UNION AND EXTENSION OF ARCS OF CYCLIC ORDER THREE

K. D. SINGH AND N. D. LANE

1. Introduction. In **(2)** Lane and Scherk discussed differentiable points of arcs in the conformal (inversive) plane. Arcs A_3 of cyclic order three were discussed in **(3; 4)**. In the present note we give necessary and sufficient conditions for the union of two A_3's to be an A_3 (Theorem 1), and for an A_3 to be extensible to a larger one (Theorem 2). The related problem of extending arcs in projective n-space was dealt with by Haupt in **(1)** and Sauter in **(5; 6)**.

2. Prerequisites. $P, Q, \ldots$ denote points in the conformal plane. C denotes an oriented circle, C_* its "*interior*", and C^* its "*exterior*", the latter region lying to the right of C. Thus C_* and C^* will depend on the sense of direction we assign to the circle C.

An *arc* A in the conformal plane is the continuous image of a real interval. (More than one function can map this interval into the same point set A, but the ordering of the points in A must be the same for all such functions under consideration.) The letters $p, q, \ldots$ will denote both the points of the parameter interval and their images in the plane. From our definition, distinct parameter values may be mapped into the same point of the plane.

An arc A is called *once conformally differentiable* at p if it satisfies the following:

CONDITION I. *There exists a point $Q \neq p$ such that if the parameter s is sufficiently close to the parameter p, $s \neq p$, then the circle $C(p, s, Q)$ through the points p, s, and Q exists. It converges if s converges to p.*

Condition I implies that there is a neighbourhood of p on the parameter interval on which the mapping which defines A is 1-1.

The limit *tangent circle* is denoted by $C(\tau, Q)$. If Condition I holds for a single point $Q \neq p$, then it holds for all such points and the set $\tau = \tau(p)$ of all the tangent circles of A at p is a parabolic pencil, i.e., any two circles of τ meet at p and nowhere else **(2**, Theorem 1**)**. At an interior point p of A which satisfies Condition I, the non-tangent circles all support or all intersect A at p according as A has or has not a cusp at p **(2**, Theorem 3**)**.

We call A *conformally differentiable* at p if it satisfies Condition I and

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CONDITION II. $\lim_{s \to p;\ s \neq p} C(\tau, s)$ *exists.*

The limit *osculating circle* is denoted by $C(p)$.

We call C a *general tangent circle* of an arc A at p if there exists a sequence of triples of mutually distinct points t, u, Q such that t and u converge on A to p and $\lim C(t, u, Q) = C$. If in addition, $Q \in A$ also converges to p, then we call C a *general osculating circle* of A at p.

The arc A is *strongly conformally differentiable* at p if it satisfies

CONDITION I′. *If* $R \neq p$, *then* $\lim_{s, t \to p;\ s \neq t;\ Q \to R} C(Q, s, t)$ *exists,*

and

CONDITION II′. $\lim_{s, t, u \to p;\ s \neq t \neq u \neq s} C(s, t, u)$ *exists.*

Remark. Condition II′ with $\lim C(s, t, u) \neq p$, or Condition II′ with p an end-point of A, implies Conditions I, I′, and II.

Let A_3 denote an arc of cyclic order three; thus no circle meets A_3 more than three times. It is well known that every A_3 satisfies Condition I′ at an interior point, and Conditions I′ and II′ at an end-point (**3**, Theorems 2 and 3). At an interior point p of A_3, the general osculating circles belong to a closed interval of $\tau(p)$ bounded by the one-sided osculating circles, and all of these circles intersect A_3 at p (**3**, 3.3). General osculating circles at distinct points of A_3 do not meet (**4**, Theorem 1).

Unless the end-points of A_3 have a common tangent circle, an open arc A_3 retains the cyclic order three when the end-points are added and also when an end-point or an interior point is counted twice on any general tangent circle at that point, and three times on any general osculating circle there (**3**, 3.3). In the exceptional case, the common tangent circle at the end-points of A_3 is the only circle that meets the closure of A_3 with a multiplicity greater than three.

