Probability in the Engineering and Informational Sciences, **30**, 2016, 492–513.
doi:10.1017/S0269964816000139

FINITE TWO LAYERED QUEUEING SYSTEMS

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We study layered queueing systems comprised two interlacing finite $M/M/\bullet$ type queues, where users of each layer are the servers of the other layer. Examples can be found in file sharing programs, SETI@home project, etc. Let L_i denote the number of users in layer i, $i = 1, 2$. We consider the following operating modes: (i) All users present in layer i join forces together to form a single server for the users in layer j ($j \neq i$), with overall service rate $\mu_j L_i$ (that changes dynamically as a function of the state of layer i). (ii) Each of the users present in layer i individually acts as a server for the users in layer j, with service rate μ_j.

These operating modes lead to three different models which we analyze by formulating them as finite level-dependent quasi birth-and-death processes. We derive a procedure based on Matrix Analytic methods to derive the steady state probabilities of the two dimensional system state. Numerical examples, including mean queue sizes, mean waiting times, covariances, and loss probabilities, are presented. The models are compared and their differences are discussed.

1. INTRODUCTION

The motivation for studying networks of queues where customers in each queue act as servers of other queues arises from distributed computer architectures labeled "peer-to-peer", designed for sharing computer resources (such as Seti@Home and others, see for example, Androutsellis-Theotokis and Spinellis [1] and references therein). A specific example can be found in Arazi, Ben-Jacob and Yechiali [2]. When activating such programs users connect into a peer-to-peer network to search for files on the computers of other users (i.e., peers) connected to the network. Files of interest can then be downloaded directly from those users. Typically, large files are broken down into smaller portions, which may be obtained from multiple peers and then reassembled by the downloader. This is done while the peer is simultaneously uploading the portions it already has to other peers. Hence, once a user activates a file sharing program, he/she operates as a server for the other connected users, and also as a customer downloading a file. Many studies on peer-to-peer apply Queueing Theory models (see e.g., Wu, Liu, and Ross [13]) to analyze the probabilistic characteristics of such systems, modeling them as layered network in which a server, while executing a service, may request a higher-layer service and wait for it to be completed. Layered queueing networks occur naturally in all kinds of information and e-commerce systems, grid systems, and real-time systems such as telecom switches; see Franks et al. [7] and references therein for an overview. Layered queues are characterized by simultaneous or separate phases where entities are no longer classified in the traditional roles of 'servers' and 'customers', but may also have a dual role of being either a server to other entities (of lower layers) or a customer to higher-layer entities. When modeling queueing systems often it is analytically convenient to assume that the queues are of infinite capacity. However, most real-world queueing systems are of finite capacity. Therefore, in this study we analyze finite-buffer queues where customers simultaneously provide and request service. Initial steps in the analysis of queues where users act as servers can be found in Perel and Yechiali [11,12], where the analysis in [11] is based on probability generating functions (PGFs), while in [12] both PGFs and Neuts's Matrix Geometric method [10] are employed. The present work extends the scope of the analysis to the case where capacities of both queues are finite, and employ other matrix analytic methods since the two methods used in [11,12] are not applicable in the current study.

Quasi birth-and-death (QBD) processes have been used extensively to model a variety of systems, mostly representing cases of unbounded populations where the service and/or inter-arrival times are given by phase-type distributions (see e.g., [10], Latouche and Ramaswami [9], and references there). For easying numerical calculations truncation methods can be used (see e.g., Bright and Taylor [3]). Finite homogeneous QBD processes were studied by Hajek [8], while finite non-homogeneous QBD processes are considered in De Nitto Personé and Grassi [5]. The QBD processes representing our models *differ* from the above processes. We exploit their special structure to employ direct matrix calculations.

Consider a system comprised of two finite connected and dependent queues (layers), where customers (users) of each queue render service to the customers of the other queue. We study three models as follows:

In Model 1 (Section 2) we assume that one queue, Q_1, operates as a multi-server finite-buffer (of size N) $M(\lambda_1)/M(\mu_1)/L_2/N$ system with Poisson arrival rate λ_1 and Exponential service time with mean $1/\mu_1$ for each individual customer, where the potential servers at Q_1 are the L_2 customers present in queue 2 (Q_2). That is, each customer present in Q_2 *individually* acts as a server for the customers in Q_1, such that, at any given moment, the actual number of active servers in Q_1 is $Min(L_1, L_2)$. The other queue (Q_2) has capacity K and operates as a single-server finite-buffer $M(\lambda_2)/M(\mu_2 L_1)/1/K$ system with Poisson arrival rate λ_2, but with dynamically changing service rate $\mu_2 L_1$ for the single served customer. That is, the L_1 customers present in Q_1 join hands together and form a *single* server with an overall service rate $\mu_2 L_1$ for the customers in Q_2.

In Model 2 (Section 3) we assume that Q_1 operates as in Model 1, namely as an $M(\lambda_1)/M(\mu_1)/L_2/N$ system, but Q_2 operates as a finite-buffer multi-server (rather than a single-server) $M(\lambda_2)/M(\mu_2)/L_1/K$ system.

In Model 3 (Section 4), Q_1 operates as a finite-buffer single-server (rather than a multi-server) $M(\lambda_1)/M(\mu_1 L_2)/1/N$ system, served by the L_2 customers present in Q_2, while Q_2 itself operates as in Model 1, namely as an $M(\lambda_2)/M(\mu_2 L_1)/1/K$ queue, drawing its servers from its opposite queue. This operating mode resembles file-sharing systems.

We formulate each of the three models described above as a finite level-dependent quasi birth-and-death (LDQBD) process and study its steady-state behavior. Utilizing the special structure of the generator matrix of the QBD processes arising in our models, we calculate each system's steady-state probabilities. We further calculate numerically the mean total number of customers in each queue, $E[L_1]$ and $E[L_2]$, mean sojourn time, $E[W_1]$ and $E[W_2]$, as well as the probability of blocking at Q_i, $i = 1, 2$. We show that in Model 2 the carried loads of the queues, namely, $\lambda_i(1 - \mathbb{P}(Q_i \text{ is blocked}))/\mu_i$, $i = 1, 2$, are equal, while in Model 3, for both queues the effective arrival rate, $\lambda_i(1 - \mathbb{P}(Q_i \text{ is blocked}))$, is smaller than the potential mean service rate, $\mu_i E[L_j]$, $i = 1, 2$, $j \neq i$ (as is the case for finite-buffer single-server Markovian queue). In Section 5, we present numerical calculations for each of the three models and discuss the results. Section 6 summarizes the paper.

