UNITARY GROUPS GENERATED BY REFLECTIONS

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1. Introduction. A *reflection* in Euclidean n-dimensional space is a particular type of congruent transformation which is of period two and leaves a prime (i.e., hyperplane) invariant. Groups generated by a number of these reflections have been extensively studied [**5**, pp. 187–212]. They are of interest since, with very few exceptions, the symmetry groups of uniform polytopes are of this type. Coxeter has also shown [**4**] that it is possible, by Wythoff's construction, to derive a number of uniform polytopes from any group generated by reflections. His discussion of this construction is elegantly illustrated by the use of a graphical notation [**4**, p. 328; **5**, p. 84] whereby the properties of the polytopes can be read off from a simple *graph* of nodes, branches, and rings.

The idea of a reflection may be generalized to unitary space U_n [**11**, p. 82]; a *p-fold reflection* is a unitary transformation of finite period p which leaves a prime invariant (2.1). The object of this paper is to discuss a particular type of unitary group, denoted here by $[p\ q;\ r]^m$, generated by these reflections. These groups are generalizations of the real groups with fundamental regions B_n, E_6, E_7, E_8, T_7, T_8, T_9 [**5**, pp. 195, 297]. Associated with each group are a number of complex polytopes, some of which are described in §6. In order to facilitate the discussion and to emphasise the analogy with the real groups, a graphical notation is employed (§3) which reduces to the Coxeter graph if the group is real.

Every polytope Π_n, whether real or complex, is associated with a configuration in projective space P_{n-1}, which may be derived by taking the centre of the polytope as origin and then interpreting the coordinates of the vertices as homogeneous coordinates in P_{n-1}. (The collineation group associated with the configuration is the group that corresponds to the symmetry group of the polytope [**11**, p. 84].) Many well-known and interesting configurations are associated with the polytopes whose symmetry groups are of the type $[p\ q;\ r]^m$ such as the configuration of 126 points in five dimensions recently investigated by Todd [**13**; **14**], Hamill [**8**], and Hartley [**9**]. We shall refer to this as the *Mitchell-Hamill configuration.*

Some of the polytopes discussed are degenerate, that is, analogous to the honeycombs of Euclidean space [**5**, p. 127]. One of these is of particular interest since its vertices are the points of a lattice associated with the extreme duodenary form K_{12} [**7**].

Throughout the paper the definitions and notation of the author's *Regular Complex Polytopes* [**11**] are assumed, but a table of notations is added for reference.

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Symbol	Meaning
E_n	Euclidean space of n dimensions.
U_n	Unitary space of n dimensions [**11**, p. 82].
P_n	Projective space of n dimensions.
Π_n	Any polytope in E_n or U_n.
$\mathfrak{G}\,(\Pi_n)$	The symmetry group of Π_n [**11**, p. 84].
$\mathfrak{G}_E(\Pi_n)$ or $\mathfrak{G}_U(\Pi_n)$	The group of (orthogonal or unitary) symmetry matrices of Π_n.
Π_n^{+1}	A polytope whose vertex figure is Π_n [**11**, pp. 85, 87].
Π_n^{+r}	A polytope whose vertex figure is $\Pi_n^{+(r-1)}$.
$\alpha_n, \beta_n, \gamma_n$	The real regular polytopes of E_n (see §6 and [**2**, p. 344]).
β_n^m, γ_n^m	The generalized cross-polytope and orthotope (§6).
$t_r\Pi_n$	The rth truncation of Π_n (§6).
$p_1(q_1)p_2 \ldots (q_{n-1})p_n$	The extended Schläfli symbol for a regular polytope [**11**, p. 88].
$[p\ q;\ r]^m$, $(p_i\ q;\ r)^m$	See §4.
$[3^{p,\,q,\,r}]$	See [**5**, p. 200].
$\{\pi_1, \pi_2\}$	See 2.3.

I must express my indebtedness to J. A. Todd and H. S. M. Coxeter for their advice and suggestions in carrying out the investigations described in this paper. I am especially grateful to the former for undertaking the formidable task of checking the abstract definitions in 4.12.

2. Reflections. Every real non-degenerate uniform polytope Π_n in E_n has a symmetry group $\mathfrak{G}(\Pi_n)$ which is generated by at most n elements. In $\mathfrak{G}_E(\Pi_n)$, the group of orthogonal symmetry matrices, it is generally possible to choose these generators as reflection matrices, that is, matrices whose characteristic roots are 1 (repeated $n - 1$ times) and -1. By reducing to diagonal form in the usual manner, a reflection matrix may be written $\mathbf{S'AS}$ where $\mathbf{S}$ is orthogonal and $\mathbf{A}$ is the matrix $\operatorname{diag}(1^{n-1}, -1)$.

The choice of the generators as reflections is not possible in a few anomalous cases, of which the most familiar are the two "snub" polyhedra [**4**, pp. 336–337].

The idea of a reflection can be extended to U_n. A p-fold reflection matrix is defined as a matrix of period p which leaves a prime of U_n invariant.

2.1. *A p-fold reflection matrix is a unitary matrix with characteristic roots* 1 (*repeated n —* 1 *times*) *and θ, a primitive pth root of unity.*

It may be written in the form $\mathbf{\bar{S}'AS}$ where $\mathbf{S}$ is unitary and $\mathbf{A}$ is the matrix diag $(1^{n-1}, \theta)$. If the invariant prime has the equation

$$\sum_{i=1}^{n} a_i x_i \equiv \mathbf{a'x} = 0,$$

then the equation of the p-fold reflection is

2.2
$$\mathbf{x}^* = (\mathbf{I} - \mathbf{ba}')\mathbf{x}$$

where $\mathbf{b}$ is chosen so as to make the transformation unitary, and

$$\mathbf{b}'\mathbf{a} = 1 - \exp(2\pi i/p).$$

Regular complex polygons [**11**, pp. 89–93] have symmetry groups generated by two reflections of this type; the polygon whose extended Schläfli symbol is $p_1(q_1)p_2$ [**11**, p. 88] has a symmetry group generated by two elements S, T corresponding to the matrices:

S which is a p_1-fold reflection permuting the vertices on an edge of the polygon cyclically, and

T which is a p_2-fold reflection permuting the vertices of a vertex figure of the polygon cyclically [cf. **11**, p. 90].

It will be readily verified that all the symmetry groups of complex regular polytopes in U_n may be generated in a similar manner by n p-fold reflections.

Suppose that

$$\mathbf{p}_1 \equiv \sum_{i=1}^{n} a_i x_i = 0, \quad \mathbf{p}_2 \equiv \sum_{i=1}^{n} b_i x_i = 0$$

are two primes of U_n. Then we define

2.3
$$\{\mathbf{p}_1, \mathbf{p}_2\} = \left|\sum_{i=1}^{n} \bar{a}_i b_i\right|.$$

This is an invariant under unitary transformations. If $\{\mathbf{p}_1, \mathbf{p}_1\} = 1$, we say that $\mathbf{p}_1$ is *normalized*, the normalization being unaffected by multiplying the equation $\mathbf{p}_1$ by any complex number of unit modulus. For two real normalized primes, $\{\mathbf{p}_1, \mathbf{p}_2\}$ is the cosine of the angle between the primes.

