THE CLOSED IDEALS IN AN ALGEBRA OF ANALYTIC FUNCTIONS

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1. Introduction. Let K and C be the closure and boundary, respectively, of the open unit disc U in the complex plane. Let $\mathfrak{A}$ be the Banach algebra whose elements are those continuous complex functions on K which are analytic in U, with norm

$$||f|| = \max_{z \in K} |f(z)| \qquad (f \in \mathfrak{A}).$$

Sometimes it will be convenient to say that $f \in \mathfrak{A}$ even if f is defined merely in U but can be extened to K, so that the extended function is a member of $\mathfrak{A}$.

In this paper, all closed ideals of $\mathfrak{A}$ will be determined (Theorem 1). Before the result can be stated, some definitions are required.

A *Blaschke product* is a function of the form

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \bar{a}_n z} \cdot \frac{|a_n|}{a_n} \qquad (z \in U), \tag{1.1}$$

where m is a non-negative integer, $0 < |a_n| < 1$, and $\sum\{1 - |a_n|\} < \infty$. The set $\{a_n\}$ may be finite, or even empty.

A *measure* is a complex-valued completely additive set function μ defined for all Borel subsets of C; μ is *singular* if μ is concentrated on a set of Lebesgue measure zero.

Following Beurling (**1**, p. 246), we call a function of the form

$$M(z) = B(z) \exp\left\{ - \int_C \frac{w + z}{w - z} \, d\lambda\,(w) \right\} \qquad (z \in U), \tag{1.2}$$

where B is a Blaschke product and λ is a non-negative singular measure, an *inner function.*

We note that a function f, analytic in U, is an inner function if and only if f is bounded in U, f has radial limits of absolute value 1 almost everywhere on C, and the first non-zero Taylor coefficient of f is positive. The necessity of these conditions follows immediately from (1.2). Conversely, if f satisfies these conditions, there is a Blaschke product B such that f/B satisfies the same conditions and has no zeros in U; the function $g = -\log(f/B)$ is analytic in U and has non-negative real part; hence (**5**, p. 185) there is a non-negative measure λ such that

$$g(z) = \int_C \frac{w + z}{w - z} \, d\lambda\,(w) \qquad (z \in U);$$

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finally, λ must be singular, since the real part of g has radial limit 0 almost everywhere on C. It follows that f can be written in the form (1.2), i.e., f is an inner function.

If E is a closed subset of C and M is an inner function, we say that M is *associated with* E if all limit points of the zeros of B are in E and if λ is concentrated on E.

If f is bounded and analytic in U, we say that the inner function M *divides* f (or f is *divisible* by M) if f/M is bounded in U. It is shown below (Lemma 7) that if $f \in \mathfrak{A}$ and M divides f, then $f/M \in \mathfrak{A}$.

The main result can now be stated:

THEOREM 1. *Choose a closed subset E of C, of Lebesgue measure zero, and choose an inner function M which is associated with E. Let $I(E, M)$ be the set of all $f \in \mathfrak{A}$ which are divisible by M and which vanish on E. Then $I(E, M)$ is a closed ideal of $\mathfrak{A}$.*

Moreover, every closed ideal of $\mathfrak{A}$ (with the exception of the null ideal) is obtained in this manner.

The restriction on the measure of E is a natural one; for if $f \in \mathfrak{A}$ and $f(z) = 0$ on a set E on C of positive Lebesgue measure, then $f(z) = 0$ for all $z \in K$.

It seems quite remarkable that the inner functions, which are in general discontinuous at the boundary, are found to play such an important role in the ideal structure of $\mathfrak{A}$. Theorem 1 has several consequences which are stated at the end of the paper.

II. Some factorization lemmas. This section contains a number of facts which will be needed in the proof of the main theorem. Although some of these are not new, it seems advisable to include at least sketches of their proofs. In particular, Lemma 2 occurs in **(1**, p. 254**)** as a consequence of an investigation of the Hilbert space H_2. The direct proof given below, based on the purely measure-theoretic Lemma 1, may be of independent interest. Lemma 5 is stated in **(1**, pp. 245–6**)**, but no explicit proof seems to exist in print.

LEMMA 1. *Let Λ be a collection of non-negative measures. There exists a measure λ_0 such that* (a) $0 \leqslant \lambda_0(E) \leqslant \lambda(E)$ *for all $\lambda \in \Lambda$ and all Borel sets E*; (b) *if λ_1 is a measure such that* (a) *is true with λ_1 in place of λ_0, then $\lambda_1(E) \leqslant \lambda_0(E)$ for all Borel sets E.*

We may call λ_0 the largest minorant of Λ.