2. MODEL 1

First we address the case where Q_1 is a multi-server finite-buffer $M(\lambda_1)/M(\mu_1)/L_2/N$ system, while Q_2 is a single-server finite-buffer $M(\lambda_2)/M(\mu_2 L_1)/1/K$ queue. All arrival and service processes are mutually independent. Without loss of generality we assume that $N \leq K$.

2.1. Balance Equations

The pair (L_1, L_2) defines a non-reducible continuous-time finite Markov process. For case (i), where $N < K$, the transition-rate diagram is depicted in Figure 1. Let $P_{nm} = \mathbb{P}(L_1 = n, L_2 = m)$, $0 \leq n \leq N$ and $0 \leq m \leq K$, denote the system's stationary probabilities. Then, the set of balance equations is given as follows:

$$\begin{aligned}
&\underline{n=0}:\\
&m=1: \quad (\lambda_1+\lambda_2)P_{01} = \mu_1 P_{11}\\
&2 \leq m < K: \quad (\lambda_1+\lambda_2)P_{0m} = \lambda_2 P_{0,m-1} + \mu_1 P_{1m}\\
&m=K: \quad \lambda_1 P_{0K} = \lambda_2 P_{0,K-1} + \mu_1 P_{1K} \qquad \textbf{(2.1)}
\end{aligned}$$

$$\begin{aligned}
&\underline{n=1}:\\
&m=0: \quad (\lambda_1+\lambda_2)P_{10} = \mu_2 P_{11}\\
&m=1: \quad (\lambda_1+\lambda_2+\mu_1+\mu_2)P_{11} = \lambda_1 P_{01} + \lambda_2 P_{10} + \mu_1 P_{21} + \mu_2 P_{12}\\
&2 \leq m < K: \quad (\lambda_1+\lambda_2+\mu_1+\mu_2)P_{1m} = \lambda_1 P_{0m} + \lambda_2 P_{1,m-1} + 2\mu_1 P_{2m} + \mu_2 P_{1,m+1}\\
&m=K: \quad (\lambda_1+\mu_1+\mu_2)P_{1K} = \lambda_1 P_{0K} + \lambda_2 P_{1,K-1} + 2\mu_1 P_{2K}. \qquad \textbf{(2.2)}
\end{aligned}$$

$$\begin{aligned}
&\underline{2 \leq n \leq N-1}:\\
&m=0: \quad (\lambda_1+\lambda_2)P_{n0} = \lambda_1 P_{n-1,0} + n\mu_2 P_{n1}\\
&1 \leq m \leq n: \quad (\lambda_1+\lambda_2+m\mu_1+n\mu_2)P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1}\\
&\qquad + m\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1}\\
&n < m < K: \quad (\lambda_1+\lambda_2+n\mu_1+n\mu_2)P_{nm} = \lambda_1 P_{n-1,m}\\
&\qquad + \lambda_2 P_{n,m-1} + (n+1)\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1}
\end{aligned}$$

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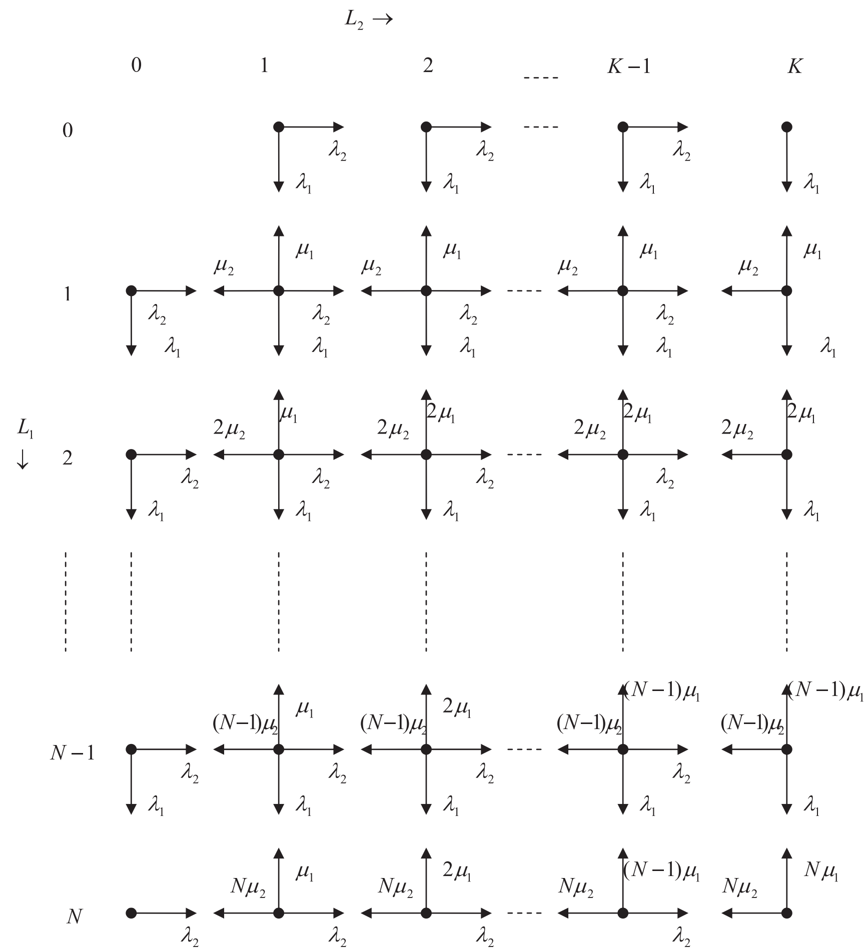

FIGURE 1. Transition rate diagram of (L_1, L_2) for Model 1.