2.4 *In U_n, the group generated by 2-fold reflections in two normalized primes $\mathbf{p}_1, \mathbf{p}_2$ is of finite order if and only if $\{\mathbf{p}_1, \mathbf{p}_2\}$ is the cosine of a rational angle, that is, a rational multiple of π. Further, if $\{\mathbf{p}_1, \mathbf{p}_2\} = \cos \pi h/k$ where h and k have no common divisor, then the order of the group is $2k$.*

If the primes are real, the result is familiar, for reflections in two primes inclined at an angle $h\pi/k$ generate a group of order $2k$. The proof of the result depends upon showing that the statement can be reduced to that of a property of real primes in E_n, by suitable choice of coordinate system.

Choose the coordinate system so that $\mathbf{p}_1$ is $x_1 = 0$, and $\mathbf{p}_2$ is $b_1 x_1 + b_2 x_2 = 0$. (To do this we have only to ensure that the intersection $\mathbf{p}_1 . \mathbf{p}_2$ is $x_1 = x_2 = 0$.) Since the equation of $\mathbf{p}_2$ may be multiplied by any complex number of unit modulus without altering the normalization, we do this in such a manner that b_1 is real. If we now change the coordinate system by writing

$$x_i^* = x_i \qquad (i = 1, 3, 4, \ldots, n),$$
$$x_2^* = (\bar{b}_2/|b_2|)x_2,$$

then the equations of $\mathbf{p}_1$ and $\mathbf{p}_2$ are both real, the matrices 2.2 are real, and the theorem follows from the result for E_n.

In order to discuss the group generated by a set of reflections it is convenient to introduce a notation for a set of reflecting primes which characterizes their geometrical relationships, but is independent of the coordinate system. The notation is the *graph* defined in §3.

3. Graphs. An elegant graphical notation for groups generated by reflections in E_n and the associated polytopes was invented by Coxeter [**5**, p.84]. Briefly, the graph for a group consists of a number of *nodes* and *branches* (called *dots* and *links* in [**4**]) constructed according to the rules:

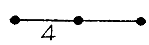
3.1 *Each reflecting prime is symbolized by a node of the graph.*

3.2 *If the angle between two primes is π/k then the corresponding nodes are joined by a branch if $k > 2$, and the branch is labelled "k" if $k \geqslant 4$.*

Conventionally branches are not numbered 3 since this type occurs most frequently.

Thus the graph for a finite group in E_n has n nodes. It may be *disconnected*, that is to say, consist of two or more separate parts which have no interconnecting branches, and then the corresponding group is the direct product of the groups represented by each part.

As an example of the graphical notation, the symmetry group of the cube is generated by reflections in three planes inclined at angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, and $\frac{1}{4}\pi$ to each other. It is denoted graphically by

3.3 

So that the graph corresponding to a given group is uniquely defined, we specify that the reflecting primes must bound a *fundamental region* of the group.

Suppose now that a unitary group is generated by reflections in a number of primes. More precisely suppose that each generator is a p_i-fold reflection in $\mathbf{p}_i$ ($i = 1, 2, \ldots, N$). If these primes are concurrent it is convenient to take the point of concurrency as the origin of the coordinate system. In any case, the graph is constructed according to the following rules:

3.4 *Each of the reflecting primes $\mathbf{p}_i$ is symbolized by a node P_i of the graph, and this node is labelled "p_i" if $p_i > 2$.*

3.5 *Each pair of nodes $\mathrm{P}_i\mathrm{P}_j$ is connected by a branch labelled $\frac{1}{2}k$, where k is the order of the group generated by reflections in $\mathbf{p}_i$ and $\mathbf{p}_j$, except that:*

if $\{\mathbf{p}_i, \mathbf{p}_j\} = 0$, the branch is omitted,
if $k = 6$, the branch is left unlabelled.

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These conventions are adopted so that 3.4 and 3.5 reduce to rules **3.1** and 3.2 if all the primes are real.

As an example, the group generated by reflections in the n primes

3.6
$$\begin{aligned} &\textit{m-fold}: \quad x_1 = 0, \\ &\textit{2-fold}: \quad x_i - x_{i-1} = 0 \qquad (i = 2, 3, \ldots, n), \end{aligned}$$

has the graph

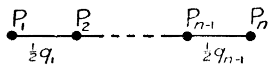
3.7 

(n nodes).

More generally, the graph of a set of reflecting primes that generate the symmetry group of the regular polytope with extended Schläfli symbol

$$p_1(q_1)p_2(q_2) \ldots (q_{n-1})p_n$$

is a simple chain



A unitary group is said to be *false* if all the matrices can be reduced to orthogonal form by suitable change of coordinate system. Otherwise the group is a *true* unitary group. Thus the group generated by two reflections in 2.4 is false, and the proof of the result depended upon this fact.

Let $A_1, A_2, \ldots, A_m$ be m nodes of a graph representing a set of primes. If the pairs of nodes $A_1 A_2, A_2 A_3, \ldots, A_{m-1} A_m, A_m A_1$ are joined by branches, then the nodes $A_1 A_2 \ldots A_m$ are said to form a *circuit* [**4**, p. 328]. All finite orthogonal groups (and therefore all false unitary groups) have graphs that do not contain any circuits [**5**, p. 297]. A connected graph without any circuits is called a *tree* [**4**, p. 328].

3.8 *The graph of a set of primes that generate a true unitary group has either a numbered node (that is to say, there is a p-fold reflection with $p > 2$) or a circuit.*

A graph with a numbered node necessarily represents a true unitary group since a matrix of type 2.2 with $p > 2$ cannot be real in any coordinate system. Suppose therefore that the group is generated by 2-fold reflections in the primes $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$, where

3.9
$$\begin{aligned} \mathbf{p}_i &\equiv \sum_{j=1}^{n} a_{ij} x_j = 0 \qquad (i = 1, 2, \ldots, n), \\ \mathbf{p}_i &\equiv \sum_{j=1}^{n} a_{ij} x_j = c_i \qquad (i = n + 1, \ldots, N). \end{aligned}$$

For the purposes of the proof we take $c_i = 0$ (all i). This corresponds to a "paral-

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lel displacement" of the prime and this does not affect the proof. Let P_i be the node of the graph corresponding to the prime $\mathbf{p}_i$ and suppose that the graph is a tree T. We prove, by induction on the number of nodes that the corresponding unitary group is false.

