ON DISJOINT CONNECTED SUBSETS OF A SQUARE CONTAINING PAIRS OF ANTIPODAL POINTS

D. Morrison, M. Kiang and J. Wright

(received May 10, 1966)

§ 1. Introduction. In [1], p. 132, it is shown that there exist two disjoint subsets of a square each connecting a pair of opposite vertices. Our interest in obtaining covers by infinite collections of such sets was prompted by a question posed by R.B. Reed in [3]. We are grateful also to Dr. M. Edelstein for his guidance and encouragement of our work on the subject.

Our first theorem presents a cover by such sets, which has the power of the continuum. The construction is shown in Figures 1, 2 and 3. In brief, the points of [abcd) (see § 2 for notation) are joined to those of [fj) by a family of lines described by a homotopic transformation of [aef] into [dlkj], which sweeps out the figure abcdlkfe as t runs from 0 to 1; each of these lines is connected to one of a family of lines similar to a graph of $y = \sin \frac{1}{x}$, which family sweeps out efkl. This construction is reflected in the origin, and corresponding lines belong to the same set S_t. Each set S_t, finally, contains one point of [mn]. Fig. 1 shows a typical set, $S_{3/16}$, as well as S_o whose construction is atypical.

P. Erdös (oral communication to M. Edelstein) has expressed the opinion that a shorter proof of mere existence of a cover as described in the theorem, could be given by transfinite methods, but that the constructive proof given is interesting.

Our second theorem shows that if there are two (or more) disjoint subsets each connecting antipodal points, then neither may be closed. It remains an open question whether under any conditions such subsets can be locally connected.

Canad. Math. Bull. vol. 9, no. 5, 1966

§2. Notation and Definitions. Let Q be the closed set in E^2 bounded by the square with vertices $(\pm 16, \pm 16)$, with the usual topology relativized. The points $p = (\alpha, \beta)$ and $-p = (-\alpha, -\beta)$ will be called antipodal to one another. In § 3, the set $\{p: -p \in A\}$ is written $-A$. The line segment from p (included) to q (excluded) will be denoted $[pq)$. We write $[pqr)$ for $[pq) \cup [qr)$. Similarly we define $(pq]$, $[pqrst]$, etc. The polygonal region whose successive vertices are p, q, r, s is pqrs if open, and $\overline{pqrs}$ if closed. The set $\{x: x+r \in [pq)\}$ will be called $[pq) - r$, where + and - are vector addition and subtraction. The letter t represents a scalar parameter. The set $\{(t\alpha, t\beta): (\alpha, \beta) \in [pq)\}$ will be called $(t)[pq)$. Similarly we define $(1 - t)([pq) - r)$, etc. Certain points of Q are named as in the figures.

§3. THEOREM 1. There exists a family $\mathscr{A}$ of continuum-many, pairwise disjoint, connected subsets of Q such that each $S \in \mathscr{A}$ contains two and only two points of bd(Q), which moreover are antipodal; and $\bigcup \mathscr{A} = Q$.

The detailed construction is as follows:

(3.1) For $0 \leq t < 1/4$, define

$$A_t = (b + (1 - 4t)([aef] - b)) \cup (f + (4t)([bg] - f)).$$

$$A_o = [aef]; \text{ and as } t \to 1/4, \quad A_t \to [bg].$$

For $1/4 \leq t < 3/4$, define

$$A_t = [pq], \text{ where } p = b + (2t - 1/2)(c - b), \text{ and}$$
$$q = g + (2t - 1/2)(h - g).$$

$$A_{1/4} = [bg]; \text{ and as } t \to 3/4, \quad A_t \to [ch].$$

For $3/4 \leq t \leq 1$, define

$$A_t = [pq] \cup (c + (4t - 3)([dlk] - c)), \text{ where}$$
$$p = c + (4t - 3)(k - c), \text{ and}$$
$$q = h + (4t - 3)(j - h).$$

$$A_{3/4} = [ch]; \text{ and } A_1 = [dlkj].$$

https://doi.org/10.4153/CMB-1966-076-8 Published online by Cambridge University Press

Each A_t is connected.