$$m = K: \quad (\lambda_1 + n\mu_1 + n\mu_2)P_{nK} = \lambda_1 P_{n-1,K} + \lambda_2 P_{n,K-1} + (n+1)\mu_1 P_{n+1,K} \tag{2.3}$$

$\underline{n = N}$:

$$m = 0: \quad \lambda_2 P_{N0} = \lambda_1 P_{N-1,0} + N\mu_2 P_{N1}$$

$$1 \le m \le N: \quad (\lambda_2 + m\mu_1 + N\mu_2)P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1}$$

$$N < m < K: \quad (\lambda_2 + N\mu_1 + N\mu_2)P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1}$$

$$m = K: \quad (N\mu_1 + N\mu_2)P_{NK} = \lambda_1 P_{N-1,K} + \lambda_2 P_{N,K-1}. \tag{2.4}$$

Define (where $P_{00} = 0$) the marginal probabilities

$$\mathbb{P}(L_1 = n) \equiv P_{n\bullet} = \sum_{m=0}^{K} P_{nm} \text{ for } 0 \le n \le N,$$

$$\mathbb{P}(L_2 = m) \equiv P_{\bullet m} = \sum_{n=0}^{N} P_{nm} \text{ for } 0 \le m \le K.$$

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Then for every $0 \le m \le K-1$, summing Eqs. (2.1)–(2.4) over n yields

$$\lambda_2 P_{\bullet m} = \mu_2 P_{\bullet m+1} \mathbb{E}[L_1 | L_2 = m+1] \tag{2.5}$$

By summing (2.5) over m we get

$$\lambda_2(1 - P_{\bullet K}) = \mu_2(\mathbb{E}[L_1] - P_{\bullet 0}\mathbb{E}[L_1 | L_2 = 0]) = \mu_2 \left(\mathbb{E}[L_1] - \sum_{n=1}^{N} n P_{n0} \right). \tag{2.6}$$

That is, the effective arrival rate to Q_2, $\lambda_2(1 - P_{\bullet K})$, equals the actual service rate $\mu_2 \left(\mathbb{E}[L_1] - \sum_{n=1}^{N} n P_{n0} \right)$. Note that $\sum_{n=1}^{N} n P_{n0}$ is the mean number of customers in Q_1 that stay idle because $L_2 = 0$.

From (2.6) we get an expression for $\mathbb{E}[L_1]$:

$$\mathbb{E}[L_1] = (1 - P_{\bullet K})\lambda_2/\mu_2 + \sum_{n=1}^{N} n P_{n0}. \tag{2.7}$$

Alternatively, by summing Eqs. (2.1)–(2.4) over m, and then over $0 \le n \le N-1$, we arrive at

$$\mathbb{E}[L_1] = (1 - P_{N\bullet})\lambda_1/\mu_1 + \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{nm}. \tag{2.8}$$

The first term in the right-hand side (RHS) of (2.8) is the mean number of customers being served in Q_1, while the second term is the mean number of customers waiting to be served there.

2.2. Deriving $(P_{nm})_{0 \le n \le N, 0 \le m \le K}$

This model can be described as a queueing system with $N+1$ "phases", where phase n indicates that the service rate in Q_2 is $n\mu_2$. State (n, m) denotes that there are m jobs in Q_2, $0 \le m \le K$, and the system is in phase n, $0 \le n \le N$.

We construct a finite non-homogeneous QBD process with generator Q given by

$$Q = \begin{pmatrix}
A_1^0 & A_0^0 & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbf{0} \\
A_2^1 & A_1^1 & A_0 & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \vdots \\
\mathbf{0} & A_2 & A_1^2 & A_0 & \mathbf{0} & \cdots & \cdots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & A_2 & A_1^N & A_0 & \mathbf{0} & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & A_2 & A_1^N & A_0 & \mathbf{0} & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & A_2 & A_1^N & A_0 \\
\mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbf{0} & A_2 & A_1
\end{pmatrix},$$

where $\mathbf{0}$ is a matrix of zeros, and starting from the upper diagonal, A_0^0, A_0; A_1^0, $A_1^1, \ldots, A_1^N$, A_1; A_2^1, A_2 are the following matrices: A_0^0 is of size $N \times (N+1)$; A_0 is of size $(N+1) \times$

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$(N+1)$; A_1^0 is of size $N \times N$; $A_1^1, \ldots, A_1^N$ and A_1 are each of size $(N+1) \times (N+1)$; A_2^1 is of size $(N+1) \times N$; and A_2 is of size $(N+1) \times (N+1)$. They are given by

$$A_0^0 = \begin{pmatrix} 0 & \lambda_2 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \lambda_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda_2 \end{pmatrix}, \quad A_0 = \mathrm{diag}(\lambda_2),$$

$$A_1^0 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \cdots & 0 \\ 0 & -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \lambda_1 \\ \vdots & \ddots & \ddots & 0 & -\lambda_2 \end{pmatrix}.$$

For all $1 \le m \le N$,

$$(A_1^m)_{ij} = \begin{cases} -(\lambda_1+\lambda_2+i\mu_1+i\mu_2) & j=i=0,1,\ldots,m \\ -(\lambda_1+\lambda_2+m\mu_1+i\mu_2) & j=i=m+1,\ldots,N-1 \\ -(\lambda_2+m\mu_1+N\mu_2) & j=i=N \\ \lambda_1 & j=i+1,\ i=0,1,\ldots,N-1 \\ i\mu_1 & j=i-1,\ i=1,\ldots,m \\ m\mu_1 & j=i-1,\ i=m+1,\ldots,N \\ 0 & \text{otherwise,} \end{cases}$$

$$A_1 = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & \cdots & 0 \\ \mu_1 & -(\lambda_1+\mu_1+\mu_2) & \lambda_1 & 0 & \cdots & \vdots \\ 0 & 2\mu_1 & -(\lambda_1+2\mu_1+2\mu_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & 3\mu_1 & -(\lambda_1+3\mu_1+3\mu_2) & \lambda_1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda_1 \\ 0 & \cdots & 0 & 0 & N\mu_1 & -(N\mu_1+N\mu_2) \end{pmatrix},$$

$$A_2^1 = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \mu_2 & 0 & \cdots & \cdots & \vdots \\ 0 & 2\mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & 3\mu_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & N\mu_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \mu_2 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 2\mu_2 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & N\mu_2 \end{pmatrix}.$$