If there is only one node the group is certainly false, so we assume shat the result has been established for the group generated by reflections in $r - 1$ primes $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{r-1}$ such that the corresponding nodes $P_1, P_2, \ldots, P_{r-1}$ and branches form a sub-tree T_1 of T. Choose a new coordinate system so that $\mathbf{p}_1$ is the prime $x_1 = 0$, the intersection $\mathbf{p}_1 \cdot \mathbf{p}_2$ is $x_1 = x_2 = 0$, and generally the intersection $\mathbf{p}_1 \cdot \mathbf{p}_2 \cdot \ldots \cdot \mathbf{p}_l$ is $x_1 = x_2 = \ldots = x_l = 0$. In this system

$$\mathbf{p}_i \equiv \sum_{j=1}^{r-1} b_{ij} x_j = 0,$$

where $b_{ij} = 0$ $(j > i)$ and, by the induction hypothesis all the other coefficients are real. Regarding these primes as normalized,

$$\sum_{j=1}^{r-1} b_{ij}^2 = 1$$

and

$$\sum_{j=1}^{r-1} b_{ij} b_{kj} = |\sum_{j=1}^{n} \bar{a}_{ij} a_{kj}| = \{\mathbf{p}_i, \mathbf{p}_k\}.$$

Add to the tree T_1 another node P_r of T in such a manner that the nodes $P_1, P_2, \ldots, P_r$ and the associated branches form another sub-tree of T. P_r is connected by one and only one branch to a node of T for if there were more than one branch the resultant graph would contain a circuit. Suppose that the nodes are numbered so that P_r is joined to P_1 only. Then if

$$\mathbf{p}_r \equiv \sum_{j=1}^{r} b_{rj} x_j = 0$$

is the normalized equation of the prime, it follows that

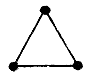
3.10 $$|\sum_{j=1}^{r} \bar{b}_{rj} b_{1j}| = |\sum_{j=1}^{n} \bar{a}_{rj} a_{1j}|,$$

3.11 $$\sum_{j=1}^{n} b_{rj} b_{ij} = 0 \qquad (i = 2, 3, \ldots, r-1).$$

But 3.10 determines b_{r1} as a real quantity since the only non-vanishing term on the left is $b_{r1}b_{11}$ and the right-hand side is real. Equations 3.11 determine $b_{r2}, b_{r3}, \ldots, b_{r(r-1)}$ successively as real quantities, and b_{rr} may be made real by multiplying the x_r coordinate by a suitable complex number of unit modulus.

Hence the result is established for connected graphs. It follows for disconnected graphs since the different parts of the graph are independent, representing primes whose intersections are absolutely orthogonal subspaces of U_n. The theorem is therefore true.

The converse of the result is false, since a graph with a circuit may represent an infinite discrete reflection group in E_n. For example the graph



is P_3, a subgroup of index 2 in the symmetry group of the degenerate polyhedron forming the plane honeycomb of hexagons. This example also shows that a graph with a circuit may represent two or more different sets of reflecting primes. (By the proof of 3.8, a graph with no circuit represents a unique set of real primes, within a parallel displacement.) In order to make the correspondence between the graphs and sets of primes unique it is necessary to label the circuits of the graph according to the rule:

3.12 *If the nodes* $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_r$ *of a graph form a circuit and* P_i *corresponds to the prime* $\mathbf{p}_i$, *where*

$$\mathbf{p}_i \equiv \sum_j a_{ij} x_j = 0 \qquad (i = 1, 2, \ldots, r),$$

then the circuit is labelled with the number k where

3.13 $$e^{\pi i/k} = \prod_{i=1}^{r} \Big(\sum_j a_{ij}\bar{a}_{(i+1)j}\Big) \Big/ \prod_{i=1}^{r} \{\mathbf{p}_i, \mathbf{p}_{i+1}\}.$$

Here, for convenience of notation $a_{(r+1)j} = a_{1j}$ (all j). In the cases we consider in §4, k will be an integer. If $k = 1$ the circuit will be left unlabelled.

The meaning of 3.13 is more easily understood if we arrange that $\sum_j a_{ij}\bar{a}_{(i+1)j}$ is real for all i except possibly $i = r$, and then

3.14 $$\sum_j a_{rj}\bar{a}_{1j} = \{\mathbf{p}_r, \mathbf{p}_1\}\, e^{\pi i/k}.$$

Determining the equations of the primes as in the proof of 3.8, it will be seen that a graph with nodes, branches, and circuits labelled according to rules 3.4, 3.5, and 3.12 now represents a set of reflecting primes uniquely (within a parallel displacement) and so represents a unitary group completely. It is not, however, always possible to find a set of primes corresponding to any given graph.

If the normalized equations of the primes $\mathbf{p}_i$ are taken as in 3.9, we write $\mathbf{A}$ for the matrix (a_{ij}) and then

3.15 $$D \equiv \det(\bar{\mathbf{A}}\,\mathbf{A}') = |\det \mathbf{A}|^2 \geqslant 0.$$

This is evidently a necessary condition for the set of primes to exist. The determinant D is called the *Schläfli determinant* [**3**, p. 137; **5**, pp. 134–135] and its importance lies in the fact that if all the reflections are 2-fold it can be written down from the graph. If d_{ij} is the (i, j)th term of D, then

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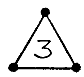
3.16 $d_{ii} = 1$ (all i),
$d_{ij} = 0$ if the corresponding nodes P_iP_j of the graph are not connected by a branch,
$d_{ij} = \cos(\pi/k)$ if the corresponding nodes P_iP_j of the graph are connected by a branch labelled "k".

The only other condition is that where the nodes $P_1, P_2, \ldots, P_r$ form a circuit labelled h, the factor $e^{\pi i/h}$ must be put before one of the terms $d_{12}, d_{23}, \ldots, d_{r1}$ and the factor $e^{-\pi i/h}$ must be put before the corresponding term with suffixes reversed (see equation 3.14).

By way of an example, consider the graph



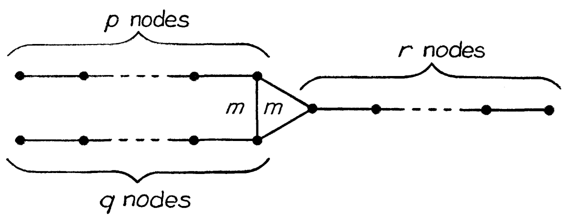
The Schläfli determinant is then

$$\begin{vmatrix} 1 & \frac{1}{2} & -\frac{1}{2}\omega \\ \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2}\omega^2 & \frac{1}{2} & 1 \end{vmatrix}$$

and its value is $\frac{1}{8}$, and so the graph satisfies the condition 3.15.