Let p and e be the end-points of an open arc A_3; thus $\bar{A}_3 = p \cup A_3 \cup e$. Let τ_e denote the pencil of tangent circles of A_3 at e. We may assign to the circle through three points t, u, v of $p \cup A_3$ the orientation defined by $e \in C(t, u, v)^*$. This orientation is continuous. It can be extended to general tangent and osculating circles. It can even be extended to $\bar{A}_3$ if

$$C(\tau, e) \neq C(\tau_e, p).$$

Then we readily verify that

$$A_3 \subset C(p)^* \cap C(\tau, e)_* \cap C(\tau_e, p)^* \cap C(e)_* \qquad \text{(cf. Figure 1).}$$

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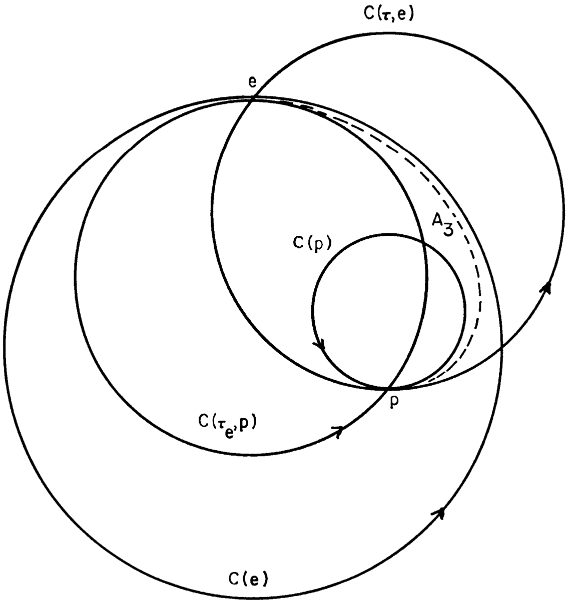

FIGURE 1

3. A property of A_3**.** Assume that $\bar{A}_3$ has cyclic order three. Let u and v be distinct points on A_3. As x runs continuously and monotonically from p to e on $\bar{A}_3$, the circle $C(x, u, v)$ moves continuously and monotonically from $C(p, u, v)$ to $C(u, v, e)$ in the family of circles through u and v. Hence if $t \in A_3$, one has

$$C(t, u, v)_* \supset C(p, u, v)_* \cap C(u, v, e)_*.$$

Here the symbol $\supset$ denotes proper inclusion. This relation remains valid if u and v are any (not necessarily distinct) points on $\bar{A}_3$, provided that we interpret $C(p, p, s)$ as $C(\tau, s)$, and so on.

Let $q, r, s \in A_3$. By a sequence of applications of the extended relation above we obtain

$$\begin{aligned} C(q, r, s)_* &\supset C(p, r, s)_* \cap C(r, s, e)_* \\ &\supset \{C(\tau, s)_* \cap C(p, s, e)_*\} \cap \{C(p, s, e)_* \cap C(\tau_e, s)_*\} \\ &= C(\tau, s)_* \cap C(p, s, e)_* \cap C(\tau_e, s)_* \\ &\supset \{C(p)_* \cap C(\tau, e)_*\} \cap \{C(\tau, e)_* \cap C(\tau_e, p)_*\} \\ &\qquad\qquad \cap \{C(\tau_e, p)_* \cap C(e)_*\} \\ &= C(p)_* \cap C(\tau, e)_* \cap C(\tau_e, p)_* \cap C(e)_*. \end{aligned}$$

Since $e \in C(p)^*$, one has $C(p)_* \subset C(\tau, e)_*$ and $C(\tau_e, p)_* \subset C(e)_*$. Hence

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$$C(q, r, s)_* \supset C(p)_* \cap C(\tau_e, p)_*.$$

The last relation remains valid if q, r, s lie on $\bar{A}_3$: cf. **(3; 4)**.