Proof. Let S be a partition of a fixed Borel set E. That is to say, S is a finite collection of disjoint Borel sets $E_1, \ldots, E_n$ whose union is E. Put

$$\lambda_S(E) = \sum_{i=1}^{n} \inf \lambda(E_i), \qquad (\lambda \in \Lambda)$$

and

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$$\lambda_0(E) = \inf \lambda_S(E),$$

the inf being taken over all partitions S of E.

A routine argument shows that the set function λ_0, so defined for all Borel sets, is additive. Complete additivity of λ_0 follows from the fact that λ_0 is majorized by every $\lambda \in \Lambda$. Thus λ_0 is a measure, and it is easy to see that (a) and (b) hold.

LEMMA 2. *Every non-empty collection $\mathfrak{F}$ of inner functions has a greatest common divisor M_0.*

More explicitly, there is an inner function M_0 which divides every $M \in \mathfrak{F}$ (so that M/M_0 is again an inner function), and every inner function which divides every $M \in \mathfrak{F}$ also divides M_0.

Proof. For every $M \in \mathfrak{F}$, write $M = M(B, \lambda)$, where B and λ determine M in accordance with (1.2). Let Λ be the collection of all measures λ which occur in this way, and let λ_0 be the measure whose existence is assured by Lemma 1. Let B_0 be the Blaschke product formed with the zeros which the functions $M \in \mathfrak{F}$ have in common, counting multiplicities. Then $M_0 = M(B_0, \lambda_0)$ is clearly the greatest common divisor of $\mathfrak{F}$, and the Lemma is proved.

We now require another definition. Suppose u is a real function which is summable on C; let β be a real constant; adopting Beurling's terminology **(1**, p. 246**)** we call the function

$$Q(z) = \exp\left\{\frac{1}{2\pi i}\int_C \frac{w+z}{w-z}\, u\,(w)\,\frac{d\,w}{w} + i\beta\right\} \qquad (z \in U) \tag{2.1}$$

an *outer function.* Writing u as the difference of two non-negative summable functions, we see that Q is the quotient of two bounded analytic functions; i.e., Q is of bounded characteristic **(5**, p. 178**)**.

Since $\log |Q(z)|$ is the Poisson integral of $u(w)$, the radial limits of $\log |Q(z)|$ are equal to $u(w)$ for almost all $w \in C$. Thus any outer function has the representation

$$Q(z) = \exp\left\{\frac{1}{2\pi i}\int_C \frac{w+z}{w-z}\log|Q\,(w)|\,\frac{dw}{w} + i\beta\right\}. \tag{2.2}$$

According to Nevanlinna **(5**, p. 190**)**, every[1] function f which is of bounded characteristic in U, can be written (uniquely) in the form

$$f(z) = \frac{B_1(z)}{B_2(z)}\exp\left\{\int_C \frac{w+z}{w-z}\, d\mu(w) + i\beta\right\} \qquad (z \in U), \tag{2.3}$$

where B_1, B_2 are Blaschke products, μ is a real-valued measure on C, and β is a real constant.

If we split off the absolutely continuous part of μ and apply the Jordan decomposition to the singular part σ of μ (i.e., $\sigma = \sigma_1 - \sigma_2$, with $\sigma_1 \geqslant 0$, $\sigma_2 \geqslant 0$, and σ_1, σ_2 mutually singular, we arrive at the following result:

[1]We assume here, and throughout this section, that $f(z)$ is not identically zero.

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LEMMA 3. *If f is of bounded characteristic in U, then*

$$f(z) = N_f(z)\, Q_f(z) \qquad (z \in U) \tag{2.4}$$

where N_f is the quotient of two inner functions without common factor[2], *and Q_f is an outer function given by*

$$Q_f(z) = \exp\left\{\frac{1}{2\pi i}\int_C \frac{w+z}{w-z}\log|f(w)|\frac{dw}{w} + i\beta\right\} \qquad (z \in U). \tag{2.5}$$

Let us recall that H_1 is the class of all functions h, analytic in U, for which the integral

$$\int_0^{2\pi} |h(re^{i\theta})|d\theta \qquad (0 \leqslant r < 1)$$

is a bounded function of r.