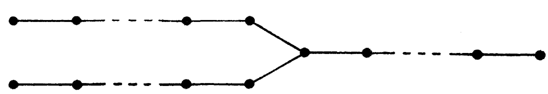
4. The groups $[p\ q;\ r]^m$. Consider the group generated by $p + q + r$ 2-fold reflections:

4.1



It is denoted by the symbol $[p\ q; r]^m$, noting that when $m = 2$, the graph becomes

4.2



so that $[p\ q;\ r]^2 \equiv [3^{p, q, r-1}]$ in the notation of [**5**, p. 200]. This alternative notation is useful since it exhibits the symmetry between the numbers p, q, $r - 1$. That is to say, the group is the same whatever the order of the indices. In general

$$[p\ q; r]^m \equiv [q\ p; r]^m$$

and if $m = 3$, the p, q, r may be permuted in any way.

A necessary condition for $[p\ q; r]^m$ to exist is given by the Schläfli determinant.

Writing $\theta = e^{\pi i/m}$,

$$D \equiv \begin{array}{c|ccc|cccc|cccc|cccc|}
 & 1 & \theta^{-1}\cos\frac{\pi}{m} & \frac12 & \frac12 & & & & & & & & & & & \\
 & \theta\cos\frac{\pi}{m} & 1 & \frac12 & & & & & \frac12 & & & & & & & \\
 & \frac12 & \frac12 & 1 & & & & & & & & & \frac12 & & & \\
\hline
 & \frac12 & & & 1 & \frac12 & & & & & & & & & & \\
p-1 & & & & \frac12 & 1 & \ddots & & & & & & & & & \\
\text{rows} & & & & & \ddots & \ddots & \frac12 & & & & & & & & \\
 & & & & & & \frac12 & 1 & & & & & & & & \\
\hline
 & & \frac12 & & & & & & 1 & \frac12 & & & & & & \\
q-1 & & & & & & & & \frac12 & 1 & \ddots & & & & & \\
\text{rows} & & & & & & & & & \ddots & \ddots & \frac12 & & & & \\
 & & & & & & & & & & \frac12 & 1 & & & & \\
\hline
 & & & \frac12 & & & & & & & & & 1 & \frac12 & & \\
r-1 & & & & & & & & & & & & \frac12 & 1 & \ddots & \\
\text{rows} & & & & & & & & & & & & & \ddots & \ddots & \frac12 \\
 & & & & & & & & & & & & & & \frac12 & 1
\end{array} \geqslant 0.$$

Direct evaluation gives (within a positive factor)

4.3 $$1 - pq\{r + 4\cos^2(\pi/m) - 1\} + p + q + r \geqslant 0.$$

When $m = 2$ this reduces to

$$pq(r - 1) \leqslant p + q + r + 1,$$

which is the condition for group $[3^{p,q,r-1}]$ to exist in a Euclidean space [**3**, p. 143].

When $D = 0$, the primes are linearly dependent, for this implies (3.15) that det $\mathbf{A} = 0$. Two cases arise according to whether we take the primes as concurrent or not. Apart from an anomalous case mentioned in a footnote to table 4.4, there are no new groups defined by concurrent primes, and so we take their equations as in 3.9 with all the c_i zero except $c_{n+1} = 1$ (or any other non-zero constant).

Table 4.4 lists all possible values of p, q, r and m satisfying 4.3. The table also gives suitable sets of reflecting primes (not necessarily in the smallest possible number of dimensions) and the order of each of the groups (computed from the abstract definitions in table 4.12).

By applying the rules 4.5–4.9, the abstract definition of each of these groups may be written down from its graph. The definitions are given in full in table 4.12.

4.4 *Table of groups* $[p\ q;\ r]^m$

m	group	reflecting primes	order
2	$[3^{1,1,n-3}] \equiv B_n$ $[3^{1,2,2}] \equiv E_6$ $[3^{1,2,3}] \equiv E_7$ $[3^{1,2,4}] \equiv E_8$ $[3^{2,2,2}] \equiv T_7$ $[3^{1,3,3}] \equiv T_8$ $[3^{1,2,5}] \equiv T_9$	$(a)_n$, (b) $(a)_6$, (c) $(a)_7$, (c) $(a)_8$, (c) $(a)_6$, (c)′, (d) $(a)_8$, (e) $(a)_9$, (c)′	$2^{n-1}.n!$ $72.6!$ $8.9!$ $192.10!$ ∞ ∞ ∞
3	$[1\ 1;\ n-2]^3$ $[2\ 1;\ 2]^3$ $[2\ 1;\ 3]^3$ $[2\ 1;\ 4]^3$	$(a)_n$, (f) $(a)_4$, (f), (g) $(a)_5$, (f), (g) $(a)_6$, (f), (g)′	$3^{n-1}.n!$ $72.6!$ $108.9!$ ∞
4	$[1\ 1;\ n-2]^4$ $[2\ 1;\ 1]^4$ $[2\ 1;\ 2]^4$ $[3\ 1;\ 1]^4$	$(a)_n$, (h) $(a)_3$, (h), (k) $(a)_4$, (h), (k)′ $(a)_3$, (h), (k), (l)[1]	$4^{n-1}.n!$ $64.5!$ ∞ ∞
m	$[1\ 1;\ n-2]^m$	$(a)_n$, (m)	$m^{n-1}.n!$

$(a)_s$ $x_i - x_{i-1} = 0$ $\quad (i = 2, 3, \ldots, s)$,
(b) $x_2 + x_1 = 0$,
(c) $2(x_1 + x_2 + x_3) - (x_4 + x_5 + x_6 + x_7 + x_8 + x_9) = 0$,
(c)′ $2(x_1 + x_2 + x_3) - (x_4 + x_5 + x_6 + x_7 + x_8 + x_9) = 1$,
(d) $(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) - 2(x_7 + x_8 + x_9) = 0$,
(e) $(x_1 + x_2 + x_3 + x_4) - (x_5 + x_6 + x_7 + x_8) = 1$,
(f) $x_1 - \omega x_2 = 0$ $\quad$ (ω a primitive cube root of unity),
(g) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$,
(g)′ $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$,
(h) $x_1 - ix_2 = 0$,
(k) $x_1 + x_2 + x_3 + x_4 = 0$,
(k)′ $x_1 + x_2 + x_3 + x_4 = 1$,
(l) $x_4 = 1$,
(m) $x_1 - \theta x_2 = 0$ $\quad$ (θ a primitive mth root of unity).

Denote the nodes of the graph by $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_r$, the node P_i being labelled p_i, that is, corresponding to a p_i-fold reflection. Let P_i denote the operation in

[1]If the primes are concurrent, that is, we take $x_4 = 0$ instead of (l), then this is a finite group of order 64.6!. It is the symmetry group of the polytope $(\frac{1}{2}\gamma_3^4)^{+1}$ described later (6.13).

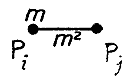
$\mathfrak{G}$ corresponding to reflection in the prime $\mathbf{p}_i$, and let E be the identity of $\mathfrak{G}$. Then $\mathfrak{G}$ is generated by $P_1, P_2, \ldots, P_r$ subject to the relations:

4.5 $(P_i)^{p_i} = 1$ $\qquad (i = 1, 2, \ldots, n)$.