4. Union of A_3's.

4.1. Let A_3 and A'_3 be open arcs of cyclic order three with a common end-point p. Let

$$\overline{A'_3} = e' \cup A'_3 \cup p, \qquad \overline{A_3} = p \cup A_3 \cup e,$$

and let $A = A_3 \cup p \cup A'_3$; $\bar{A} = e' \cup A \cup e$. Assume that $\overline{A_3}$ *and* $\overline{A'_3}$ *are also of cyclic order three*; thus

$$C(\tau_e, p) \neq C(\tau, e) \quad \text{and} \quad C(\tau_{e'}, p) \neq C(\tau', e'),$$

where τ' is the family of tangent circles of A'_3 at p. Let $C(p)$ and $C'(p)$ denote the osculating circles of A_3 and A'_3 respectively at p. We may assume that $e \in C(p)^*$. This induces a continuous orientation on all the circles through three points of $\overline{A_3}$ and also on τ.

If A has cyclic order three, the following conditions will hold; cf. Figure 2.

(i) *A satisfies Condition* I *at p*. Thus the two pencils of tangent circles of A_3 and A'_3 at p coincide. We denote this common pencil by τ.

(ii) $C(\tau, e')_* \subset C'(p)_* \subseteq C(p)_* \subset C(\tau, e)_*$. Thus $A'_3 \subset C'(p)_* \subseteq C(p)_*$ and $A_3 \subset C(p)^* \subseteq C'(p)^*$.

(iii) *$A_3[A'_3]$ does not meet $C(\tau_{e'}, p)[C(\tau_e, p)]$.*

(iv) *$A_3 \cup p[A'_3 \cup p]$ does not meet $C(\tau_{e'}, e)[C(\tau_e, e')]$.*

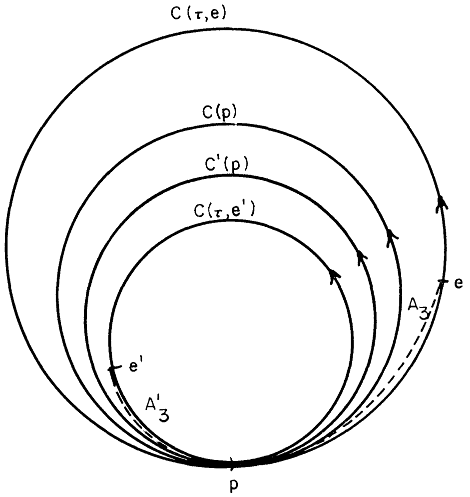

FIGURE 2

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We wish to prove

THEOREM 1. *Conditions* (i)–(iv) *are not only necessary but are also sufficient for A to have cyclic order three.*

We observe that $\bar{A}$ will also have cyclic order three if we add the condition $C(\tau_{e'}, e) \neq C(\tau_e, e')$.

Remark. It is clear that Condition (ii) implies Condition (i). However, Conditions (ii), (iii), and (iv) are independent, as may be shown by examples.

4.2. LEMMA. *Assume Conditions* (i), (ii), *and* (iv). *Then Condition* (iii) *is equivalent to A having no cusp at p.*

Proof. The following discussion is easiest to follow if we designate p as the point at infinity. Then $C(\tau, e')$, $C'(p)$, $C(p)$, and $C(\tau, e)$ will be represented by four parallel straight lines. By (ii) and §2, $A_3 \subset C(p)^* \cap C(\tau, e)_*$ and $A'_3 \subset C'(p)_* \cap C(\tau, e')^*$. Thus by (ii)

$$A_3 \cup A'_3 \subset C(\tau, e)_* \cap C(\tau, e')^* = R, \text{ say} \qquad \text{(cf. Figures 3, 4).}$$

Since $C(\tau_e, p) \neq C(\tau, e)$, they intersect at e. Hence $C(\tau_e, e')$ also intersects $C(\tau, e)$ at e. Furthermore, since $C(\tau_e, e')$ does not meet $A'_3 \cup p$ and since $\overline{A'_3}$ is of cyclic order three, $C(\tau_e, e')$ will intersect $C(\tau, e')$ at e'. Symmetrically $C(\tau_{e'}, e)$ intersects $C(\tau, e')$ at e' and $C(\tau, e)$ at e.

