximation we may assume Newtonian motion and additional expressions induced by the dipole fields are not necessary.

If in (11.1) we assume Newtonian motion, then (11.1) can be integrated, because the surface integral of $\underset{4}{\Lambda}_{mn}$ vanishes then. But if we do this, we introduce Newtonian motion into $\underset{6}{\Lambda}_{mn}$. This is admissible because any difference between $\underset{6}{\Lambda}$ calculated this way and $\underset{6}{\Lambda}$ calculated with the proper motion is of order $\underset{8}{\Lambda}$. Thus since we do not propose to go beyond $\underset{6}{\Lambda}$ we may ignore the additional dipole fields. It is for this reason that the previous special calculations in [1] were correct, but the general theory was not.

We shall now solve

$$\underset{4}{\gamma}_{mn,rr} = 2\underset{4}{\Lambda}_{mn} = -\underset{3}{\gamma}_{0m,\underset{1}{0}n} - \underset{3}{\gamma}_{0n,\underset{1}{0}m} + 2\delta_{mn}\varphi_{,00} - 2\varphi\varphi_{,mn} - \varphi_{,m}\varphi_{,n} + \tfrac{3}{2}\delta_{mn}\varphi_{,s}\varphi_{,s} \tag{11.2}$$

assuming the Newtonian equation of motion, i.e. (10.13).

We can ignore the dipole expressions because we are interested only in the equations of motion to the next approximation. But, for the same reason, we are interested only in those expressions in $\underset{4}{\gamma}_{mn}$ which give a contribution to the corresponding surface integral of $\underset{6}{\Lambda}_{mn}$.

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An inspection of $\underset{6}{\Lambda}_{mn}$ (**A.12**) shows that we need only the knowledge of $\underset{4}{\gamma}_{mn}$ in the neighbourhood of the singularities, and we may ignore in it the terms which do not become infinite as $\overset{s}{r} \to 0$, since the surface integral due to these terms must vanish (see **A.12**). On the other hand $\underset{4}{\gamma}_{ss}$ which also appears in $\underset{6}{\Lambda}$ should and will be calculated in the entire space.

In the equation (11.2) we have, on the right-hand side "cross products," that is, products belonging to different singularities. Because of them (11.2) can only be integrated in the neighbourhood of the first singularity, say. The expression arising from the second singularity can be expanded into a power series near the first singularity. Retaining all the expressions that may give some contribution to the surface integral and those only, we have in the neighbourhood of the *first* singularity:

$$(11.3)\qquad \begin{cases} \underset{4}{\gamma}_{mn} = \left\{ f\left[(x^n - \eta^n)\dot{\eta}^m + (x^m - \eta^m)\dot{\eta}^n - \delta_{mn}(x^s - \eta^s)\dot{\eta}^s\right]\right\}_{,0} \\ \qquad + \left\{ g\left[(x^n - \zeta^n)\dot{\zeta}^m + (x^m - \zeta^m)\dot{\zeta}^n - \delta_{mn}(x^s - \zeta^s)\dot{\zeta}^s\right]\right\}_{,0} \\ \qquad + \tfrac{7}{4}\overset{1}{r}{}^2 f_{,m} f_{,n} + \tfrac{7}{4}\overset{2}{r}{}^2 g_{,m} g_{,n} \\ \qquad - f_{,m}(x^n - \eta^n)\tilde{g} \\ \qquad + \alpha_{mn} f + \beta_{mn} g. \end{cases}$$

Here only the expression

$$- f_{,m}(x^n - \eta^n)\tilde{g}\ ;\ \tilde{g} = g \text{ for } x^s = \eta^s,$$

is due to the interaction terms. The two last expressions are the additive harmonic functions (dipoles are excluded) and they are determined by the co-ordinate condition

$$(11.4)\qquad \underset{4}{\gamma}_{mr,r} = 0.$$

The result is:

$$(11.5)\qquad \begin{cases} \alpha_{mn} = 2\dot{\eta}^m\dot{\eta}^n + \delta_{mn}\tilde{g} \\ \beta_{mn} = 2\dot{\zeta}^m\dot{\zeta}^n + \delta_{mn}\tilde{f}, \\ \tilde{f} = f(r);\ \tilde{g} = g(r);\ r^2 = (\eta^s - \zeta^s)(\eta^s - \zeta^s). \end{cases}$$

But, let us say once more, that all this is true only if the Newtonian motion is assumed.

Finally, as we mentioned before, $\underset{4}{\gamma}_{rr}$ can be calculated rigorously. The result is:

$$(11.6)\qquad \underset{4}{\gamma}_{rr} = -2\overset{1}{m}\overset{1}{r}_{,00} - 2\overset{2}{m}\overset{2}{r}_{,00} + \tfrac{7}{4}\varphi^2 + \alpha f + \beta g.$$

Here the α and β are determined so that near the singularity (11.6) will be consistent with (11.3) for $m = n = r$. The result is:

$$(11.7)\qquad \begin{aligned} \alpha &= 2\dot{\eta}^s\dot{\eta}^s + \tfrac{1}{2}\tilde{g} \\ \beta &= 2\dot{\zeta}^s\dot{\zeta}^s + \tfrac{1}{2}\tilde{f}. \end{aligned}$$

Thus our transition steps are accomplished.