LEMMA 4. *If*

$$\int_0^{2\pi} |f(e^{i\theta})|\, d\theta < \infty$$

and Q_f is defined by (2.5), *then $Q_f \in H_1$.*

Proof. Let $P_r(\theta)$ be the Poisson kernel

$$P_r(\theta) = \frac{1}{2\pi}\cdot\frac{1-r^2}{1-2r\cos\theta + r^2}.$$

Taking absolute values in (2.5), the well-known inequality between the geometric and arithmetic means yields

$$|Q_f(re^{i\theta})| = \exp\left\{\int_0^{2\pi} P_r(\theta-\phi)\log|f(e^{i\phi})|d\phi \leqslant \int_0^{2\pi} P_r(\theta-\phi)\,|f(e^{i\phi})|d\phi,\right.$$

so that

$$\int_0^{2\pi} |Q_f(re^{i\theta})|d\theta \leqslant \int_0^{2\pi} |f(e^{i\phi})|d\phi,$$

and $Q_f \in H_1$.

LEMMA 5. *If $f \in H_1$, then $f = M_f Q_f$, where M_f is an inner function, and Q_f is given by* (2.5).

That is to say, the denominator of N_f in (2.4) reduces to 1. It is easy to deduce that the Lemma is true if $f \in H_p$, for $0 < p < \infty$, but the case $p = 1$ is the one which is needed later.

The representation of f in the form $M_f Q_f$ will be called the *canonical factorization* of f.

Proof. Put $\log^+ = \max(\log, 0)$, $\log^- = -\min(\log, 0)$.

[2]To say that two inner functions have no common factor means that the constant 1 is the only inner function which divides both of them.

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Choose $R < 1$, and replace $f(z)$ by $f(Rz)$. Since $\log |f|$ is subharmonic in U, we have, putting $z = re^{i\theta}$,

$$\log |f(Rz)| \leqslant \int_0^{2\pi} P_r(\theta - \phi) \log^+ |f(Re^{i\phi})| d\phi \tag{2.6}$$
$$- \int_0^{2\pi} P_r(\theta - \phi) \log^- |f(Re^{i\phi})| d\phi.$$

By virtue of the inequality

$$|\log^+ s - \log^+ t| \leqslant |s - t| \qquad (s, t \geqslant 0),$$

and the fact that $f(Re^{i\theta})$ tends to $f(e^{i\theta})$ in the norm of L_1, the first integral in (2.6) tends to

$$\int_0^{2\pi} P_r(\theta - \phi) \log^+ |f(e^{i\phi})| d\phi$$

as $R \to 1$. Fatou's lemma, applied to the second integral in (2.6), then leads to

$$\log |f(z)| \leqslant \int_0^{2\pi} P_r(\theta - \phi) \log |f(e^{i\phi})| d\phi \qquad (z = re^{i\theta}). \tag{2.7}$$

But (2.5) shows that $\log |Q_f(z)|$ is equal to the right member of (2.7). Hence f/Q_f is bounded in U, and the radial limits of f/Q_f have absolute value 1 almost everywhere on C (by (2.5)). It follows that f/Q_f is an inner function, if β is properly chosen in (2.5).

LEMMA 6. *Suppose $f \in \mathfrak{A}$. Let E_f be the set of all $z \in C$ such that $f(z) = 0$. Let $f = M_f Q_f$ be the canonical factorization of f. Then M_f is associated with E_f, $Q_f \in \mathfrak{A}$, and $Q_f(z) = 0$ on E_f.*

Proof. The zeros of M_f in U are the same as those of f, hence their limit points lie in E_f. If the measure λ of (1.2) were not concentrated on E_f, then λ, considered as a function of bounded variation, would have $+\infty$ as derivative at some point $z_0 \in C - E_f$ (7, p. 128); taking absolute values in (1.2), the resulting Poisson integral shows that $M(rz_0) \to 0$ as $r \to 1$. But $|Q_f(z)| \leqslant |f(z)|$, so that Q_f is bounded in U. This implies that $f(z_0) = 0$, a contradiction. Thus M_f is associated with E_f.

It follows that M_f is continuous and different from 0 at every point of $C - E_f$, so that $Q_f = f/M_f$ is continuous on $K - E_f$. The Poisson integral representation of $\log |Q_f(z)|$ (the right member of (2.7)) shows that $Q_f(z) \to 0$ as $z \to z_0 \in E_f$, and the lemma follows.

LEMMA 7. *If $f \in \mathfrak{A}$ and M is an inner function which divides f, then $f/M \in \mathfrak{A}$.*

Proof. Put $g = f/M$, $f = M_f Q_f$, $g = M_g Q_g$. By the uniqueness of the canonical factorizations, $Q_g = Q_f$ and $M_g = M_f/M$. Since M_f is associated with E_f, so is M_g. By Lemma 6, $Q_g \in \mathfrak{A}$ and $Q_g(z) = 0$ on E_f. Hence $M_g Q_g \in \mathfrak{A}$.