4.6 $P_iP_j = P_jP_i$ *for every pair of nodes* P_i, P_j *not connected by a branch.*

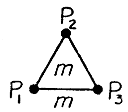
4.7 $(P_iP_j)^k = 1$ *for every pair of nodes* P_i, P_j *connected by a branch labelled* k, *and with* $p_i = p_j = 2$.

4.8 *A relation is required connecting* P_iP_j *when* p_i *or* p_j *is not* 2. This cannot be read off from the graph immediately, but the required relation for



is $(P_i P_j)^2 = (P_j P_i)^2$.

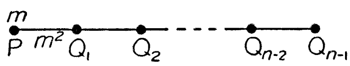
4.9 *If* $P_1, P_2, \ldots, P_l$ *form a circuit, one further relation is required connecting the operations* $P_1, P_2, \ldots, P_l$. In the case of a circuit of the form



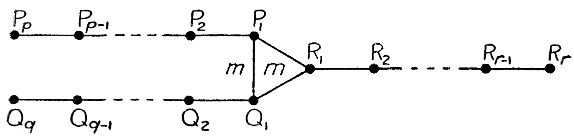
a suitable relation is $(P_2 P_3 P_1)^2 = (P_3 P_1 P_2)^2$.

Table 4.12 gives the abstract definitions of the groups $\mathfrak{G}(\beta^m{}_n)$ (see §6) and the finite groups $[p\ q;\ r]^m$; 4.10 and 4.11 indicate the method of labelling the nodes. Abstract definitions for the groups $[3^{p,q,r}]$ are given in [**3**, pp. 144–151].

4.10



4.11



4.12 *Table of abstract definitions.*

group	generators	relations
$\mathfrak{G}(\gamma_2^m)$	P, Q_1	(a)
$\mathfrak{G}(\gamma_n^m)$	$P, Q_1, Q_2, \ldots, Q_{n-1}$	(a), (b)
$\mathfrak{G}(\frac{1}{m}\gamma_3^m) \equiv [1\ 1;\ 1]^m$	P_1, Q_1, R_1	$(c)_m$
$\mathfrak{G}(\frac{1}{m}\gamma_n^m) \equiv [1\ 1;\ n-2]^m$	$P_1, Q_1, R_1, R_2, \ldots, R_{n-2}$	$(c)_m$, (d)
$[2\ 1;\ 2]^3$	P_2, P_1, Q_1, R_1, R_2	$(c)_3$, (e)
$[3\ 1;\ 2]^3$	$P_3, P_2, P_1, Q_1, R_1, R_2$	$(c)_3$, (e), (f)
$[2\ 1;\ 1]^4$	P_2, P_1, Q_1, R_1	$(c)_4$, (g)

(a) $P^m = Q_1^2 = 1;\ (PQ_1)^2 = (Q_1P)^2,$

(b) $Q_i^2 = (Q_i Q_{i-1})^3 = 1 \qquad (i = 2, 3, \ldots, n-1);$
$PQ_i = Q_iP \qquad (i = 2, 3, \ldots, n-1);$
$(Q_iQ_j)^2 = 1 \qquad (i, j = 1, 2, \ldots, n-1;\ |i-j| \geqslant 2),$

$(\mathrm{c})_m$ $P_1^2 = Q_1^2 = R_1^2 = (Q_1R_1)^3 = (R_1P_1)^3 = (P_1 Q_1)^m = 1;$
$(P_1 Q_1 R_1)^2 = (R_1 P_1 Q_1)^2,$

(d) $R_i^2 = (P_1 R_i)^2 = (Q_1 R_i)^2 = (R_i R_{i-1})^3 = 1 \qquad (i = 2, 3, \ldots, n-2);$
$R_i R_j = R_j R_i \qquad (i, j = 1, 2, \ldots, n-2;\ |i-j| \geqslant 2),$

(e) $P_2^2 = R_2^2 = (P_2P_1)^3 = (P_2Q_1)^2 = (P_2R_1)^2 = (P_2R_2)^2 = (P_1R_2)^2 = (Q_1R_2)^2 = (R_1R_2)^3 = 1,$

(f) $P_3^2 = (P_3P_2)^3 = (P_3P_1)^2 = (P_3Q_1)^2 = (P_3R_1)^2 = (P_3R_2)^2 = 1,$

(g) $P_2^2 = (P_2P_1)^3 = (P_2Q_1)^2 = (P_2R_1)^2 = 1.$

The above definitions have been checked by the Todd-Coxeter method [**15**]. In the case of the larger groups the work can be simplified by considering a polytope Π_n whose symmetry group is being examined, and taking as the generating subgroup, the symmetry group of one of the vertex figures of Π_n. The vertices of Π_n are then in 1–1 correspondence with the cosets of this subgroup, and the work in the coset tables can be continually checked.

I am indebted to J. A. Todd for the following remark about the group $[3\ 1; 2]^3$. The given relations $(\mathrm{c})_3$, (e), (f) imply

$$(P_3P_2P_1QR_1R_2)^{42} = 1.$$

If, however, we postulate $(P_3P_2P_1QR_1R_2)^7 = 1$, the resulting factor group is of order $18 \cdot 9!$, being the *collineation* group [**11**, p. 84] corresponding to $[3\ 1; 2]^3$, viz, the group of the Mitchell-Hamill configuration in five dimensions [**8**].

5. Graphs for polytopes. In order to represent a polytope graphically we add to the graph of its symmetry group one or more rings round the nodes [**4**, p. 329]. Of particular interest are the polytopes denoted by a graph with only one ring, and we define these in 5.1.

First, we suppose that the reflecting primes lie in space of n dimensions, are n in number and are linearly independent. Let O be the point of concurrency of the primes. Define G_i $(i = 1, 2, \ldots, n)$ as being any point at unit distance from O and lying on the line of intersection of $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n$, and let P_i be the node corresponding to $\mathbf{p}_i$.

5.1 *If the node* P_i *of the graph is ringed, then the new graph represents the polytope of which one vertex is* G_i *and the other vertices are the images of* G_i *under the operations of the group.*

For example, 3.3 represents a cube if the left node is ringed, and an octahedron if the right node is ringed.

If, on the other hand, the group is generated by reflections in $n + 1$ non-concurrent primes, then G_i is defined as the point of intersection of all the primes

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except the ith, and rule 5.1 still applies. Only in these two cases will the graphical notation for a polytope be employed.