If $t_1 \neq t_2$, then by the homotopic nature of the construction, $A_{t_1} \cap A_{t_2} = \emptyset$. As t runs from 0 to 1, $\{A_t\}$ sweeps out successively the figures abgfe, bchg, and cdlkh, and parts of their boundaries, so that $\{A_t\}$ sweeps out $\overline{abcdlkfe}$.

(3.2) For $0 \leq t \leq 1$, define

$$B_t = B_{1,t} \cup B_{2,t} \cup \ldots \cup B_{n,t} \cup \ldots, \text{ where:}$$

$$B_{1,t} = (j + (1-t)([yf] - j)) \cup ([yz] + (t)(j-y)) \cup (z + (t)([kr] - z))$$
$$\cup ([zy] + (t)(s-y)) \cup (s + (1-t)([yx] - s))$$

and for $n > 1$,

$B_{n,t} = \{(1/4\ \alpha, \beta) : (\alpha, \beta) \in B_{n-1,t}\}$. $B_{1,t}$ is connected. This implies that each $B_{n,t}$ is connected. Furthermore, B_t is connected; for let $(8, \eta_t) = j + (1-t)(f-j)$. Then

$$(2^{5-2n}, \eta_t) \in B_{n,t} \cap B_{n-1,t}.$$

If $t_1 \neq t_2$, then by the homotopic nature of the construction, $B_{t_1} \cap B_{t_2} = \emptyset$. As t runs from 0 to 1, $\{B_{n,t}\}$ sweeps out the region $\{(4^{1-n}\alpha, \beta) : (\alpha, \beta) \in \overline{fkrx}\}$. Therefore $\{B_t\}$ sweeps out $\overline{efkl} - [el]$.

$A_t \cup B_t$ is connected, for $A_t \cap B_t = \{(8, \eta_t)\}$.

If $0 \neq t_1 \neq 1$ and $0 \neq t_2 \neq 1$ and $t_1 \neq t_2$, then $A_{t_1} \cap B_{t_2} = \emptyset$.

(3.3) Define $i_t = m + (t)(n - m)$.

Define $S_o = A_o \cup B_o \cup (-A_1) \cup (-B_1) \cup [em] \cup A_1 \cup B_1 \cup (-A_o)$
$\cup (-B_o) \cup [ln]$.

https://doi.org/10.4153/CMB-1966-076-8 Published online by Cambridge University Press

For $0 < t < 1$, define

$$S_t = A_t \cup B_t \cup \{i_t\} \cup (-A_t) \cup (-B_t).$$

Then $\mathscr{S} = \{S_t : 0 \leq t < 1\}$ is the required family of sets.

(3.4) Proof. $\mathscr{S}$ has the power of the continuum, for it contains a set corresponding to each $t \in [0, 1)$.

The construction is such that if $t_1 \neq t_2$ then $S_{t_1} \cap S_{t_2} = \emptyset$. That is, members of $\mathscr{S}$ are pairwise disjoint.

Each S_t is connected. To prove S_o connected, note that:

$$A_o \cap B_o = (ef], \ (-A_1) \cap (-B_1) = -(lkj], \text{ and}$$

$$A_o \cap (-A_1) \cap [em] = \{e\}.$$

Therefore $A_o \cup B_o \cup (-A_1) \cup (-B_1) \cup [em]$ is connected.

Similarly, $A_1 \cup B_1 \cup (-A_o) \cup (-B_o) \cup [ln]$ is connected. The former contains m, an accumulation point of the latter, so their union S_o is connected.

For $t \neq 0$, to prove S_t is connected, note that $A_t \cup B_t$ is connected, and its closure contains $[mn]$, to which i_t belongs. Then $A_t \cup B_t \cup \{i_t\}$ lies between a connected set and its closure, and so is connected ([4], p. 13). Similarly $(-A_t) \cup (-B_t) \cup \{i_t\}$ is connected. These have a common point, so their union S_t is connected.

Each S_t, by construction, contains two and only two points of bd(Q), and these are antipodal.

$\cup \mathscr{S} = Q$; for every point in Q either lies in one of the regions swept out by $\{A_t\}$, $\{B_t\}$, $\{-A_t\}$ or $\{-B_t\}$, or is in $[em] \cup [ln] \cup S_o$, or is an i_t.

https://doi.org/10.4153/CMB-1966-076-8 Published online by Cambridge University Press

§4. THEOREM 2. Let A and B be connected subsets of Q such that $\{c, -c\} \subset A$ and $\{b, -b\} \subset B$, and suppose A is closed. Then $A \cap B \neq \emptyset$.