Orient $C(\tau_e, e')$ and $C(\tau_{e'}, e)$ such that $p \in C(\tau_e, e')^* \cap C(\tau_{e'}, e)^*$; thus

$$A'_3 \subset C(\tau_e, e')^* \cap C(\tau, e')^* \cap C'(p)_*$$

and

$$A_3 \subset C(\tau_{e'}, e)^* \cap C(\tau, e)_* \cap C(p)^*.$$

Hence $A_3 \cup A'_3$ has no points in common with

$$C(\tau_e, e')_* \cap C(\tau_{e'}, e)_* \cap R = R_0, \text{ say.}$$

The boundary of R_0 decomposes R into three disjoint regions of which R_0 is one. Let R_1 and R_2 be the other two; thus $A_3 \cup A'_3 \cup R_1 \cup R_2$.

Case 1. *A has a cusp at p*; cf. Figure 3. Then A_3 and A'_3 both lie in R_1 or both

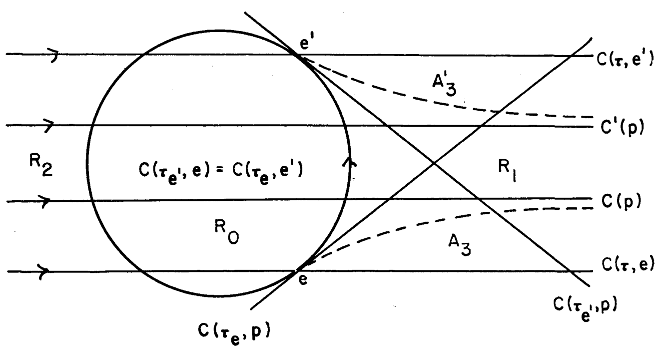

FIGURE 3

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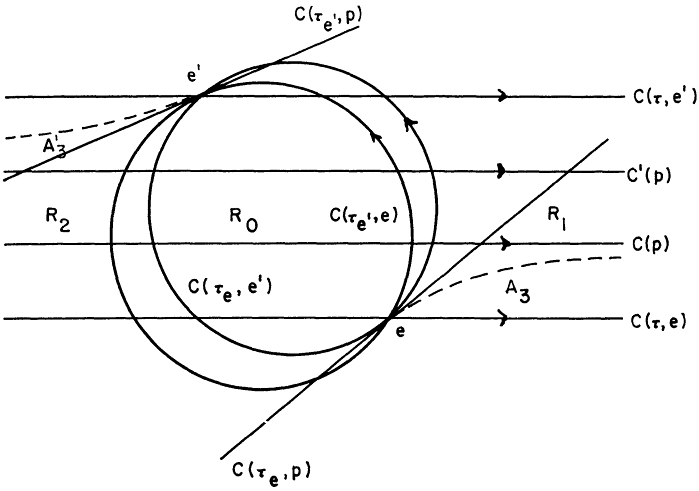

FIGURE 4

of them lie in R_2. Say they lie in R_i; then both e and e' will lie on the boundary of R_i. Since $C(\tau_e, p)$ and $C(\tau_e, e')$ are tangent circles at e, $C(\tau_e, p)$ will decompose R_i into two disjoint regions and A'_3 will have points in both of them. Hence $C(\tau_e, p)$ will intersect A'_3.

Remark. If Conditions (i), (ii), and (iv) are satisfied and A has a cusp at p, then $C(\tau_e, e') = C(\tau_{e'}, e)$.