Define the steady state probability vectors $\vec{P}_0 = (P_{10}, \ldots, P_{N0})$ and $\vec{P}_m = (P_{0m}, P_{1m}, \ldots, P_{Nm})$ for all $1 \leq m \leq K$. Then the steady state probability vectors satisfy

$$
\begin{aligned}
&\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0}\\
&\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2 = \vec{0}\\
&\vec{P}_1 A_0 + \vec{P}_2 A_1^2 + \vec{P}_3 A_2 = \vec{0}\\
&\vdots\\
&\vec{P}_{N-1} A_0 + \vec{P}_N A_1^N + \vec{P}_{N+1} A_2 = \vec{0}\\
&\vdots\\
&\vec{P}_{K-2} A_0 + \vec{P}_{K-1} A_1^N + \vec{P}_K A_2 = \vec{0}\\
&\vec{P}_{K-1} A_0 + \vec{P}_K A_1 = \vec{0}.
\end{aligned}
\tag{2.9}
$$

Clearly, one can solve directly (numerically) the set (2.9) (including the normalization equation, $\sum_{m=0}^{K} \vec{P}_m \cdot \vec{e} = 1$, where $\vec{e}$ is a vector of 1's). This requires some computational effort. We wish to present an alternative algorithmic-type method to ease the required computational effort. For further discussion on computational issues for finite models and infinite models the reader is referred to [9], Elhafsi and Molle [6], [3,5], to mention a few.

Theorem 2.1: *The following equations hold:*

$$
\vec{P}_{N-i} = -\vec{P}_N A_2 A_0^{-1} C_{11}(i-2) + \vec{P}_{N-1}\left(C_{21}(i-2) - A_1^{N-1} A_0^{-1} C_{11}(i-2)\right), \quad 2 \leq i \leq N-1 \tag{2.10}
$$

$$
\vec{P}_{K-j} = -\vec{P}_K A_2 A_0^{-1} D_{11}^{(j-2)} + \vec{P}_{K-1}\left(D_{21}^{(j-2)} - A_1^N A_0^{-1} D_{11}^{(j-2)}\right), \quad 2 \leq j \leq K-N+1 \tag{2.11}
$$

where $C_{11}(i-2)$ and $C_{21}(i-2)$ are the $(N+1)\times(N+1)$ sub-matrices of the $2(N+1)\times 2(N+1)$ product matrix $C(i-2)$ defined as

$$
C(i) = \begin{cases} I_{2(N+1)}, & i = 0,\\ C(i-1)\begin{pmatrix} -A_1^{N-i-1} A_0^{-1} & I_{N+1}\\ -A_2 A_0^{-1} & \mathbf{0}\end{pmatrix}, & i > 0\ . \end{cases} \tag{2.12}
$$

$D_{11}^{(j-2)}$ and $D_{21}^{(j-2)}$ are the $(N+1)\times(N+1)$ sub-matrices of the $2(N+1)\times 2(N+1)$ power matrix $D^{(j-2)}$ defined as

$$
D = \begin{pmatrix} -A_1^N A_0^{-1} & I_{N+1}\\ -A_2 A_0^{-1} & \mathbf{0}\end{pmatrix}, \quad D^{(j)} = \begin{cases} I_{2(N+1)}, & j = 0,\\ D^{(j-1)} D, & j > 0.\end{cases} \tag{2.13}
$$

where, I_n is the n-dimensional identity matrix.

Proof: The inductive proof uses the special structure of the model in which the off-diagonal block matrix, A_0, is non-singular (which is not the common case in LDQBD models). We show first that (2.11) holds for every $2 \le j \le K - N + 1$, and then that (2.10) holds for every $2 \le i \le N - 1$.

By the definition of the matrix $D^{(j)}$ we have

$$D^{(j)} = D^{(j-1)}D = \begin{pmatrix} D_{11}^{(j-1)} & D_{12}^{(j-1)} \\ D_{21}^{(j-1)} & D_{22}^{(j-1)} \end{pmatrix} \begin{pmatrix} -A_1^N A_0^{-1} & I_{N+1} \\ -A_2 A_0^{-1} & \mathbf{0} \end{pmatrix}$$
$$= \begin{pmatrix} -D_{11}^{(j-1)} A_1^N A_0^{-1} - D_{12}^{(j-1)} A_2 A_0^{-1} & D_{11}^{(j-1)} \\ -D_{21}^{(j-1)} A_1^N A_0^{-1} - D_{22}^{(j-1)} A_2 A_0^{-1} & D_{21}^{(j-1)} \end{pmatrix}. \quad \mathbf{(2.14)}$$

Therefore

$$D_{11}^{(j)} = -D_{11}^{(j-1)} A_1^N A_0^{-1} - D_{12}^{(j-1)} A_2 A_0^{-1}\ ,\ D_{12}^{(j)} = D_{11}^{(j-1)},$$
$$D_{21}^{(j)} = -D_{21}^{(j-1)} A_1^N A_0^{-1} - D_{22}^{(j-1)} A_2 A_0^{-1}\ ,\ D_{22}^{(j)} = D_{21}^{(j-1)}. \quad \mathbf{(2.15)}$$

For $j = 2$, we get from Eq. (2.9)

$$\vec{P}_{K-2} = -\vec{P}_K A_2 A_0^{-1} - \vec{P}_{K-1} A_1^N A_0^{-1}.$$

From (2.13), $D_{11}^{(0)} = I_{N+1}$ and $D_{21}^{(0)} = \mathbf{0}$. Hence,

$$\vec{P}_{K-2} = -\vec{P}_K A_2 A_0^{-1} D_{11}^{(0)} + \vec{P}_{K-1} \left(D_{21}^{(0)} - A_1^N A_0^{-1} D_{11}^{(0)} \right).$$