Proof. Let R denote the unbounded component of $E^2 - A$, and suppose F is the boundary of R. Then F is connected. See [2], p. 124.

Moreover, $F \subset A$. Indeed, if we assume $F \not\subset A$, then let $x \in F - A$. Then because A is closed, there is an open disc O such that $x \in O \subset E^2 - A$. Because F is the boundary of R, there are $y, z \in O$ such that $y \in R$, $z \in E^2 - R$. Because O is convex, $[yz] \subset O$. $[yz] \cup R$ is connected. But then R is not a maximal connected subset (component) of $E^2 - A$, as postulated.

Clearly $\{c, -c\} \subset F$ and $\{b, -b\} \subset R$. Construct in R a cross-cut $L = [c(2b)] \cup [(-c)(2b)]$. See Fig. 4. This decomposes R into two components ([4], p. 110), whose common boundary $G \subset F \cup L$. Evidently b and $-b$ lie in different components, say $b \in C_1$, $-b \in C_2$. Because B is connected, $B \cap G \neq \emptyset$ ([2], p. 73). But $B \cap L = \emptyset$, so $B \cap F \neq \emptyset$. Therefore $B \cap A \supset B \cap F \neq \emptyset$.

The coordinates below refer to points in the following figures:

a	(0, 16)	l	(0, -8)
b	(16, 16)	m	(0, 4)
c	(16, -16)	n	(0, -4)
d	(0, -16) = -a	o	(0, 0)
e	(0, 8)	r	(2, -8)
f	(8, 8)	s	(2, 4)
g	(8, 7)	x	(2, 8)
h	(8, 5)	y	(4, 8)
j	(8, 4)	z	(4, -4)
k	(8, -8)		

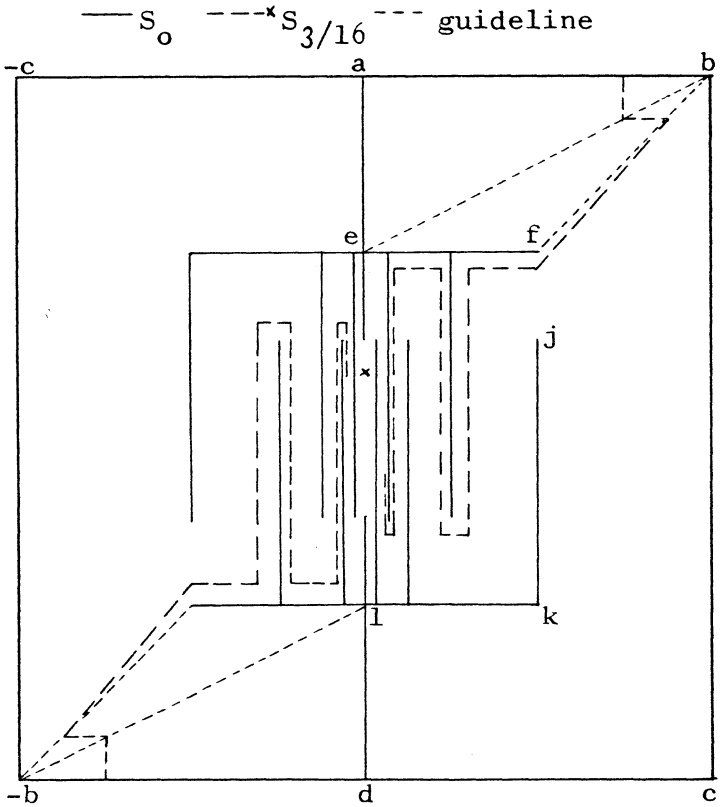

Figure 1.