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12. Beyond the Newtonian approximation. We write down the next field equations:

(12.1a) $$\underset{4}{\gamma}_{00,rr} = 2\underset{4}{\Lambda}_{00} = -\tfrac{3}{2}\varphi_{,r}\varphi_{,r}$$

(12.1b) $$\underset{5}{\gamma}_{0m,rr} = 2\underset{5}{\Lambda'}_{0m} = \varphi_{,s}\gamma_{0s,m} - \varphi_{,sm}\gamma_{0s} - 3\varphi_{,0}\varphi_{,m}$$

(12.1c) $$\underset{6}{\gamma}_{mn,rr} = 2\underset{6}{\Lambda}_{mn}.$$

The explicit expressions for $\underset{6}{\Lambda}_{mn}$ are quoted in **A.12**. The solution of (12.1a) is simple:

(12.2) $$\underset{4}{\gamma}_{00} = -\tfrac{3}{4}\varphi^2 - 4\overset{1}{\underset{4}{m}}\overset{1}{\psi} - 4\overset{2}{\underset{4}{m}}\overset{2}{\psi}.$$

As we know from the general theory, the arbitrary harmonic functions have to be determined in such a way as to make (12.1b) self-consistent, that is, the corresponding surface integral must vanish.

The co-ordinate conditions, are here, as before,

(12.3a) $$\underset{5}{\gamma}_{0r,r} - \underset{4}{\gamma}_{00,\underset{1}{0}} = 0$$

(12.3b) $$\underset{6}{\gamma}_{mr,r} = 0.$$

Because of this, the conditions for solvability of (12.1b, c) are:

(12.4a) $$\frac{1}{4\pi}\int^{s}\left\{2\underset{5}{\Lambda'}_{0m} - \underset{4}{\gamma}_{00,\underset{1}{0}m}\right\}n_m\,dS = 0$$

(12.4b) $$\frac{1}{4\pi}\int^{s}2\underset{6}{\Lambda}_{mr}n_r\,dS = 0.$$

We have in (12.4a) the equations that determine $\underset{4}{m}$. The result of evaluating the surface integrals in (12.4a), (see **A.12**) is:

(12.5) $$\left\{\begin{aligned} \overset{1}{\underset{4}{m}} &= \frac{1}{2}\overset{1}{m}\left\{\dot{\eta}^s\dot{\eta}^s + \frac{1}{2}\tilde{g}\right\} = \frac{1}{2}\left(\overset{1}{m}\dot{\eta}^s\dot{\eta}^s - \overset{1}{m}\overset{2}{m}\frac{1}{r}\right) \\ \overset{2}{\underset{4}{m}} &= \frac{1}{2}\overset{2}{m}\left\{\dot{\zeta}^s\dot{\zeta}^s + \frac{1}{2}\tilde{f}\right\} = \frac{1}{2}\left(\overset{2}{m}\dot{\zeta}^s\dot{\zeta}^s - \overset{1}{m}\overset{2}{m}\frac{1}{r}\right) \\ \overset{1}{m} &= \overset{1}{\underset{2}{m}}\,;\ \overset{2}{m} = \overset{2}{\underset{2}{m}}\,;\ r^2 = (\eta^s - \zeta^s)(\eta^s - \zeta^s). \end{aligned}\right.$$

The next step, after the self-consistency of (12.1b) has been insured is to calculate the $\underset{5}{\gamma}_{0s}$. We need them, because they enter into the next surface

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integral. Including only relevant terms that can influence the surface integral we find near the first singularity:

$$(12.6)\quad \left\{\begin{aligned} \gamma_{\underset{5}{0m}} = & - \tfrac{7}{4}\, \overset{1}{r} f_{,m} f_{,r}\, \dot{\eta}^r + \tfrac{3}{4} f^2\, \ddot{\eta}^m \\ & + \tfrac{3}{2}\, (x^s - \eta^s)(\dot{\eta}^s - \dot{\zeta}^s)\, f \tilde{g}_{,m} \\ & - (x^m - \eta^m) f \tilde{g}_{,s} (\dot{\eta}^s - \dot{\zeta}^s) \\ & + \tfrac{1}{2}\, (x^s - \eta^s) f_{,m} \dot{\zeta}^s \left\{\tilde{g} + \tilde{g}_{,r}(x^r - \eta^r)\right\} \\ & + \tfrac{1}{2}\, (x^s - \eta^s) \left\{ f \tilde{g}_{,s} \dot{\zeta}^m + f_{,m} \tilde{g} \dot{\zeta}^s \right\} \\ & + a_{0m} f. \end{aligned}\right.$$

Again a_{0m} is determined from the co-ordinate condition (12.3a) and the result is:

$$(12.7)\qquad a_{0m} = - \dot{\eta}^s \dot{\eta}^s \dot{\eta}^m + \tilde{g} \dot{\eta}^m - \tilde{g} \dot{\zeta}^m .$$

Now the scene is set for the last and most difficult calculation:

$$(12.8)\qquad \overset{1}{\underset{6}{C}}_m = \frac{1}{4\pi} \int^1 2 \underset{6}{\Lambda}_{mn}\, n_m\, dS.$$

Some remarks about this calculation are made in **A.12**, and partial results given. We obtain:

$$\overset{1}{\underset{6}{C}}_m = - 4 \overset{1}{m} \overset{2}{m} \left\{ \left[\dot{\eta}^s \dot{\eta}^s + \frac{3}{2} \dot{\zeta}^s \dot{\zeta}^s - 4 \dot{\eta}^s \dot{\zeta}^s - 4 \frac{\overset{2}{m}}{r} - 5 \frac{\overset{1}{m}}{r} \right] \frac{\partial}{\partial \eta^m} \left(\frac{1}{r} \right) \right.$$

$$\left. + \left[4 \dot{\eta}^s (\dot{\zeta}^m - \dot{\eta}^m) + 3 \dot{\eta}^m \dot{\zeta}^s - 4 \dot{\zeta}^s \dot{\zeta}^m \right] \frac{\partial}{\partial \eta^s} \left(\frac{1}{r} \right) + \frac{1}{2} \frac{\partial^3 r}{\partial \eta^s \partial \eta^r \partial \eta^m} \dot{\zeta}^s \dot{\zeta}^r \right\}.$$

Thus the equation of motion belonging to this stage of approximation is:

$$(12.10)\qquad \lambda^4 \overset{1}{\underset{4}{C}}_m + \lambda^6 \overset{1}{\underset{6}{C}}_m = 0.$$

We can now re-absorb the λ's by substituting new units for τ and $\overset{1}{m}$, $\overset{2}{m}$:

old $\tau = \lambda \cdot$ new τ; old mass $= \lambda^{-2} \cdot$ new mass.