So far, the choice of reflecting primes for a given unitary group has not been restricted in any way, so that a number of different graphs may correspond to the same group. It is now convenient to discuss some of the restrictions that may be imposed, so that the graph of the polytope has a number of additional properties. Select the primes so that:

5.2 *Reflections in* $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ (*or* $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n+1}$) *generate the group.*

5.3 *The points* $G_1, G_2, \ldots, G_n$ (*or* $G_1, G_2, \ldots, G_{n+1}$) *are not equivalent, that is to say, one cannot be transformed into another by an operation of the group.*

5.4 *If there is more than one set of primes satisfying* 5.3, *choose that set which makes as many as possible of the polytopes given by ringing one node different.*

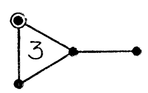
5.5 *The image of* G_i *under reflection in* $\mathbf{p}_i$ *is at least as near* G_i *as any other point equivalent to* G_i. Thus the vertex G_i of the polytope with the ith node ringed is transformed by reflection in $\mathbf{p}_i$ into a point of its vertex figure.

The primes denoted by the graphs 4.1 have all these properties, and conversely for a group of this type, the set of primes satisfying 5.2, 5.3, 5.4, and 5.5 is unique, within an operation of the group. There is reason to suppose that selection according to the above rules is possible for any finite n-dimensional unitary group generated by n reflections, or any discrete infinite group generated by $n + 1$ reflections in non-concurrent primes, but this has not been proved. With this choice of primes the rule given by Coxeter [**5**, p. 198] for obtaining the graph of the vertex figure of a polytope still holds:

5.6 *If the ringed node belongs to only one branch, we obtain the vertex figure by removing that node (along with its branch) and transferring the ring to the node to which that branch was connected.*

For example the vertex figure of 6.7 is 6.9 and the vertex figure of 6.9 is

5.7



If the polytope is a polygon, application of this rule leaves us with a single ringed node, labelled "p" say. This is to be interpreted as a p-line [**11**, p. 85].

5.8 *In order to determine the number of vertices of any given polytope* Π_n, *consider the group* $\mathfrak{G}^*$ *which corresponds to the graph formed by removing the ringed node (and any branches connected to it) from the graph of* Π_n. *The number of vertices is then the quotient of the order of* $\mathfrak{G}(\Pi_n)$ *by the order of* $\mathfrak{G}^*$ [cf. **4**, p. 329].

For example, in the case of the polytope 6.10, the group $\mathfrak{G}^*$ has the graph

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This is the symmetry group of the regular simplex α_4, and so is of order 5!. The symmetry group of 6.10 is $[2\ 1;\ 2]^3$ of order 72.6!, and so the number of vertices is $72.6!/5! = 432$.

Some of the bounding figures of a complex polytope may be determined from its graph in the same manner as for a real polytope [**4**, p. 334]. There are, however, other bounding figures not given by this procedure. A simple example is the icosahedron $2(6)2(10)2$, which, in addition to twenty triangles contains twelve pentagons of the same edge length. These are not counted as bounding figures of the real polyhedron since they lie inside the figure. There is no distinction between *interior* and *exterior* of a complex polyhedron [**11**, p. 83] and so in this case these pentagons (which are not given by the graph) must be included.

In order to facilitate reference, it is convenient to define a symbol for the polytope whose graph is given by ringing one of the nodes of the graph of $[p\ q;\ r]^m$. Referring to 4.11, if P_i is ringed, the polytope is denoted by $(p_i\ q;\ r)^m$ and similarly, suffixes are added to the q or r if nodes Q_i or R_i are ringed [cf. **4**, p. 331]. For example, $(2\ 1;\ 3_3)^3$, which is the same as $(3_3\ 2;\ 1)^3$, is the polytope represented by graph 6.7.

6. Fractional γ polytopes. In n dimensions $(n > 4)$ there are three real regular non-degenerate polytopes [**2**, p. 344]. These are the regular simplex, the cross polytope, and the measure polytope or orthotope. They are denoted by α_n, β_n, and γ_n respectively. In U_n $(n > 4)$, in addition to the simplex there are two series of regular polytopes: the generalized cross polytopes and the generalized orthotopes [**11**, p. 96]. The first of these, which is denoted by $\beta^m{}_n$, has mn vertices:

$$({}_m1, 0, 0, \ldots, 0)'$$

in the abbreviated notation. (The pre-suffix m implies that the 1 may be multiplied by any mth root of unity, and the prime means that the coordinates are to be permuted in every way. For a fuller explanation see [**11**, p. 96].) The extended Schläfli symbol for this polytope is $2(6)2(6)\ \ldots\ (6)2(2m^2)m$.

The reciprocal polytope is the generalized orthotope denoted by $\gamma^m{}_n$ and first described by Coxeter [**4a**, p. 287]. It has m^n vertices:

6.1 $$(\theta^{k_1}, \theta^{k_2}, \ldots, \theta^{k_n})$$

where $k_1, k_2, \ldots, k_n$ take any integral values and θ is a primitive mth root of unity. The symmetry groups of $\beta^m{}_n$ and $\gamma^m{}_n$ are identical, of order $m^n.n!$, and their abstract definitions are given as $\mathfrak{G}(\beta^m{}_n)$ in table 4.12. The group is generated by reflections in the primes 3.6 and so may be represented by the graph 3.7. The polytope $\beta^m{}_n$ is denoted by ringing the node furthest to the right, and $\gamma^m{}_n$ is denoted by ringing the node on the left.

Polytopes corresponding to the same graph but with any other node ringed are what may be called truncations of $\beta^m{}_n$ or $\gamma^m{}_n$. If the node Q_μ (see 4.10) is

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ringed we may denote the resulting polytope by $t_p\gamma^m{}_n$ or $t_{n-p-1}\beta^m{}_n$ by analogy with the corresponding notation in the theory of real polytopes [**2**, p. 354; **5**, pp. 145–148]. The vertices of this polytope are

6.2 $$({}_m1, {}_m1, \ldots, {}_m1, 0, 0, \ldots, 0)'$$

with p terms zero and n coordinates in all. They are the centres of the α_{n-p-1} that bound $\beta^m{}_n$ or of the $\gamma^m{}_p$ that bound $\gamma^m{}_n$.

Evidently $\beta^2{}_n$ and $\gamma^2{}_n$ are the real polytopes β_n and γ_n respectively, so that we conventionally omit the superscript if its value is 2.

The eight vertices of γ_3 may be divided into two sets of four such that each set consists of the vertices of a regular tetrahedron α_3. The same process may be applied to real γ polytopes of higher dimension, and we write $\frac{1}{2}\gamma_n$ (which is the $h\gamma_n$ of Coxeter [**2**, p. 362]) for the polytope whose vertices are the "alternate" vertices of γ_n; that is, we select half the vertices of γ_n in such a manner that no two are joined by an edge of γ_n. For example $\frac{1}{2}\gamma_4 = \beta_4$ and in general in the notation of Coxeter [4, p. 331; **2**, p. 372],

$$\tfrac{1}{2}\gamma_n = 1_{(n-3)1} = (1_1\ 1\ (n-3)).$$

In a similar manner we can select a subset of the vertices of $\gamma^m{}_n$ so that the points of this subset are equivalent. Writing the coordinates of the vertices as in 6.1, instead of allowing $k_1, k_2, \ldots, k_n$ to take any integral values we consider the points for which

$$\sum k_i \equiv 0 \pmod{m}.$$

There are precisely m^{n-1} such vertices and there are m similar subsets in all, given by the m congruence classes of $\sum k_i$ modulo m. Taking these m^{n-1} points as vertices, and lines joining pairs of vertices whose distance apart is $\sqrt{2}$ as edges we obtain a polytope which will be denoted by $\frac{1}{m}\gamma^m{}_n$.