Assuming that the proposition holds for $j - 1$, we now show that it holds for $j \le K - N + 1$. From (2.9) we have

$$\vec{P}_{K-j} = -\vec{P}_{K-j+2} A_2 A_0^{-1} - \vec{P}_{K-j+1} A_1^N A_0^{-1}.$$

Substituting the values of $\vec{P}_{K-j+2}$ and $\vec{P}_{K-j+1}$ we get

$$\begin{aligned} \vec{P}_{K-j} = &- \left(-\vec{P}_K A_2 A_0^{-1} D_{11}^{(j-4)} + \vec{P}_{K-1} \left(D_{21}^{(j-4)} - A_1^N A_0^{-1} D_{11}^{(j-4)} \right) \right) A_2 A_0^{-1} \\ &- \left(-\vec{P}_K A_2 A_0^{-1} D_{11}^{(j-3)} + \vec{P}_{K-1} \left(D_{21}^{(j-3)} - A_1^N A_0^{-1} D_{11}^{(j-3)} \right) \right) A_1^N A_0^{-1} \\ = &\ \vec{P}_K A_2 A_0^{-1} \left(D_{11}^{(j-4)} A_2 A_0^{-1} + D_{11}^{(j-3)} A_1^N A_0^{-1} \right) \\ &- \vec{P}_{K-1} \left(\left(D_{21}^{(j-4)} - A_1^N A_0^{-1} D_{11}^{(j-4)} \right) A_2 A_0^{-1} + \left(D_{21}^{(j-3)} - A_1^N A_0^{-1} D_{11}^{(j-3)} \right) A_1^N A_0^{-1} \right). \end{aligned} \quad \mathbf{(2.16)}$$

Substituting (2.15) in (2.16) yields

$$\begin{aligned} \vec{P}_{K-j} = &\ \vec{P}_K A_2 A_0^{-1} \left(D_{12}^{(j-3)} A_2 A_0^{-1} + D_{11}^{(j-3)} A_1^N A_0^{-1} \right) \\ &- \vec{P}_{K-1} \left(\left(D_{22}^{(j-3)} - A_1^N A_0^{-1} D_{12}^{(j-3)} \right) A_2 A_0^{-1} + \left(D_{21}^{(j-3)} - A_1^N A_0^{-1} D_{11}^{(j-3)} \right) A_1^N A_0^{-1} \right) \\ = &\ \vec{P}_K A_2 A_0^{-1} \left(D_{12}^{(j-3)} A_2 A_0^{-1} + D_{11}^{(j-3)} A_1^N A_0^{-1} \right) \\ &- \vec{P}_{K-1} \left(D_{22}^{(j-3)} A_2 A_0^{-1} + D_{21}^{(j-3)} A_1^N A_0^{-1} - A_1^N A_0^{-1} \left(D_{12}^{(j-3)} A_2 A_0^{-1} + D_{11}^{(j-3)} A_1^N A_0^{-1} \right) \right) \\ = &-\vec{P}_K A_2 A_0^{-1} D_{11}^{(j-2)} + \vec{P}_{K-1} \left(D_{21}^{(j-2)} - A_1^N A_0^{-1} D_{11}^{(j-2)} \right). \end{aligned} \quad \mathbf{(2.17)}$$

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By the definition of the matrix $C(i)$ we have

$$C(i) = C(i-1)\begin{pmatrix} -A_1^{N-i-1}A_0^{-1} & I_{N+1} \\ -A_2A_0^{-1} & \mathbf{0} \end{pmatrix}$$
$$= \begin{pmatrix} C_{11}(i-1) & C_{12}(i-1) \\ C_{21}(i-1) & C_{22}(i-1) \end{pmatrix}\begin{pmatrix} -A_1^{N-i-1}A_0^{-1} & I_{N+1} \\ -A_2A_0^{-1} & \mathbf{0} \end{pmatrix}$$
$$= \begin{pmatrix} -C_{11}(i-1)A_1^{N-i-1}A_0^{-1} - C_{12}(i-1)A_2A_0^{-1} & C_{11}(i-1) \\ -C_{21}(i-1)A_1^{N-i-1}A_0^{-1} - C_{22}(i-1)A_2A_0^{-1} & C_{21}(i-1) \end{pmatrix} \quad \mathbf{(2.18)}$$

Therefore

$$C_{11}(i) = -C_{11}(i-1)A_1^{N-i-1}A_0^{-1} - C_{12}(i-1)A_2A_0^{-1}\ ,\ C_{12}(i) = C_{11}(i-1)$$
$$C_{21}(i) = -C_{21}(i-1)A_1^{N-i-1}A_0^{-1} - C_{22}(i-1)A_2A_0^{-1}\ ,\ C_{22}(i) = C_{21}(i-1). \quad \mathbf{(2.19)}$$

For $i = 2$, using Eq. (2.9) leads to

$$\vec{P}_{N-2} = -\vec{P}_N A_2 A_0^{-1} - \vec{P}_{N-1}A_1^{N-1}A_0^{-1}.$$

From (2.12), $C_{11}(0) = I_{N+1}$ and $C_{21}(0) = \mathbf{0}$. Therefore,

$$\vec{P}_{N-2} = -\vec{P}_N A_2 A_0^{-1}C_{11}(0) + \vec{P}_{N-1}\left(C_{21}(0) - A_1^{N-1}A_0^{-1}C_{11}(0)\right).$$

Assuming that the proposition holds for $i-1$, we now show that it holds for $i \le N-1$. From (2.9) we have

$$\vec{P}_{N-i} = -\vec{P}_{N-i+2}A_2A_0^{-1} - \vec{P}_{N-i+1}A_1^{N-i+1}A_0^{-1}.$$