Preserving the old symbols for the new units we have for the equations of motion of the first particle:

$$(12.11)\quad \begin{aligned} \ddot{\eta}^m - \overset{2}{m} \frac{\partial (1/r)}{\partial \eta^m} = \overset{2}{m} & \left\{ \left[\dot{\eta}^s \dot{\eta}^s + \tfrac{3}{2} \dot{\zeta}^s \dot{\zeta}^s - 4 \dot{\eta}^s \dot{\zeta}^s - 4 \frac{\overset{2}{m}}{r} - 5 \frac{\overset{1}{m}}{r} \right] \frac{\partial}{\partial \eta^m} (1/r) \right. \\ & + \left[4 \dot{\eta}^s (\dot{\zeta}^m - \dot{\eta}^m) + 3 \dot{\eta}^m \dot{\zeta}^s - 4 \dot{\zeta}^s \dot{\zeta}^m \right] \frac{\partial}{\partial \eta^s} (1/r) \\ & \left. + \frac{1}{2} \frac{\partial^3 r}{\partial \eta^s \partial \eta^r \partial \eta^m} \dot{\zeta}^s \dot{\zeta}^r \right\}. \end{aligned}$$

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The equations of motion for the other particle are obtained by replacing

$$\overset{1}{m}, \overset{2}{m}, \eta, \zeta, \text{ by } \overset{2}{m}, \overset{1}{m}, \zeta, \eta,$$

respectively.

These are the equations of motion of two particles. They can be integrated and conclusions concerning perihelion motion of a double star can be drawn from them [5]. The entire method can also be adapted for the case of a charged particle in an electromagnetic field [4].

13. The equations of motion and the co-ordinate condition. The contents of the last three sections are not new. Its presentation, however, is different than that given before in [1] and [2], since it has been adjusted to the new theory. There is one more question that we wish to answer and which we did not treat before. It is possible to do so only now after the general theory has been perfected. We ask: To what extent do the equations of motion as formulated in (12.11) depend on the particular choice of the co-ordinate system?

We reject any particular choice of co-ordinate system and write the first two equations:

(13.1) $$\underset{2}{\Phi_{00}} + 2\underset{2}{\Lambda_{00}} = -\underset{2}{\gamma_{00,rr}} = 0$$

(13.2) $$\underset{3}{\Phi_{0m}} + 2\underset{3}{\Lambda_{0m}} = -\underset{3}{\gamma_{0m,rr}} + \underset{3}{\gamma_{0r,mr}} - \underset{2}{\gamma_{00,m\underset{1}{0}}} = 0.$$

We *assume* that we start our approximation procedure with the same $\underset{2}{\gamma_{00}}$ and $\underset{3}{\gamma_{0m}}$ functions as *we did before.* But from now on, while dealing with the rest of the equations we shall look for *general* solutions not restricted by any additional co-ordinate conditions.

Thus the equations that we wish to consider now are:

(13.3a) $$\underset{4}{\Phi_{mn}} + 2\underset{4}{\Lambda_{mn}} = 0$$

(13.3b) $$\underset{4}{\Phi_{00}} + 2\underset{4}{\Lambda_{00}} = 0$$

(13.3c) $$\underset{5}{\Phi_{0m}} + 2\underset{5}{\Lambda_{0m}} = 0.$$

In the previous three sections we solved these equations, using special co-ordinate conditions. Let us *now* call the special solutions that we obtained there:

(13.4) $$\underset{4}{\gamma^*_{mn}}, \underset{4}{\gamma^*_{00}}, \underset{5}{\gamma^*_{0m}}.$$

Knowing them, as we do, we can find the general solution of (13.3). The procedure is similar to that outlined in Sec. **9**, only slightly different, because we have now a set of equations of order $(2l)$, $(2l)$, and $(2l + 1)$, whereas before

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we had a set of order $(2l-2)$, $(2l-1)$, and $(2l)$. But a straightforward substitution shows, that because of the linear expressions in (13.3), (and they alone enter the argument), the general solution of (13.3) is:

$$\underset{4}{\gamma}_{mn} = \underset{4}{\gamma}^*_{mn} + \underset{4}{a}_{m,n} + \underset{4}{a}_{n,m} - \delta_{mn}\underset{4}{a}_{r,r} \tag{13.5a}$$

$$\underset{4}{\gamma}_{00} = \underset{4}{\gamma}^*_{00} + \underset{4}{a}_{r,r} \tag{13.5b}$$

$$\underset{5}{\gamma}_{0m} = \underset{5}{\gamma}^*_{0m} + \underset{5}{a}_{0,m} + \underset{4}{a}_{m,\underset{1}{0}} \tag{13.5c}$$

where a_μ are arbitrary. The question then is: If we substitute these new expressions into the Λ's do we change the integrals

$$\int \underset{4}{\Lambda}_{mr} n_r\, dS,\ \int \underset{5}{\Lambda}_{0r} n_r\, dS,\ \int \underset{6}{\Lambda}_{mr} n_r\, dS\ ? \tag{13.6}$$