For example, $\frac{1}{3}\gamma^3{}_3$ has nine vertices:

$$(\omega^{k_1}, \omega^{k_2}, \omega^{k_3})$$

where $\omega^3 = 1$, $k_1 + k_2 + k_3 \equiv 0 \pmod 3$, or

$$_3(1, 1, 1),\ (1, \omega, \omega^2)'.$$

These nine points are the vertices of $\beta^3{}_3$, and may be reduced to the more familiar form $({}_31, 0, 0)'$ [**11**, p. 96] by the transformation

$$\mathbf{x}^* = \tfrac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}\mathbf{x}.$$

In general $\frac{1}{m}\gamma^m{}_n$ will not be regular, but there are four exceptional cases:

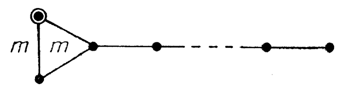
6.3 $$\tfrac{1}{2}\gamma_3 = \alpha_3, \qquad \tfrac{1}{3}\gamma^3{}_3 = \beta^3{}_3,$$
$$\tfrac{1}{2}\gamma_4 = \beta_4, \qquad \tfrac{1}{m}\gamma^m{}_2 = \{m\},$$

where $\{m\}$ is the real regular m-gon.

The symmetry group of $\frac{1}{m}\gamma^m{}_n$ is of order $m^{n-1} \, . \, n!$ and is generated by reflections in n primes,

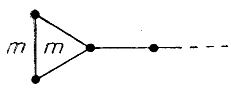
6.4
$$\textit{2-fold:} \quad x_1 - \theta x_2 = 0 \qquad (\theta^m = 1),$$
$$\textit{2-fold:} \quad x_i - x_{i-1} = 0 \qquad (i = 2, 3, \dots, n),$$

from which we deduce that it is the group $[1\ 1;\ n-2]^m$. Hence $\frac{1}{m}\gamma^m{}_n$ is the polytope $(1_1\ 1;\ n-2)^m$ and its graph is

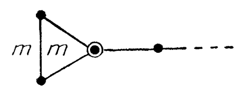


with n nodes. The polytopes corresponding to the same graph but with other nodes ringed are identical with the truncations of $\gamma^m{}_n$ or $\beta^m{}_n$, with vertices 6.2. This may be stated in the form of a rule:

6.5 *A polytope is unaltered if its graph is changed by replacing*

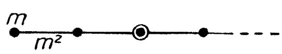


or



by

or



respectively.

When $m = 2$ the rule is already known [**4**, p. 333]. It is only the polytope which is unchanged by the above replacements; the order of the group is increased by a factor m.

Another polytope whose vertices are a selection of the vertices of a γ polytope is that which we shall denote by $\frac{1}{2}\gamma^4{}_n$. It has 2.4^{n-1} vertices:

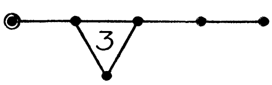
$$(i^{k_1}, i^{k_2}, \dots, i^{k_n}), \qquad \sum k_j \equiv 0 \pmod 2,$$

where $i^2 = -1$. The symmetry group of this polytope is not generated by n reflections (see the note to table 4.4), and so cannot be symbolized graphically. We shall refer to this polytope later as the vertex figure of a lattice.

Polytopes associated with the groups $[p\ q;\ r]^m$ may be derived from the polytopes $\frac{1}{m}\gamma^m{}_n$ by using the latter as their vertex figures. In fact,

$$(p_p\ 1;\ n-2)^m = (\tfrac{1}{m}\gamma^m{}_n)^{+p-1}.$$

For example, $(2_2\ 1;\ 3)^3$, with graph



has, as its vertex figure (see 5.6), the polytope $(1_1\ 1; 3)^3 = \frac{1}{3}\gamma^3{}_5$, and so we may write

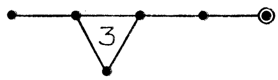
$$(2_2\ 1; 3)^3 = (\tfrac{1}{3}\gamma^3{}_5)^{+1}.$$

It has 4032 vertices (by 5.8), whose coordinates, in the abbreviated notation, are

6.6
$$\begin{array}{ll} {}_6(3, 0, 0, 0, 0, 0)'; & \pm\lambda({}_31, {}_31, {}_31, 0, 0, 0)'; \\ {}_6(2, -\omega^{k_1}, -\omega^{k_2}, -\omega^{k_3}, -\omega^{k_4}, -\omega^{k_5}) & \qquad \sum k_i \equiv 0 \pmod 3, \end{array}$$

where $\lambda = 1 - \omega$. The edge length is $\sqrt{6}$ and the vertex distance is 3. The related projective configuration (in which the coordinates of the vertices are interpreted as homogeneous coordinates in P_5) consists of the 672 points known as the H-points (the vertices of 112 *α-hexahedra*) of the Mitchell-Hamill configuration [**8**; **9**; **13**; **14**].

Another polytope having the same symmetry group is $(3_3\ 1; 2)^3$ or $(2\ 1; 3_3)^3$ or $(\frac{1}{3}\gamma^3{}_4)^{+2}$, with the graph

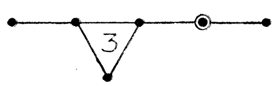
6.7


It has 756 vertices,

6.8
$$\begin{array}{ll} \pm\lambda({}_31, -{}_31, 0, 0, 0, 0)'; & \\ \pm(\omega^{k_1}, \omega^{k_2}, \omega^{k_3}, \omega^{k_4}, \omega^{k_5}, \omega^{k_6}), & \qquad \sum k_i \equiv 0 \pmod 3, \end{array}$$

and its edge length and vertex distance are both $\sqrt{6}$. The related projective configuration consists of the 126 centres of homologies of the Mitchell-Hamill configuration.

