SEPARATION AXIOMS AND DIRECT LIMITS

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A topological space X is called a direct limit of a family (X_α) of subspaces of X if and only if

(1) $\bigcup X_\alpha = X$

(2) a subset of X is closed in X provided that its intersection with each X_α is closed in X_α.

If X is a direct limit of an increasing sequence (X_n) of closed subspaces then it is well known and easy to prove that X is a T_1-space resp. a T_4-space provided all X_n are T_1-spaces resp. T_4-spaces. The following example shows that the corresponding statement is false for any of the properties T_2, Urysohn (i.e., for any two different points there exist disjoint closed neighbourhoods); T_3, or completely regular.

All spaces considered in this paper are assumed to be T_1-spaces.

Example. Let (C_n) be a sequence of pairwise disjoint, completely regular, non-normal spaces and let (A_n, B_n) be a pair of disjoint closed subsets of C_n which cannot be separated by open sets in C_n. Add to the union of all C_n two points a and b and topologize this set X in the following manner:

$$U \subset X \text{ open} \Leftrightarrow \begin{cases} 1. & U \cap C_n \text{ is open in } C_n \text{ for all } n; \\ 2. & a \in U \Rightarrow A_n \subset U \text{ for all but a finite number of} \\ 3. & b \in U \Rightarrow B_n \subset U \text{ for all but a finite number of} \end{cases}$$

Then X is not a T_2-space but is a direct limit of the increasing sequence of completely regular, closed subspaces

$$X_n = \{a, b\} \cup \bigcup_1^\infty A_m \cup \bigcup_1^\infty B_m \cup \bigcup_1^n C_m .$$

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Remark. (B. Banaschewski), $X \times X$ is not a direct limit of $(X_n \times X_n)$ since the diagonal ΔX of $X \times X$ is not closed in $X \times X$ but, for all n, $\Delta X \cap (X_n \times X_n) = \Delta X_n$ is closed in $X_n \times X_n$.

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