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III. Proof of Theorem 1. We now assume that E is a closed subset of C, of Lebesgue measure zero, that M is an inner function associated with E, and that $I(E, M)$ is the set of all $f \in \mathfrak{A}$ which are divisible by M and which vanish on E.

It is clear that $I(E,M)$ is an ideal; we have to prove that $I(E,M)$ is closed.

Choose any $g \in \mathfrak{A}$ which is in the closure of $I(E, M)$. Fix r, $0 < r < 1$, such that $M(z) \neq 0$ if $|z| = r$. Put

$$\delta = \min_{\theta} |M(re^{i\theta})|. \tag{3.1}$$

Then $0 < \delta \leqslant 1$. By our choice of g, the definition of $I(E, M)$, and Lemma 7, there exists an $f \in \mathfrak{A}$ such that

$$\| Mf - g \| < \delta. \tag{3.2}$$

This implies

$$\| f \| = \| Mf \| < \| g \| + \delta. \tag{3.3}$$

If $|z| = r$, (3.1) and (3.2) lead to

$$\left| f(z) - \frac{g(z)}{M(z)} \right| < 1, \tag{3.4}$$

so that

$$\left| \frac{g(z)}{M(z)} \right| < 1 + |f(z)| < 1 + \|g\| + \delta \leqslant 2 + \|g\| \qquad (|z| = r). \tag{3.5}$$

Since (3.5) is true for all r such that $M(z) \neq 0$ on $|z| = r$, g/M is bounded in U, so that M divides g. Also, being in the closure of $I(E, M)$, $g(z) = 0$ if $z \in E$. Hence $g \in I(E, M)$, so that $I(E, M)$ is closed, and the first part of Theorem 1 is proved.

To prove the second part, let J be a closed ideal of $\mathfrak{A}$, distinct from the null ideal. Consider the canonical factorizations $f = M_f Q_f$ for all $f \in J$ which do not vanish identically, and let M be the greatest common divisor of the functions M_f so obtained (Lemma 2). Let E_f be the subset of C on which $f(z) = 0$, and let E be the intersection of these sets E_f.

It is then clear that J is contained in $I(E, M)$. We have to prove that these two ideals are actually equal.

Let J_1 be the set of all functions f/M, with $f \in J$. Since M divides M_f, Lemma 7 shows that J_1 is a subset of $\mathfrak{A}$. Hence it is clear that J_1 is an ideal of $\mathfrak{A}$; and since $\|f/M\| = \|f\|$, J_1 is a closed ideal. If we can prove that $J_1 = I(E, 1)$, it will therefore follow that $J = I(E, M)$.

Thus it suffices to prove the theorem for the case in which M is the constant 1, i.e., there is no non-constant inner function which divides every $f \in J$.

Let μ be a measure which annihilates J, in the sense that

$$\int_C f(w)d\mu(w) = 0 \qquad (f \in J) \tag{3.6}$$

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Since J is an ideal,

$$\int_C k(w)\,f(w)d\mu(w) = 0 \quad (k \in \mathfrak{A}, f \in J). \tag{3.7}$$

By a well-known theorem of F. and M. Riesz **(6)**, (3.7) implies that for every $f \in J$ there is a function $h_f \in H_1$ whose boundary values give rise to the following equality between measures on C:

$$f\,(w)\,d\,\mu(w) = h_f\,(w)\,dw. \tag{3.8}$$

If S is a closed subset of $C - E_f$, of Lebesgue measure zero, (3.8) shows that the total variation of μ on S is zero. This is true for all $f \in J$, so that the singular part σ of μ is concentrated on E. Put $\alpha = \mu - \sigma$. Then there exists $\phi \in L_1$ on C such that

$$d\alpha(w) = \phi(w)\,dw. \tag{3.9}$$

Since $f(w) = 0$ on E and σ is concentrated on E, we have, by (3.8) and (3.9),

$$h_f(w)\,dw = f(w)\,d\mu(w) = f(w)\,d\alpha(w) = f(w)\,\phi(w)\,dw,$$

so that

$$h_f(w) = f(w)\,\phi(w) \tag{3.10}$$

for all $f \in J$ and almost all $w \in C$. Put

$$g_f(z) = \frac{h_f(z)}{f(z)} \qquad (f \in J, z \in U). \tag{3.11}$$

The functions g_f, meromorphic in U, have the same non-tangential boundary values, namely $\phi(w)$, for almost all $w \in C$. Hence **(4**; p. 159**)** $g_f = g$, a function independent of f. Since the functions $f \in J$ have no common zeros in U, g is analytic in U. Being of bounded characteristic, g is of the form (2.4), i.e., $g = N_g Q_g$, with $Q_g \in H_1$ (Lemma 4), unless g is identically zero.