As far as the first two integrals are concerned the answer is easy; $\underset{4}{\Lambda}$ is not changed; only linear expressions in $\underset{5}{\Lambda}$ are affected, but the surface integral of the additional expressions disappears because of the lemma. But it is different with the third surface integral. In $\underset{6}{\Lambda}$ new terms appear containing the a's. They appear both through the linear and the non-linear expressions. But these additional expressions—quoted in the last appendix—are such that their surface integral vanishes. Thus in the sense explained here the equations of motion do not depend on the choice of the co-ordinate system. This dependence would appear probably in the next approximation steps $(\underset{8}{\Lambda})$, but it does not enter into the surface integral of $\underset{6}{\Lambda}$. This is a satisfying result, because it is difficult to see the meaning of our co-ordinate conditions

$$\begin{aligned} &\underset{4}{\gamma}_{mr,r} = 0 \\ &\underset{5}{\gamma}_{0r,r} - \underset{4}{\gamma}_{00,\underset{1}{0}} = 0 \\ &\underset{6}{\gamma}_{mr,r} = 0 \end{aligned} \tag{13.7}$$

and it is good to know that our equations of motion are independent of it. This result is general. If we have a system

$$\begin{aligned} &\underset{2l}{\Phi}_{mn} + 2\underset{2l}{\Lambda}_{mn} = 0 \\ &\underset{2l}{\Phi}_{00} + 2\underset{2l}{\Lambda}_{00} = 0 \\ &\underset{2l+1}{\Phi}_{0m} + 2\underset{2l+1}{\Lambda}_{0m} = 0, \end{aligned} \tag{13.8}$$

then the surface integral of $\underset{2l+2}{\Lambda}_{mn}$ is independent of the co-ordinate conditions introduced in this particular approximation stage. This is so, because the $\underset{2l}{a}$'s combine with the φ's in the same way in each approximation step.

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APPENDICES

A.2

The field equations are:

(A.2, 1) $$R_{\mu\nu} = -\left\{ {\rho \atop \mu\nu} \right\}_{|\rho} + \left\{ {\rho \atop \mu\rho} \right\}_{|\nu} + \left\{ {\rho \atop \mu\sigma} \right\}\left\{ {\sigma \atop \rho\nu} \right\} - \left\{ {\rho \atop \mu\nu} \right\}\left\{ {\sigma \atop \rho\sigma} \right\}.$$

Introducing here the h's as defined in (2.3) and splitting (A.2, 1) into linear and non-linear terms we have

(A.2, 2a) $$R_{00} = -\tfrac{1}{2} h_{00|ss} + h_{0s|0s} - \tfrac{1}{2} h_{ss|00} + L'_{00}$$

(A.2, 2b) $$R_{0n} = -\tfrac{1}{2} h_{0n|ss} + \tfrac{1}{2} h_{0s|ns} + \tfrac{1}{2} h_{ns|0s} - \tfrac{1}{2} h_{ss|n0} + L'_{0n}$$

(A.2, 2c) $$R_{mn} = -\tfrac{1}{2} h_{mn|ss} + \tfrac{1}{2} h_{ms|ns} + \tfrac{1}{2} h_{ns|ms} - \tfrac{1}{2} h_{ss|mn} + \tfrac{1}{2} h_{mn|00} - \tfrac{1}{2} h_{m0|n0} - \tfrac{1}{2} h_{n0|m0} + \tfrac{1}{2} h_{00|mn} + L'_{mn}.$$

Here $L'_{\mu\nu}$ are the non-linear expressions. We form now:

(A.2, 3) $$-2(R_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}R_{\alpha\beta}) = 0.$$

Substituting the γ's for the h's, we see that (A.2, 3) written out is (2.10)—(2.18), where

(A.2, 4) $$\Lambda'_{\mu\nu} = L'_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}L_{\alpha\beta}.$$

A.4

In calculating the surface integrals we need to take into account only expressions that go to infinity like r^{-2}, because only such expressions will give finite contributions. Since all the field functions are finite (outside of the singularity), and since the contributions do not depend on the shape of the surface, we may ignore all other expressions. But we have to keep the surface fixed, because in our calculations a complicated expression whose surface integral does not depend on the shape of the surface, is split into partial expressions with non-vanishing divergence. Thus in our calculations the surface is always a two-dimensional "sphere" with radius shrinking to zero. Let us assume, for the sake of simplicity, that the space co-ordinate of the singularity is (0, 0, 0). We shall first give some examples of the surface integrals formed around such a singularity.

Example 1. We calculate:

$$\int^0 \psi_{,s} n_s dS\ ;\ \psi = r^{-1};\ r^2 = x^s x^s,$$

We have:

$$\int^0 \psi_{,s} n_s dS = -\int^0 \frac{x^s x^s}{r^4} r^2 \sin\theta d\theta d\varphi = -4\pi.$$

Example 2. We calculate:

$$\int^0 \psi_{,s} n_r dS = -\int^0 \frac{x^s x^r}{r^4} r^2 \sin\theta d\theta d\varphi = -\frac{4\pi}{3}\delta_{sr}.$$

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Example 3. We calculate:

$$\int^{0} \psi_{,mn} n_n \chi(r) dS.$$

To find such a surface integral we expand $\chi(r)$ as a power series in the neighbourhood of the singularity:

$$\chi = \chi(0) + \chi_{,s}(0) x_s + \ldots .$$

The only contribution is from the second expression, that is, we have to calculate:

$$\chi_{,s}(0) \int^{0} \psi_{,mn} n_n x^s dS = - \chi_{,s}(0) \int^{0} \frac{x^m x^s}{r^4} r^2 \sin \theta d\theta d\varphi$$

$$+ 3\chi_{,s}(0) \int^{0} \frac{x^m x^s}{r^4} r^2 \sin \theta d\theta d\varphi$$

$$= \frac{8\pi}{3} \chi_{,m}(0) .$$

In the course of our calculations we shall have to find more complicated surface integrals and the following table will prove to be useful:

Table of Surface Integrals

I. $$\frac{1}{4\pi} \int^{0} \psi_{,n} n_n dS = -1.$$

II. $$\frac{1}{4\pi} \int^{0} \psi_{,s} n_n dS = -\tfrac{1}{3} \delta_{sn}.$$

III. $$\frac{1}{4\pi} \int^{0} x^r \psi_{,ns} n_n = \tfrac{2}{3} \delta_{rs}.$$

IV. $$\frac{1}{4\pi} \int^{0} x^r \psi_{,ms} n_n dS = -\tfrac{1}{15} \{2\delta_{rn}\delta_{ms} - 3\delta_{rm}\delta_{ns} - 3\delta_{rs}\delta_{mn}\}.$$

V. $$\frac{1}{4\pi} \int^{0} x^r \psi_{,rs} n_n dS = \tfrac{2}{3} \delta_{ns}.$$

VI. $$\frac{1}{4\pi} \int^{0} x^n \psi_{,ms} n_n dS = 0.$$

VII. $$\frac{1}{4\pi} \int^{0} x^n x^s \psi_{,mrl} n_n = 0.$$

VIII. $$\frac{1}{4\pi} \int^{0} x^m x^s \psi_{,nrl} n_n dS = \tfrac{2}{5} \delta_{ms}\delta_{lr} - \tfrac{3}{5} (\delta_{ml}\delta_{rs} + \delta_{mr}\delta_{ls}).$$

A.8

The linear terms of (8.26) give the following contribution to $2\Lambda_{\underset{2l}{mn}}$:

(A.8, 1) $$\sum_{s=1}^{p} \left\{ \underset{2l-2}{\overset{s}{S}}{}_m \overset{s}{\psi}_{,n} + \underset{2l-2}{\overset{s}{S}}{}_n \overset{s}{\psi}_{,m} - \delta_{mn} \underset{2l-2}{\overset{s}{S}}{}_r \overset{s}{\psi}_{,r} \right\}_{,\underset{2}{00}}.$$

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The non-linear terms can be found in the following way: Inspecting the terms in $\underset{6}{\Lambda}_{mn}$(A.12, 3) we see products of $\underset{4}{\gamma}_{00}$ and $\underset{2}{\gamma}_{00}$ or, as it is there called 2φ. Thus, if we put there the expression in (8.18) in place of $\underset{4}{\gamma}_{00}$ and write for brevity:

(A.8, 2) $$(S_r\psi) = \sum_s \overset{s}{\underset{2l-2}{S_r}} \overset{s}{\psi}$$

we get five new terms. Thus with the abbreviation (A.8, 2) we have in *every* approximation the following additional terms:

(A.8, 3) $$\left\{\begin{aligned} &\{(S_m\psi)_{,n}+(S_n\psi)_{,m}-\delta_{mn}(S_r\psi)_{,r}\}_{,00} \\ &+\varphi(S_r\psi)_{,rmn}+\tfrac{1}{2}\varphi_{,n}(S_r\psi)_{,rm} \\ &+\tfrac{1}{2}\varphi_{,m}(S_r\psi)_{,rn}-\tfrac{3}{2}\delta_{mn}\varphi_{,s}(S_r\psi)_{,rs}+\varphi_{,mn}(S_r\psi)_{,r}\,. \end{aligned}\right.$$

Only three of the linear terms give us a contribution to the surface integral. It is more difficult to see that the non-linear terms do not give any contribution, since it requires some knowledge of how to deal with surface integrals which is outlined in **A.4**, and which we shall here assume. We can write the non-linear terms in (A.8, 3), in the following way:

(A.8, 4) $$\begin{aligned} &\{\varphi_{,mn}(S_r\psi)\}_{,r}-\{\varphi_{,mr}(S_n\psi)\}_{,r} \\ &+\varphi_{,mn}(S_n\psi)_{,r} \\ &+\tfrac{1}{2}\{\varphi_{,n}(S_r\psi)_{,m}\}_{,r}-\tfrac{1}{2}\{\varphi_{,r}(S_n\psi)_{,m}\}_{,r} \\ &+\tfrac{1}{2}\varphi_{,r}(S_n\psi)_{,mr} \\ &-\tfrac{3}{2}\delta_{mn}\varphi_{,r}(S_s\psi)_{,sr}. \end{aligned}$$

These are the non-linear expressions, and their divergence vanishes because φ is a harmonic function. The expressions written out in pairs in (A.8, 4) do not give any contribution to the surface integrals, because of our lemma in Sec. **3**. Thus the only contribution could come from the terms:

(A.8, 5) $$\tfrac{3}{2}\varphi_{,s}S_n\psi_{,ms}-\tfrac{3}{2}\delta_{mn}\varphi_{,s}S_r\psi_{,rs}\,.$$

Here only the "cross products" could give contributions and we find with the help of the table in **A.4**, that the result is zero.

A.10

In the $l = 2$ approximation we have:

(A.10, 1) $$\left\{\begin{aligned} \underset{2}{\gamma_{00}} &= 2\varphi = 2f+2g \\ \underset{3}{\gamma_{0n}} &= -2f\dot{\eta}^n-2g\dot{\zeta}^n = \underset{3}{h_{0n}} \\ \underset{2}{h_{00}} &= \varphi = f+g \\ \underset{2}{h^{00}} &= -h_{00} = -\varphi \\ \underset{3}{h^{0n}} &= \underset{3}{h_{0n}} = \underset{3}{\gamma_{0n}} \\ \underset{2}{h_{mn}} &= -\underset{2}{h^{mn}} = \delta_{mn}\varphi\,. \end{aligned}\right.$$