Substituting the values of $\vec{P}_{N-i+2}$ and $\vec{P}_{N-i+1}$ we get

$$\begin{aligned}\vec{P}_{N-i} = &-\left(-\vec{P}_N A_2A_0^{-1}C_{11}(i-4) + \vec{P}_{N-1}\left(C_{21}(i-4) - A_1^{N-1}A_0^{-1}C_{11}(i-4)\right)\right)A_2A_0^{-1} \\ &-\left(-\vec{P}_N A_2A_0^{-1}C_{11}(i-3) + \vec{P}_{N-1}\left(C_{21}(i-3) - A_1^{N-1}A_0^{-1}C_{11}(i-3)\right)\right)A_1^{N-i+1}A_0^{-1} \\ = &\ \vec{P}_N A_2A_0^{-1}\left(C_{11}(i-4)A_2A_0^{-1} + C_{11}(i-3)A_1^{N-i+1}A_0^{-1}\right) \\ &- \vec{P}_{N-1}\left(\left(C_{21}(i-4) - A_1^{N-1}A_0^{-1}C_{11}(i-4)\right)A_2A_0^{-1}\right) \\ &- \vec{P}_{N-1}\left(\left(C_{21}(i-3) - A_1^{N-1}A_0^{-1}C_{11}(i-3)\right)A_1^{N-i+1}A_0^{-1}\right). \end{aligned} \quad \mathbf{(2.20)}$$

Substituting (2.19) into (2.20) we have

$$\begin{aligned}\vec{P}_{N-i} = &\ \vec{P}_N A_2A_0^{-1}\left(C_{12}(i-3)A_2A_0^{-1} + C_{11}(i-3)A_1^{N-i+1}A_0^{-1}\right) \\ &- \vec{P}_{N-1}\left(C_{22}(i-3) - A_1^{N-1}A_0^{-1}C_{12}(i-3)\right)A_2A_0^{-1} \\ &+ \vec{P}_{N-1}\left(C_{21}(i-3) - A_1^{N-1}A_0^{-1}C_{11}(i-3)\right)A_1^{N-i+1}A_0^{-1} \\ = &\ \vec{P}_N A_2A_0^{-1}\left(C_{12}(i-3)A_2A_0^{-1} + C_{11}(i-3)A_1^{N-i+1}A_0^{-1}\right) \\ &- \vec{P}_{N-1}\left(C_{22}(i-3)A_2A_0^{-1} + C_{21}(i-3)A_1^{N-i+1}A_0^{-1}\right) \\ &+ \vec{P}_{N-1}\left(A_1^{N-1}A_0^{-1}\left(C_{12}(i-3)A_2A_0^{-1} + C_{11}(i-3)A_1^{N-i+1}A_0^{-1}\right)\right) \\ = &-\vec{P}_N A_2A_0^{-1}C_{11}(i-2) + \vec{P}_{N-1}\left(C_{21}(i-2) - A_1^{N-1}A_0^{-1}C_{11}(i-2)\right). \end{aligned} \quad \mathbf{(2.21)}$$

This completes the proof. ■

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From Theorem 2.1 we conclude that $\vec{P}_m$, $1 \le m \le K-2$, can be expressed in terms of the four boundary probability vectors $\vec{P}_{N-1}$, $\vec{P}_N$, $\vec{P}_{K-1}$ and $\vec{P}_K$. Therefore, the solution of (2.9) can be calculated by solving only the following reduced linear system:

$$\begin{aligned}
&\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0},\\
&\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2 = \vec{0},\\
&\vec{P}_{N-1} A_0 + \vec{P}_N A_1^N + \Big(-\vec{P}_K A_2 A_0^{-1} D_{11}^{(K-N-1)}\\
&\quad + \vec{P}_{K-1}\left(D_{21}^{(K-N-1)} - A_1^N A_0^{-1} D_{11}^{(K-N-1)}\right)\Big) A_2 = \vec{0}\\
&\vec{P}_N A_0 - \vec{P}_K A_2 A_0^{-1}\left(D_{11}^{(K-N-1)} A_1^N + D_{11}^{(K-N-2)} A_2\right) + \vec{P}_{K-1}\Big(D_{21}^{(K-N-1)} A_1^N\\
&\quad + D_{21}^{(K-N-2)} A_2 - A_1^N A_0^{-1}\left(D_{11}^{(K-N-1)} A_1^N + D_{11}^{(K-N-2)} A_2\right)\Big) = \vec{0},\\
&\vec{P}_{K-1} A_0 + \vec{P}_K A_1 = \vec{0}.
\end{aligned} \tag{2.22}$$

From (2.10) we can express $\vec{P}_1$ and $\vec{P}_2$ in terms of $\vec{P}_{N-1}$ and $\vec{P}_N$. Hence the system (2.22) becomes

$$\begin{aligned}
&\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0},\\
&\vec{P}_0 A_0^0 - \vec{P}_N A_2 A_0^{-1}\left(C_{11}(N-3) A_1^1 + C_{11}(N-4) A_2\right) + \vec{P}_{N-1}\big(C_{21}(N-3) A_1^1\\
&\quad + C_{21}(N-4) A_2 - A_1^{N-1} A_0^{-1}\left(C_{11}(N-3) A_1^1 + C_{11}(N-4) A_2\right)\big) = \vec{0},\\
&\vec{P}_{N-1} A_0 + \vec{P}_N A_1^N + \Big(-\vec{P}_K A_2 A_0^{-1} D_{11}^{(K-N-1)} + \vec{P}_{K-1}\Big(D_{21}^{(K-N-1)}\\
&\quad - A_1^N A_0^{-1} D_{11}^{(K-N-1)}\Big)\Big) A_2 = \vec{0},\\
&\vec{P}_N A_0 - \vec{P}_K A_2 A_0^{-1}\left(D_{11}^{(K-N-1)} A_1^N + D_{11}^{(K-N-2)} A_2\right) + \vec{P}_{K-1}\Big(D_{21}^{(K-N-1)} A_1^N\\
&\quad + D_{21}^{(K-N-2)} A_2 - A_1^N A_0^{-1}\left(D_{11}^{(K-N-1)} A_1^N + D_{11}^{(K-N-2)} A_2\right)\Big) = \vec{0},\\
&\vec{P}_{K-1} A_0 + \vec{P}_K A_1 = \vec{0}.
\end{aligned} \tag{2.23}$$