Polytopes symbolized by ringing other single nodes of the graph have the same symmetry group, $[3\ 1; 2]^3$ or $[2\ 1; 3]^3$, so that the related projective configuration will again be connected with the Mitchell-Hamill configuration. For example, $(2\ 1; 3_2)^3$ or



has 30,240 vertices:

$$\pm({}_33, {}_33, 0, 0, 0, 0)'; \quad \pm\lambda({}_32, -{}_31, -{}_31, 0, 0, 0)';$$
$$\left.\begin{array}{l} \pm\lambda(\omega^{k_1}, \omega^{k_2}, \omega^{k_3}, -\omega^{k_4}, -\omega^{k_5}, -\omega^{k_6})', \\ \pm(2\omega^{k_1}, 2\omega^{k_2}, 2\omega^{k_3}, 2\omega^{k_4}, -\omega^{k_5}, -\omega^{k_6})', \\ \pm(2-\omega)\omega^{k_1}, (2-\omega^2)\omega^{k_2}, \omega^{k_3}, \omega^{k_4}, \omega^{k_5}, \omega^{k_6})', \end{array}\right\} \quad \sum k_i \equiv 0 \pmod 3.$$

Its edge length and circumradius are $3\sqrt{2}$. The related projective configuration consists of 5040 points lying by threes on the 1680 κ-lines [**8**, p. 403] of the

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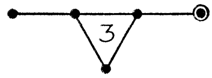
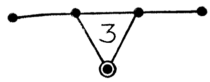
Mitchell-Hamill configuration. Each point is the harmonic conjugate of one of the 126 points with respect to the other two that lie on the κ-line.

The symmetry group of $(2\ 1;\ 2_2)^3$ is $[2\ 1;\ 2]^3$ of order 51,840. Its graph is

6.9 

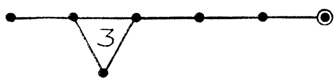
and so it may be written $(\frac{1}{3}\gamma^3{}_4)^{+1}$. It has 80 vertices, the points 6.8 that lie on $\sum x_i = 3$. Evidently the points 6.6 or 6.8 lying on primes parallel to this will have the same symmetry group. In particular $(2\ 1_1;\ 2)^3$ or

6.10 

has 432 vertices, the points 6.6 on $\sum x_i = 3$.

The collineation group corresponding to $[2\ 1;\ 2]^3$ is the simple group of order 25,920 which is familiar as the collineation group of the *Baker configuration* [**1**; **12**]. Consequently the related projective configurations are associated with the Baker figure. For example the intersections of the 40 κ-lines through any point of the Mitchell-Hamill configuration meet the prime polar to that point [8, p. 402] in 40 points forming the projective figure related to $(2\ 1;\ 2_2)^3$.

The only other group with $m = 3$ to be discussed is the degenerate group generated by reflections in $n + 1$ primes, $[2\ 1;\ 4]^3$. The degenerate polytope $(2\ 1;\ 4_4)^3$ or $(\frac{1}{3}\gamma^3{}_4)^{+3}$ or

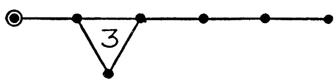


is of exceptional interest since its vertices form the lattice associated with the extreme duodenary form K_{12} of [7]. The simplest way of exhibiting the vertices (discovered by Todd and Coxeter) is

6.11 $$(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)$$

where the x_i are integers of the field $R(\omega)$ mutually congruent modulo λ, and whose sum is congruent to zero modulo 3.

The polytope $(2_2\ 1;\ 4)^3$ or $(\frac{1}{3}\gamma^3{}_6)^{+1}$ with graph



has vertices which do not form a lattice but are a subset of the points 6.11. The coordinate vectors of the vertices are the aggregate of vectors

$$\mathbf{a} + \lambda\mathbf{b}$$

where $\mathbf{b}$ is any vector of the set 6.11 and $\mathbf{a}$ may be any one of

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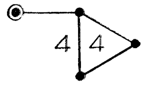
$$(0, 0, 0, 0, 0, 0); \quad \lambda({}_31, {}_31, {}_31, 0, 0, 0)',$$
$$\left.\begin{array}{l}(\omega^{k_1}, \omega^{k_2}, \omega^{k_3}, \omega^{k_4}, \omega^{k_5}, \omega^{k_6}),\\ (2\omega^{k_1}, 2\omega^{k_2}, -\omega^{k_3}, -\omega^{k_4}, -\omega^{k_5}, -\omega^{k_6})',\end{array}\right\} \qquad \sum k_i \equiv 0 \pmod 3.$$

Now consider polytopes associated with the groups $[p\ q;\ r]^4$. The only finite group is $[2\ 1;\ 1]^4$, and the polytope $(2_2\ 1;\ 1)^4$ has the graph



It is $(\frac{1}{4}\gamma^4{}_3)^{+1}$ and has 80 vertices:

6.12
$$\begin{array}{ll} {}_4(2, 0, 0, 0)'; & \\ (i^{k_1}, i^{k_2}, i^{k_3}, i^{k_4}), & \qquad \sum k_j \equiv 0 \pmod 4.\end{array}$$

The edge length and vertex distance are both 2. The related projective configuration consists of 20 points in three dimensions which form the vertices of five tetrahedra selected out of the 15 tetrahedra that form the *Klein configuration* [**10**, p. 48], the selection being made in such a manner that no three of the five belong to a desmic system.

The 60 vertices of the complete Klein configuration are related to the 240 vertices of the polytope $(\frac{1}{2}\gamma^4{}_3)^{+1}$:

6.13
$$\begin{array}{ll} {}_4(2, 0, 0, 0)'; & \\ ({}_4(1 + i), {}_4(1 + i), 0, 0)'; & \\ (i^{k_1}, i^{k_2}, i^{k_3}, i^{k_4}), & \qquad \sum k_j \equiv 0 \pmod 2.\end{array}$$

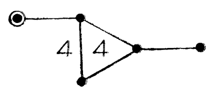
The symmetry group of this figure is of order 46,080. The corresponding real 8-dimensional polytope is $(PA)_8$ or 4_{21} [**2**, p. 385; **5**, pp. 201, 204].

The polytope $(\frac{1}{2}\gamma^4{}_3)^{+2}$ is degenerate with vertices

6.14
$$(x_1, x_2, x_3, x_4)$$

where the x_j are integers of the field $R(i)$ mutually congruent modulo $(1 - i)$ and whose sum is congruent to zero modulo 2. The vertices form a lattice. (Both this and the lattice $(\frac{1}{3}\gamma^3{}_4)^{+3}$ bear a remarkable resemblance to the lattices 3_{31}, 5_{21}, and 2_{22} of [**6**, pp. 420–421].)

The degenerate polytope $(2_2\ 1;\ 2)^4$, $(\frac{1}{4}\gamma^4{}_4)^{+1}$, or



has vertices with coordinate vectors

6.15
$$\mathbf{a} + (1 - i)\mathbf{b}$$

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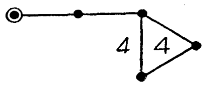
where **b** is any coordinate vector of type 6.14 and **a** is any vector

$$(0, 0, 0, 0), \quad (i^{k_1}, i^{k_2}, i^{k_3}, i^{k_4}), \qquad \sum k_j \equiv 0 \pmod 4.$$

The degenerate polytope $(3_3\ 1;\ 1)^4$, $(\frac{1}{4}\gamma^4{}_3)^{+2}$, or



has coordinates of type 6.15 with **b** of type 6.14 and **a** any vector of the set 6.12.

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