Case 2. *A has no cusp at p*; cf. Figure 4. Here A_3 lies in R_1, say, and A'_3 lies in R_2. Thus $e[e']$ lies on the boundary of $R_1[R_2]$.

Then $C(\tau_e, p) \cap R$ lies in R_1 and $C(\tau_{e'}, p) \cap R$ lies in R_2. Hence

$$C(\tau_e, p)\ [C(\tau_{e'}, p)]$$

does not meet $A'_3[A_3]$.

COROLLARY. *Conditions* (i)–(iv) *imply that A has no cusp at p.*

Remark. Conditions (i), (ii), (iii) do not imply (iv), even if we assume A has no cusp at p.

4.3. We shall assume Conditions (i)–(iv) in this and the next section. Let $\{q, r, \ldots\} \subset \overline{A_3}$ and $\{q', r', \ldots\} \subset \overline{A'_3}$. The orientation of circles through three points of A_3 $[A'_3]$ is that induced by $e \in C(p)^*$ $[e' \in C'(p)_*]$.

LEMMA. $A'_3 \cup e' \subset C(q, r, s)_*$ *and* $A_3 \cup e \subset C(q', r', s')^*$.

Here, two or all three of q, r, s or of q', r', s' can coincide so that these relations also apply to tangent and general osculating circles of A_3 and A'_3.

Proof. Since $e \in C(p)^*$, §3 implies that

$$C(q, r, s)_* \supset C(p)_* \cap C(\tau_e, p)_*.$$

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Since $C(\tau_e, p) \notin \tau$, the Corollary of 4.2 implies that $C(\tau_e, p)$ intersects A at p. Now $A_3 \subset C(\tau_e, p)^*$, and by (iii) and (iv), $A'_3 \cup e'$ does not meet $C(\tau_e, p)$. Hence $A'_3 \cup e' \subset C(\tau_e, p)_*$. Altogether,

$$A'_3 \cup e' \subset C(p)_* \cap C(\tau_e, p)_* \subset C(q, r, s)_*.$$

Symmetry yields the other relation.

4.4. *Proof of Theorem* 1. Let $t, u \in A_3 \cup p$, $t' \in A'_3 \cup p$. Using 4.3 and assumption (iv), we prove successively that the following circles do not meet A elsewhere: $C(\tau_{e'}, t)$ and symmetrically $C(\tau_e, t')$; $C(e', t, e)$ and $C(e', t', e)$; $C(e', t', t)$ and $C(t', t, e)$; $C(t', t, u)$. Put

$$C_i(t) = \begin{cases} C(\tau_{e'}, t) \\ C(e', t, e) \\ C(t', t, e) \\ C(t', t, u) \end{cases} \quad \text{if } i = \begin{cases} 1, \\ 2, \\ 3, \\ 4. \end{cases}$$

Now $C_i(p)$ does not meet A again. If t moves continuously on A_3 from p to e, $C_i(t)$ cannot pass through p, cannot increase the multiplicity with which it meets e or e', and cannot support $A_3 \cup A'_3$ at a new point. Hence $C_i(t)$ does not meet A elsewhere.

5. Extension of an A_3.

We wish to prove

THEOREM 2. *An open arc A_3 can be extended through an end-point p to a larger arc of cyclic order three if and only if $C(p) \neq p$ and $\overline{A_3}$ is of cyclic order three.*

5.1. Sufficiency. (The necessity part of the proof follows from properties of arcs of cyclic order three discussed at the end of §2.) A reflection in a circle followed by a reflection in an orthogonal circle is a conformal transformation which leaves both of these circles invariant. Given an A_3 with end-points p and e, $C(p) \neq p$, and $C(\tau, e) \neq C(\tau_e, p)$, we construct an arc A'_3 by first reflecting A_3 in $C(p)$ and then reflecting the resulting arc in any circle D through p which is orthogonal to $C(p)$, and which does not meet $A_3 \cup e$. We finally choose a suitable subarc $B_3 \subset A_3$ with image $B'_3 \subset A'_3$ and verify that $A = B'_3 \cup p \cup A_3$ is of cyclic order three.