The normalization equation now becomes

$$\begin{aligned}
&\vec{P}_0 \vec{e} + \vec{P}_{N-1}\left(\sum_{i=1}^{N-2} C_{21}(N-i-2) - A_1^{N-1} A_0^{-1} \sum_{i=1}^{N-2} C_{11}(N-i-2) + I_{N+1}\right)\vec{e}\\
&\quad - \vec{P}_N\left(A_2 A_0^{-1} \sum_{i=1}^{N-2} C_{11}(N-i-2) + I_{N+1}\right)\vec{e}\\
&\quad + \vec{P}_{K-1}\left(\sum_{j=N-1}^{K-2} D_{21}^{(K-j-2)} - A_1^N A_0^{-1} \sum_{j=N-1}^{K-2} D_{11}^{(N-i-2)} + I_{N+1}\right)\vec{e}\\
&\quad - \vec{P}_K\left(A_2 A_0^{-1} \sum_{j=N-1}^{K-2} D_{11}^{(K-j-2)} + I_{N+1}\right)\vec{e} = 1.
\end{aligned} \tag{2.24}$$

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Therefore, instead of solving tediously the set of linear Eqs. (2.9), it is enough to calculate the matrices $C(i)$, $0 \le i \le N-3$ and $D^{(j)}$, $0 \le j \le K-N-1$, and solve the set of linear Eqs. (2.23) and (2.24), which yields the set of sought-for probability vectors $\vec{P}_0, \vec{P}_1, \ldots, \vec{P}_K$.

Moreover, in our case, due to the structure of A_0 and A_2, the computational effort can be further reduced as follows: $A_0^{-1} = \operatorname{diag}(1/\lambda_2)$ and $A_2 = \mu_2 \vec{Z} I_{(N+1)}$, where $\vec{Z} = (0, 1, 2, \ldots, N)$. Thus, $A_2 A_0^{-1} = \frac{\mu_2}{\lambda_2} \vec{Z} I_{(N+1)}$. It follows that

$$C(i) = \begin{cases} I_{2(N+1)}, & i = 0 \\ C(i-1) \begin{pmatrix} -\frac{1}{\lambda_2} A_1^{N-i-1} & I_{N+1} \\ -\frac{\mu_2}{\lambda_2} \vec{Z} I_{(N+1)} & \mathbf{0} \end{pmatrix}, & i > 0. \end{cases}, D = \begin{pmatrix} -\frac{1}{\lambda_2} A_1^{N} & I_{N+1} \\ -\frac{\mu_2}{\lambda_2} \vec{Z} I_{(N+1)} & \mathbf{0} \end{pmatrix}.$$

The mean total number of customers in Q_2, $\mathbb{E}[L_2]$, is given by

$$\begin{aligned} \mathbb{E}[L_2] = \sum_{m=1}^{K} m \vec{P}_m \vec{e} = \vec{P}_{N-1} &\left(\sum_{i=1}^{N-2} i C_{21}(N-i-2) - A_1^{N-1} A_0^{-1} \sum_{i=1}^{N-2} i C_{11}(N-i-2) \right) \vec{e} \\ &- \vec{P}_N A_2 A_0^{-1} \sum_{i=1}^{N-2} i C_{11}(N-i-2) \vec{e} \\ &+ \vec{P}_{K-1} \left(\sum_{j=N-1}^{K-2} j D_{21}^{(K-j-2)} - A_1^N A_0^{-1} \sum_{j=N-1}^{K-2} j D_{11}^{(K-j-2)} \right. \\ &\left. + (K-1) I_{N+1} \right) \vec{e} \\ &- \vec{P}_K \left(A_2 A_0^{-1} \sum_{j=N-1}^{K-2} j D_{11}^{(K-j-2)} + K I_{N+1} \right) \vec{e}. \end{aligned} \quad \mathbf{(2.25)}$$

With $\vec{Z}_0 = (1, 2, \ldots, N)$, Eq. (2.7) can now be expressed as

$$\mathbb{E}[L_1] = \left(1 - \vec{P}_K \vec{e}\right) \lambda_2/\mu_2 + \vec{P}_0 \vec{Z}_0 \quad \mathbf{(2.26)}$$

Applying Little's law (see Cooper [4]), the mean sojourn time in Q_i (layer i) for $i = 1, 2$, is given by (same for all models):

$$\mathbb{E}[W_i] = \frac{\mathbb{E}[L_i]}{\lambda_i^{eff}} = \frac{\mathbb{E}[L_i]}{\lambda_i \left(1 - \mathbb{P}\left(Q_i \text{ is blocked}\right)\right)}. \quad \mathbf{(2.27)}$$

In Sections 5 and 6, we will compare and discuss the results of Model 1 with the results of Model 2 (Section 3) and Model 3 (Section 4).

3. MODEL 2

In this model, we modify the service scheme for Q_2: while Q_1 remains a multi-server $M(\lambda_1)/M(\mu_1)/L_2/N$ system, Q_2 now also operates as a multi-server $M(\lambda_2)/M(\mu_2)/L_1/K$ queue. The corresponding transition rate diagram is depicted in Figure 2.