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A straightforward calculation gives:

$$
\text{(A.10, 2)}\qquad
\left\{
\begin{aligned}
2\underset{2}{\Lambda}_{00} &= 0\\
2\underset{3}{\Lambda}_{0m} &= -\underset{2}{\gamma}_{00,\underset{1}{m0}}\\
2\underset{4}{\Lambda}_{mn} &= -\underset{3}{\gamma}_{0m,\underset{1}{0n}} - \underset{3}{\gamma}_{0n,\underset{1}{0m}} + 2\delta_{mn}\varphi_{,\underset{2}{00}}\\
&\quad - 2\varphi\varphi_{,mn} - \varphi_{,m}\varphi_{,n} + \tfrac{3}{2}\,\delta_{mn}\varphi_{,s}\varphi_{,s}.
\end{aligned}
\right.
$$

The contributions to the surface integrals are (for the first singularity):

$$
\begin{array}{lcl}
-\gamma_{0m,0n} & \rightarrow & 4\,\overset{1}{m}\ddot{\eta}^{m}.4\pi\\
-\gamma_{0n,0m} & \rightarrow & \frac{4}{3}\,\overset{1}{m}\ddot{\eta}^{m}.4\pi\\
2\delta_{mn}\varphi_{,00} & \rightarrow & -\frac{4}{3}\,\overset{1}{m}\ddot{\eta}^{m}.4\pi\\
-2\varphi\varphi_{,mn} & \rightarrow & \frac{8}{3}\,\overset{1}{m}\tilde{g}_{,m}.4\pi\\
-\varphi_{,m}\varphi_{,n} & \rightarrow & -\frac{8}{3}\,\overset{1}{m}\tilde{g}_{,m}.4\pi\\
\frac{3}{2}\,\delta_{mn}\varphi_{,s}\varphi_{,s} & \rightarrow & 2\,\overset{1}{m}\tilde{g}_{,m}.4\pi
\end{array}
$$

$$
(\overset{1}{m} = \underset{2}{\overset{1}{m}}).
$$

A.12

A straightforward calculation of $\underset{4}{\Lambda}$, $\underset{5}{\Lambda}$, $\underset{6}{\Lambda}$ gives

$$
\text{(A.12, 1)}\qquad 2\underset{4}{\Lambda}_{00} = -\tfrac{3}{2}\,\varphi_{,s}\varphi_{,s}
$$

$$
\text{(A.12, 2)}\qquad 2\underset{5}{\Lambda'}_{0m} = \varphi_{,s}\underset{3}{\gamma}_{0s,m} - \varphi_{,sm}\underset{3}{\gamma}_{0s} - 3\varphi_{,0}\varphi_{,m}
$$

$$
\text{(A.12, 3)}\qquad
\begin{aligned}
2\underset{6}{\Lambda}_{mn} = &-\underset{5}{\gamma}_{0m,0n} - \underset{5}{\gamma}_{0n,0m} + \delta_{mn}\underset{4}{\gamma}_{00,00} + \underset{4}{\gamma}_{mn,00} - \varphi\underset{4}{\gamma}_{00,mn}\\
&- \varphi\underset{4}{\gamma}_{ss,mn} - \varphi_{,mn}\underset{4}{\gamma}_{00} - \varphi_{,mn}\underset{4}{\gamma}_{ss} + \varphi_{,ms}\underset{4}{\gamma}_{ns}\\
&+ \varphi_{,ns}\underset{4}{\gamma}_{ms} - \delta_{mn}\varphi_{,sr}\underset{4}{\gamma}_{sr} - 2\varphi_{,s}\underset{4}{\gamma}_{mn,s} + \varphi_{,s}\underset{4}{\gamma}_{ms,n}\\
&+ \varphi_{,s}\underset{4}{\gamma}_{ns,m} - \tfrac{1}{2}\,\varphi_{,m}\underset{4}{\gamma}_{ss,n} - \tfrac{1}{2}\,\varphi_{,n}\underset{4}{\gamma}_{ss,m} - \tfrac{1}{2}\,\varphi_{,n}\underset{4}{\gamma}_{00,m}\\
&- \tfrac{1}{2}\,\varphi_{,m}\underset{4}{\gamma}_{00,n} + \tfrac{3}{2}\,\delta_{mn}\varphi_{,s}\underset{4}{\gamma}_{rr,s} + \tfrac{3}{2}\,\delta_{mn}\varphi_{,s}\underset{4}{\gamma}_{00,s}\\
&- \underset{3}{\gamma}_{0s}\underset{3}{\gamma}_{0n,ms} - \underset{3}{\gamma}_{0s}\underset{3}{\gamma}_{0m,ns} + 2\underset{3}{\gamma}_{0s}\underset{3}{\gamma}_{0s,mn}\\
&+ \tfrac{1}{2}\,\delta_{mn}\underset{3}{\gamma}_{0s,r}\underset{3}{\gamma}_{0r,s} - \tfrac{3}{2}\,\delta_{mn}\underset{3}{\gamma}_{0s,r}\underset{3}{\gamma}_{0s,r} + \underset{3}{\gamma}_{0s,m}\underset{3}{\gamma}_{0s,n}\\
&+ \underset{3}{\gamma}_{0m,s}\underset{3}{\gamma}_{0n,s} - \varphi_{,0n}\underset{3}{\gamma}_{0m} - \varphi_{,0m}\underset{3}{\gamma}_{0n} + 2\delta_{mn}\underset{3}{\gamma}_{0s}\varphi_{,0s}\\
&- \varphi_{,0}\underset{3}{\gamma}_{0m,n} - \varphi_{,0}\underset{3}{\gamma}_{0n,m} - \varphi_{,n}\underset{3}{\gamma}_{0m,0} - \varphi_{,m}\underset{3}{\gamma}_{0n,0}\\
&+ 2\varphi\underset{3}{\gamma}_{0m,0n} + 2\varphi\underset{3}{\gamma}_{0n,0m} - 2\delta_{mn}\varphi\varphi_{,00}\\
&+ 2\varphi\varphi\varphi_{,mn} - \varphi\varphi_{,m}\varphi_{,n} + \tfrac{3}{2}\,\delta_{mn}\varphi\varphi_{,s}\varphi_{,s}\\
&+ \tfrac{1}{2}\,\delta_{mn}\varphi_{,0}\varphi_{,0}.
\end{aligned}
$$