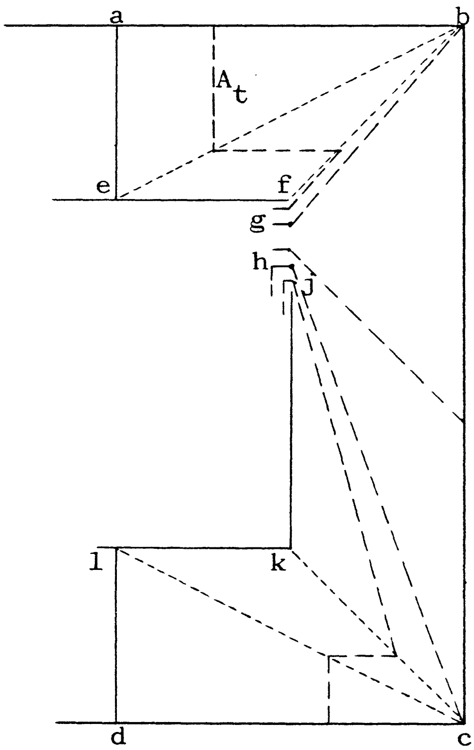

Figure 2.

https://doi.org/10.4153/CMB-1966-076-8 Published online by Cambridge University Press

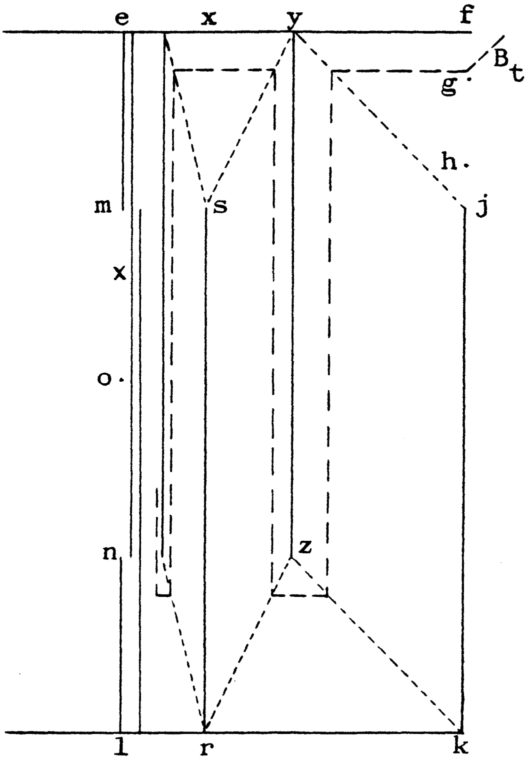

Figure 3.

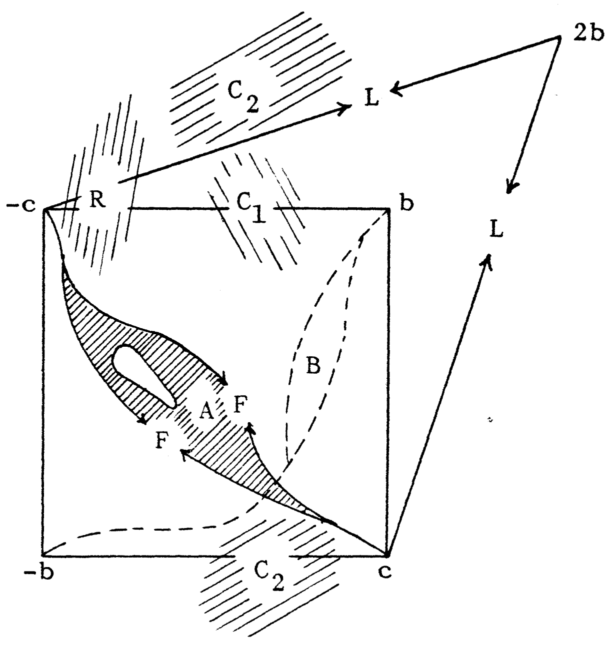

Figure 4.

https://doi.org/10.4153/CMB-1966-076-8 Published online by Cambridge University Press

REFERENCES

1. B.R. Gelbaum and J.M.H. Olmstead, Counter-examples in analysis. Holden-Day, San Francisco (1964).

2. M.H.A. Newman, Elements of the topology of plane sets of points. 2nd. ed. University Press, Cambridge (1951).

3. R.B. Reed, Problem P 108, Canadian Mathematical Bulletin, 8 (1965), page 837.

4. G.T. Whyburn, Analytic topology. Amer. Math. Soc., Colloquium publications, XXVIII (1942). Amer. Math. Soc., New York City.

Dalhousie University, Halifax

https://doi.org/10.4153/CMB-1966-076-8 Published online by Cambridge University Press