Since both reflections leave $C(p)$ invariant, the arc A is conformally differentiable at p. It is clear that Conditions (i) and (ii) will hold. It remains to show that Conditions (iii) and (iv) hold.

In the following, e, e', f, and f' are the end-points $\neq p$ of A_3, A'_3, B_3, and B'_3 respectively. We may assume that $e \in D^*$ and use the orientation of D to orient the family of circles which are tangent to D at p. The orientation of the circles through three points of $\overline{A_3}$ [$\overline{A'_3}$] is the usual one induced by $e \in C(p)^*$ [$e' \in C'(p)_* = C(p)_*$]. Thus

$$A_3 \cup e \subset C(p)^* \cap D^* \quad \text{and} \quad A'_3 \cup e' \subset C(p)_* \cap D_*.$$

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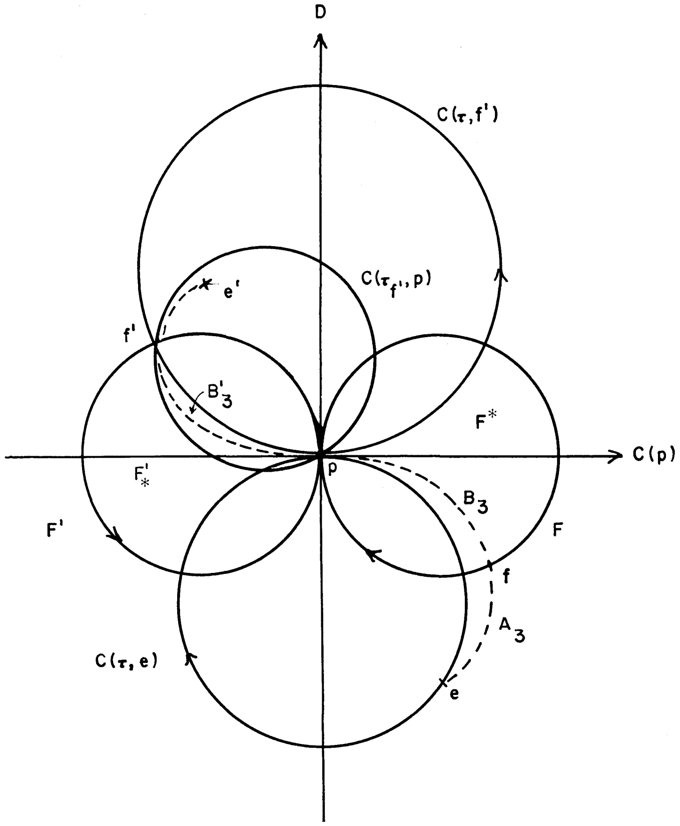

FIGURE 5

5.2. Condition (iii) (cf. Figure 5). Let F be the circle orthogonal to τ through p and a point of A_3, and let F' be its image under the reflections in $C(p)$ and D; thus $F'_* \subset D_* \subset F_*$. If F is sufficiently small, it will meet A_3 at only one point f and it will intersect A_3 there. Let $B_3 = A_3 \cap F^*$; thus $\overline{B_3} = p \cup B_3 \cup f$. Hence $B'_3 \subset A'_3 \cap F'_*$. Furthermore, $B'_3 \subset C(\tau, f')^* \cap F'_*$. Hence

$$C(\tau_{f'}, p) \subset C(\tau, f')_* \cup F'_* \cup f' \cup p \subset C(p)_* \cup D_* \cup p.$$

Since $A_3 \subset C(p)^* \cap D^*$, A_3 does not meet $C(\tau_{f'}, p)$.

By shortening B'_3, if necessary (e.g., by choosing B_3 in $C(\tau_{e'}, p)^*$), we can assume B'_3 does not meet $C(\tau_e, p)$.