Writing $h_f = h = M_h Q_h$, $f = M_f Q_f$, (3.11) gives

$$g = N_g Q_g = \frac{M_h Q_h}{M_f Q_f}.$$

It follows that $M_h = M_f N_g$, so that the denominator of N_g divides M_f, for *every* $f \in J$. Since we are dealing with the case $M = 1$, this implies that the denominator of N_g also reduces to 1, so that N_g is an inner function. Hence $g \in H_1$. This is of course true *a fortiori* in the excluded case (g identically zero).

Since g has boundary values $\phi(w)$, we have

$$\int_C k(w)\;\phi\;(w)dw = 0 \qquad (k \in \mathfrak{A}).$$

Now if $k \in \mathfrak{A}$ and k vanishes on E, that is to say, if $k \in I\,(E, 1)$, then

$$\int_C k(w)d\,\mu\,(w) = \int_C k(w)d\,\alpha\,(w) = \int_C k(w)\;\phi\;(w)dw = 0.$$

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Thus every measure which annihilates J also annihilates $I\ (E, 1)$. Hence J cannot be a proper subspace of $I\ (E, 1)$, and the theorem is proved.

IV. Consequences of Theorem 1. A closed ideal J in a Banach algebra is said to be *principal* if it is generated by a single element g, i.e., if J is the smallest closed ideal which contains g.

THEOREM 2. *Every closed ideal of $\mathfrak{A}$ is principal.*

Proof. Consider a closed ideal $I\ (E, M)$. Construct a negative summable function u on C such that u has a bounded derivative on every closed subarc of $C - E$, and such that $u(w) \to -\infty$ as $w \to w_0$, for all $w_0 \in E$. Let Q be the outer function obtained from this u by formula (2.1), with $\beta = 0$. Then $Q \in \mathfrak{A}$ (compare (**2**, pp. 342–4, 360–1)).

If $g = MQ$, then $g \in \mathfrak{A}$, since M is associated with E and Q vanishes on E. By Theorem 1, $I\ (E, M)$ is equal to the closed ideal generated by g, so that $I\ (E, M)$ is principal. The theorem follows.

Among other things, Theorem 1 implies the well-known theorem that the maximal ideals of $\mathfrak{A}$ consist of all members of $\mathfrak{A}$ which vanish at a given point of K. In fact, $I\ (E, M)$ is maximal in exactly the following three cases:

(a) E is empty, $M(z) = z$.

(b) E is empty, $M(z) = \dfrac{a - z}{1 - \bar{a}z} \cdot \dfrac{|a|}{a}$ for some $a \in U$, $a \neq 0$.

(c) E consists of one point of C, $M(z) = 1$.

The question often arises in the study of Banach alegbras whether every closed ideal is the intersection of maximal ideals (or whether every closed ideal is the kernel of its hull, in Segal's terminology (**3**, p. 56)). Theorem 1 shows that in $\mathfrak{A}$ the following is the case:[3]

THEOREM 3. *$I\ (E, M)$ is the intersection of maximal ideals of $\mathfrak{A}$ if and only if M is a Blaschke product without multiple zeros; in particular, the measure λ in the representation* (1.2) *of M must be zero.*

A closed ideal in a Banach algebra is said to be primary if it is contained in only one maximal ideal. There are two kinds of primary ideals in $\mathfrak{A}$. First, if $a \in U$ and m is a positive integer, the ideal $J\ (a, m)$ generated by $(z - a)^m$ is primary; with each $a \in U$ there is thus associated a countable set of primary ideals. Secondly, if $a \in C$ and t is a non-negative real number, the ideal $J\ (a, t)$ generated by

$$(z - a) \exp \left\{ -t \frac{a + z}{a - z} \right\}$$

[3]Whenever the symbol $I\ (E, M)$ appears, it is understood that M is associated with E.

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is primary. For $a \in U$, $J(a, m) \subset J(a, n)$ if and only if $m \geqslant n$; for $a \in C$, $J(a, t) \subset J(a, s)$ if and only if $t \geqslant s$. There are no other inclusion relations between primary ideals.

In some Banach algebras every closed ideal is the intersection of primary ideals, although there are closed ideals which are not intersections of maximal ideals (for references, see (**3**, p. 182)). Theorem 1 shows that in $\mathfrak{A}$ the situation is as follows:

THEOREM 4. *$I(E, M)$ is the intersection of primary ideals of $\mathfrak{A}$ if and only if the measure λ in the representation* (1.2) *of M has no continuous component* (i.e., *if λ is the sum of an at most countable number of point-measures concentrated on E*).

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