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The surface integral (12.4a) for $s = 1$ is, because of (12.2), and (12.1b):

$$\frac{1}{4\pi}\int \left(\varphi_{,r}\underset{3}{\gamma}_{0r,m} - \varphi_{,rm}\underset{3}{\gamma}_{0n} - \tfrac{3}{2}\,\varphi_{,0}\varphi_{,m} + \tfrac{3}{2}\,\varphi\varphi_{,0m} + 4(\overset{1}{m}\underset{4}{\psi})_{,0m}\right) n_m dS = 0.$$

The contributions of these five expressions are respectively:

$$\begin{aligned}
(1) &\rightarrow -\frac{4\overset{1}{m}}{3}\,\tilde{g}_{,s}\dot{\zeta}^s - 4\overset{1}{m}\tilde{g}_{,s}\dot{\eta}^s\\
(2) &\rightarrow -\frac{8\overset{1}{m}}{3}\,\tilde{g}_{,s}\dot{\zeta}^s\\
(3) &\rightarrow \quad 3\overset{1}{m}\,\tilde{g}_{,s}\dot{\zeta}^s + \overset{1}{m}\tilde{g}_{,s}\dot{\eta}^s\\
(4) &\rightarrow \quad 2\overset{1}{m}\tilde{g}_{,s}\dot{\eta}^s\\
(5) &\rightarrow -4\underset{4}{\overset{1}{\dot{m}}}\,.
\end{aligned}$$

Therefore:

$$\begin{aligned}
-4\underset{4}{\overset{1}{\dot{m}}} &= \overset{1}{m}\tilde{g}_{,s}\dot{\zeta}^s + \overset{1}{m}\tilde{g}_{,s}\dot{\eta}^s = 2\overset{1}{m}\tilde{g}_{,s}\dot{\eta}^s - \overset{1}{m}\tilde{g}_{,s}\dot{\eta}^s + \overset{1}{m}\tilde{g}_{,s}\dot{\zeta}^s\\
&= -\overset{1}{m}(2\dot{\eta}^s\dot{\eta}^s + \tilde{g})_{,0}\,.
\end{aligned}$$

From the last equation (12.5) follows immediately.

The last step is to calculate the surface integrals due to $\underset{6}{\Lambda}$. Here a skilful use of the lemma may save the calculation of many surface integrals. Indeed, $2\underset{6}{\Lambda}_{mn}$ can be written in the following form:

$$\begin{aligned}
2\underset{6}{\Lambda}_{mn} = {} & (\varphi_{,n}\underset{4}{\gamma}_{sm} - \varphi_{,s}\underset{4}{\gamma}_{nm})_{,s} + (\varphi\underset{4}{\gamma}_{ms,n} - \varphi\underset{4}{\gamma}_{mn,s})_{,s}\\
& + (\delta_{ms}\varphi_{,r}\underset{4}{\gamma}_{rn} - \delta_{mn}\varphi_{,r}\underset{4}{\gamma}_{rs})_{,s} + (\delta_{mn}\varphi_{,s}\underset{4}{\gamma}_{rr} - \delta_{ms}\varphi_{,n}\underset{4}{\gamma}_{rr})_{,s}\\
& + \tfrac{1}{2}(\delta_{mn}\varphi\underset{4}{\gamma}_{rr,s} - \delta_{ms}\varphi\underset{4}{\gamma}_{rr,n})_{,s} + (\delta_{mn}\underset{5}{\gamma}_{0s,0} - \delta_{ms}\underset{5}{\gamma}_{0n,0})_{,s}\\
& + \tfrac{1}{2}(\underset{3}{\gamma}_{0s,m}\underset{3}{\gamma}_{0n} - \underset{3}{\gamma}_{0n,m}\underset{3}{\gamma}_{0s})_{,s} + (\delta_{mn}\varphi_{,0}\underset{3}{\gamma}_{0s} - \delta_{ms}\varphi_{,0}\underset{3}{\gamma}_{0n})_{,s}\\
& + (\underset{3}{\gamma}_{0n}\underset{3}{\gamma}_{0m,s} - \underset{3}{\gamma}_{0s}\underset{3}{\gamma}_{0m,n})_{,s} + \tfrac{1}{2}(\delta_{mn}\underset{3}{\gamma}_{0s,r}\underset{3}{\gamma}_{0r} - \delta_{ms}\underset{3}{\gamma}_{0n,r}\underset{3}{\gamma}_{0r})_{,s}\\
& + (\delta_{ms}\underset{3}{\gamma}_{0r,n}\underset{3}{\gamma}_{0r} - \delta_{mn}\underset{3}{\gamma}_{0r,s}\underset{3}{\gamma}_{0r})_{,s}\\
& - \underset{5}{\gamma}_{0m,0n} + \underset{4}{\gamma}_{mn,00} + \underset{3}{\gamma}_{0s}\underset{3}{\gamma}_{0s,mn} && [a_1 + a_2 + a_3]\\
& - \tfrac{1}{2}\delta_{mn}\underset{3}{\gamma}_{0s,r}\underset{3}{\gamma}_{0s,r} - (\varphi_{,n}\underset{3}{\gamma}_{0m})_{,0} && [a_4 + a_5]\\
& - (\varphi_{,m}\underset{3}{\gamma}_{0n})_{,0} + (\varphi\underset{3}{\gamma}_{0m,n})_{,0} + (\varphi\underset{3}{\gamma}_{0n,m})_{,0} && [a_6 + a_7 + a_8]\\
& - \tfrac{3}{2}\delta_{mn}\varphi_{,0}\varphi_{,0} - \tfrac{1}{2}\varphi\underset{4}{\gamma}_{ss,mn} && [a_9 + a_{10}]\\
& + \tfrac{1}{2}\varphi_{,n}\underset{4}{\gamma}_{ss,m} + \tfrac{1}{2}\varphi_{,n}\underset{4}{\gamma}_{00,m} && [a_{11} + a_{12}]\\
& + \tfrac{1}{2}\varphi_{,m}\underset{4}{\gamma}_{00,n} - \tfrac{1}{2}\delta_{mn}\varphi_{,s}\underset{4}{\gamma}_{00,s} && [a_{13} + a_{14}]\\
& - \varphi\varphi_{,00}\delta_{mn} - 2\varphi\varphi_{,m}\varphi_{,n} && [a_{15} + a_{16}]\\
& + \tfrac{11}{4}\varphi\varphi_{,s}\varphi_{,s}\delta_{mn}\,. && [a_{17}]
\end{aligned}$$