5.3. Condition (iv) (cf. Figure 6). In the following it is convenient to take e as the point at infinity. Then $C(\tau_e, f')$ and $C(\tau_e, p)$ will be represented by parallel straight lines and $C(\tau, e)$ and $C(f', p, e)$ by lines through p. As usual, the orientation of the circles through three points of A_3 induced by $e \in C(p)^*$ implies that $A_3 \subset C(\tau_e, p)^* \cap C(\tau, e)_*$. Since $C(\tau_e, p)$ does not meet B'_3, B'_3 will lie in $C(\tau_e, p)_*$. Hence A_3 will not meet $C(\tau_e, f')$. We orient $C(\tau_e, f')$ such that $A_3 \subset C(\tau_e, f')^*$.

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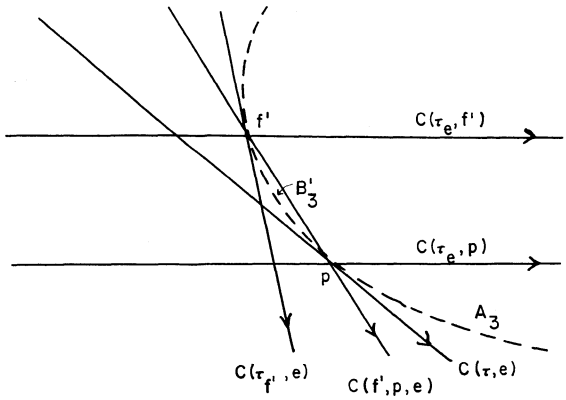

FIGURE 6

If B'_3 is chosen small enough, then $C(\tau_e, f')$ will be close to $C(\tau_e, p)$, while, by strong conformal differentiability, a circle through e and two points of $\overline{B'_3}$ will be close to $C(\tau, e)$. Since $C(\tau_e, p) \neq C(\tau, e)$, however, $C(\tau_e, f')$ does not meet $B'_3 \cup p$. Hence $B'_3 \cup p \subset C(\tau_e, f')^*$.

Next, $C(f', p, e)$ is close to $C(\tau, e)$, while a circle which meets $\overline{B'_3}$ three times is close to $C(p)$. Hence $C(f', p, e)$ cannot meet $\overline{B'_3}$ again.

Since

$$f' \subset C(\tau, e) \cap C(\tau_e, p)_*,$$

we have

$$C(f', p, e) \subset \{C(\tau, e)_* \cap C(\tau_e, p)_*\} \cup \{C(\tau, e)^* \cap C(\tau_e, p)^*\} \cup p \cup e.$$

As $A_3 \subset C(\tau, e)_* \cap C(\tau_e, p)^*$, $C(f', p, e)$ does not meet A_3. We may assume that $A_3 \subset C(f', p, e)_*$; thus $B'_3 \subset C(f', p, e)^*$.

By strong differentiability, $C(\tau_{f'}, e)$ is close to $C(\tau, e)$, while $C(\tau_e, f')$ is close to $C(\tau_e, p)$. Hence

$$C(\tau_{f'}, e) \neq C(\tau_e, f').$$

Furthermore, $C(\tau_{f'}, e) = C(f', p, e)$ would imply that $C(\tau_{f'}, e) = C(\tau_{f'}, p)$. Since $C(\tau_{f'}, p)$ is close to $C(p)$, however, we obtain $C(\tau_{f'}, e) \neq C(f', p, e)$. Since $B'_3 \subset C(f', p, e)^* \cap C(\tau_e, f')^*$, one has

$$C(\tau_{f'}, e) \subset \{C(f', p, e)^* \cap C(\tau_e, f')^*\} \cup \{C(f', p, e)_* \cap C(\tau_e, f')_*\} \cup f' \cup e.$$

Finally, as $A_3 \subset C(f', p, e)_* \cap C(\tau_e, f')^*$, we obtain that $C(\tau_{f'}, e)$ does not meet $A_3 \cup p$.

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