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TABLE OF SURFACE INTEGRALS FOR $\int_6^1 \Lambda_{ms} n_s\, dS$

No.	Expression	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	Result	Remarks
1	$\overset{1}{m} g_{,s}\dot\eta^s\ddot\eta^m$	$-\frac{16}{3}$				$-\frac{4}{3}$		$-\frac{8}{3}$	$-\frac{4}{15}$		$\frac{8}{15}$	$\frac{4}{15}$				$\frac{4}{5}$			-8	$\tilde g_{,s} = -2\overset{2}{m}\frac{\partial \overset{1}{r}}{\partial \eta^s}$
2	$\overset{1}{m}\tilde g\ddot\eta^m$	-2						-4	$-\frac{4}{3}$			$-\frac{29}{3}$	3	$\frac{11}{3}$	$-\frac{5}{3}$	$\frac{2}{3}$	$\frac{32}{3}$	$-\frac{22}{3}$	-8	$\tilde g = -\frac{2\overset{2}{m}}{r};\ \ddot\eta^m = -\frac{1}{2}\tilde g_{,m}$
3	$\overset{1}{m}\tilde g_{,m}\dot\eta^s\dot\eta^s$	1					$-\frac{4}{3}$	$-\frac{4}{5}$			$\frac{8}{5}$	$\frac{4}{5}$	1	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{4}{15}$			2	$\tilde g_{,m} = -2\overset{1}{m}\ddot\eta^m$
4	$\overset{1}{m}\tilde g_{,m}\dot\zeta^s\dot\zeta^s$											2	$\frac{1}{3}$	1	$-\frac{1}{3}$				3	
5	$\overset{1}{m}\tilde g_{,m}\tilde f$	$\frac{4}{3}$				2	$\frac{2}{3}$					$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$				5	$\tilde g_{,m}\tilde f = -\tilde g\tilde f_{,m};\ \tilde f = -\frac{2\overset{1}{m}}{r}$
6	$\overset{1}{m}\overset{2}{m}\tilde r_{,00m}$											-2							-2	$\tilde r_{,00m} = (\overset{2}{\tilde r}_{,00m})$ for $x^s = \eta^s$ *
7	$\overset{1}{m}\tilde g_{,s}\dot\zeta^m\dot\eta^s$	$\frac{16}{5}$				$\frac{8}{3}$	$\frac{4}{5}$	$\frac{4}{3}$											8	
8	$\overset{1}{m}\tilde g_{,s}\dot\zeta^s\ddot\eta^m$	$\frac{16}{5}$					$-\frac{8}{15}$	4	$\frac{4}{3}$	-2									6	
9	$\overset{1}{m}\tilde g_{,m}\dot\eta^s\dot\zeta^s$	$-\frac{32}{15}$		$-\frac{16}{3}$	$-\frac{8}{3}$		$\frac{4}{5}$		$\frac{4}{3}$										-8	
10	$\overset{1}{m}\tilde g_{,s}\dot\zeta^s\ddot\zeta^m$	$-\frac{8}{3}$				-4	$-\frac{4}{3}$												-8	

* $\tilde r_{,00m} = \frac{\partial^3 r}{\partial\eta^s\partial\eta^r\partial\eta^m}\dot\zeta^s\dot\zeta^r$, as $\frac{\partial^2 r}{\partial\eta^s\partial\eta^m}\ \frac{\partial \overset{1}{r}}{\partial\eta^s} = 0.$

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Because of the lemma we have to find now the surface integrals of only 17 expressions denoted successively by $\alpha_1, \alpha_2, \ldots, \alpha_{17}$. The result of this calculation is summarized in the table. Only ten types of expressions (or their equivalents) appear in the result. The table tells us what is the contribution of each of the α's to the final result. The only α that does not give a contribution is $\alpha_2 = \underset{4}{\gamma_{mn,00}}$.

A.13

The additional expressions in $\underset{6}{\Lambda_{mn}}$ induced through rejection of the coordinate condition are:

$$
\begin{aligned}
&2\ (\delta_{mn} a_{0,r0} - \delta_{mr} a_{0,n0})_{,r}\\
+&\ (\varphi_{,m} a_{n,r} - \varphi_{,m} a_{r,n})_{,r}\\
+&\ (\varphi_{,n} a_{m,r} - \varphi_{,r} a_{m,n})_{,r}\\
+&\ (\varphi_{,n} a_{s,m} - \varphi_{,s} a_{n,m})_{,s}\\
-&\ 2(\delta_{mn} \varphi_{,s} a_{s,r} - \delta_{mr} \varphi_{,s} a_{s,n})_{,r}\\
+&\ 2(\delta_{mn} \varphi_{,s} a_{r,r} - \delta_{ms} \varphi_{,n} a_{r,r})_{,r}.
\end{aligned}
$$

They are written in such a way, that the vanishing of each line is evident, because of the lemma.

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