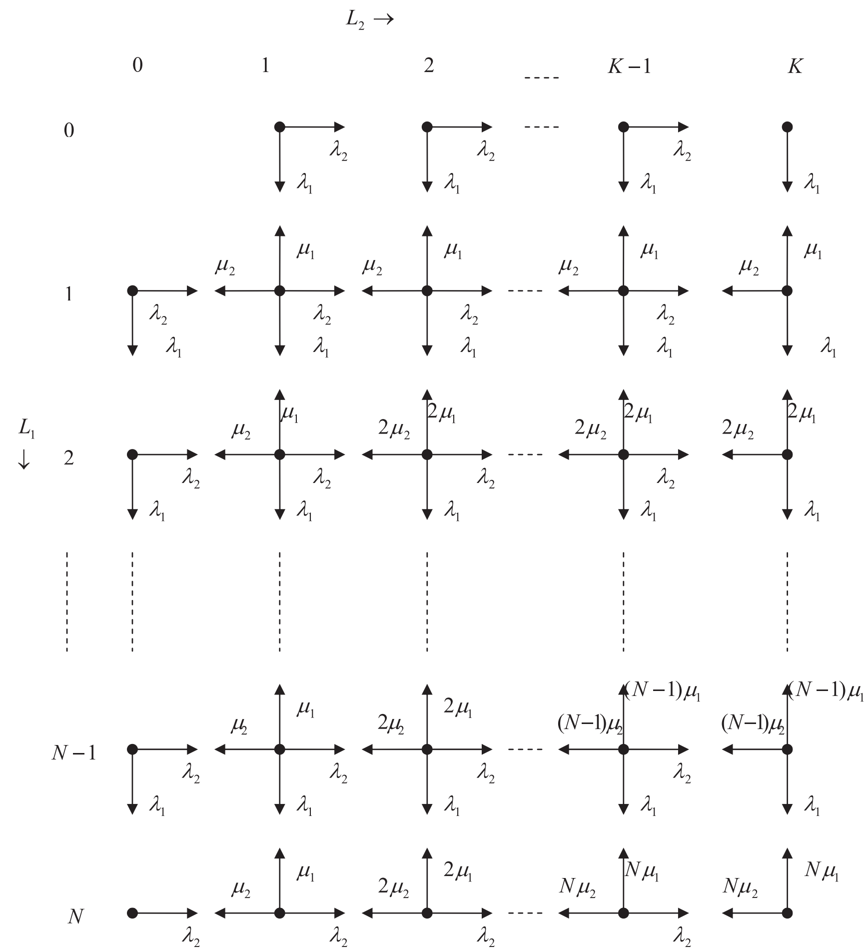

FIGURE **2.** Transition rate diagram of (L_1, L_2) for Model 2.

Similarly to Section 2.1, by algebraic manipulations on the (omitted) balance equations we arrive at

$$\mathbb{E}[L_1] = (1 - P_{N\bullet})\lambda_1/\mu_1 + \sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)P_{nm} = (1 - P_{\bullet K})\lambda_2/\mu_2 + \sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)P_{nm},$$

implying that

$$(1 - P_{N\bullet})\lambda_1/\mu_1 = (1 - P_{\bullet K})\lambda_2/\mu_2. \tag{3.1}$$

Equation (3.1) reveals an interesting result: the carried load of Q_1, namely $(1 - P_{N\bullet})\lambda_1/\mu_1$, is equal to the carried load of Q_2, $(1 - P_{\bullet K})\lambda_2/\mu_2$, independent of the capacities N and K.

3.1. Deriving $(P_{nm})_{0\le n\le N,\ 0\le m\le K}$

This model, similarly to Model 1, can be described as a queueing system with $N + 1$ phases, where phase n indicates that there are n servers available, and its generator matrix Q is

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given by

$$Q = \begin{pmatrix} A_1^0 & A_0^0 & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbf{0} \\ A_2^1 & A_1^1 & A_0 & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \vdots \\ \mathbf{0} & A_2^2 & A_1^2 & A_0 & \mathbf{0} & \cdots & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & A_2^N & A_1^N & A_0 & \mathbf{0} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & A_2^N & A_1^N & A_0 & \mathbf{0} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & A_2^N & A_1^N & A_0 \\ \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbf{0} & A_2^N & A_1 \end{pmatrix},$$

where, A_0^0, A_0, A_1^0 and A_1 are the same as in Section 2.2, while A_1^1, A_1^2, $\ldots$, A_1^N; A_2^1, A_2^2, $\ldots$, A_2^N are slightly different and are given by

For all $1 \leq m \leq N$,

$$(A_1^m)_{ij} = \begin{cases} -(\lambda_1 + \lambda_2 + i\mu_1 + i\mu_2) & j = i = 0, 1, \ldots, m \\ -(\lambda_1 + \lambda_2 + m\mu_1 + m\mu_2) & j = i = m+1, \ldots, N-1 \\ -(\lambda_2 + m\mu_1 + m\mu_2) & j = i = N \\ \lambda_1 & j = i+1\ ,\ i = 0, 1, \ldots, N-1 \\ i\mu_1 & j = i-1\ ,\ i = 1, \ldots, m \\ m\mu_1 & j = i-1\ ,\ i = m+1, \ldots, N \\ 0 & \text{otherwise} \end{cases}$$

$$A_2^1 = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \mu_2 & 0 & \cdots & \cdots & \vdots \\ 0 & \mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \mu_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \mu_2 \end{pmatrix}.$$

$$\text{For all } 2 \leq m \leq N, \quad (A_2^m)_{ij} = \begin{cases} i\mu_2 & j = i-1\ ,\ i = 1, \ldots, m-1, \\ m\mu_2 & j = i-1\ ,\ i = m, \ldots, N, \\ 0 & \text{otherwise.} \end{cases}$$

Define the steady state probability vectors $\vec{P}_0 = (P_{10}, \ldots, P_{N0})$ and $\vec{P}_m = (P_{0m}, P_{1m}, \ldots, P_{Nm})$ for all $1 \leq m \leq K$. Then the steady state probability vectors satisfy

$$\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0}$$
$$\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2^2 = \vec{0}$$
$$\vec{P}_1 A_0 + \vec{P}_2 A_1^2 + \vec{P}_3 A_2^3 = \vec{0}$$
$$\vdots$$

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$$
\begin{aligned}
&\vec{P}_{N-2}A_0 + \vec{P}_{N-1}A_1^{N-1} + \vec{P}_N A_2^N = \vec{0}\\
&\vec{P}_{N-1}A_0 + \vec{P}_N A_1^N + \vec{P}_{N+1}A_2^N = \vec{0}\\
&\vdots\\
&\vec{P}_{K-2}A_0 + \vec{P}_{K-1}A_1^N + \vec{P}_K A_2^N = \vec{0}\\
&\vec{P}_{K-1}A_0 + \vec{P}_K A_1 = \vec{0}.
\end{aligned} \tag{3.2}
$$

THEOREM 3.1: *The following equations hold:*

$$
\vec{P}_{N-i} = -\vec{P}_N A_2^N A_0^{-1} C_{11}(i-2) + \vec{P}_{N-1}\left(C_{21}(i-2) - A_1^{N-1}A_0^{-1}C_{11}(i-2)\right), \quad 2 \le i \le N-1, \tag{3.3}
$$

$$
\vec{P}_{K-j} = -\vec{P}_K A_2^N A_0^{-1} D_{11}^{(j-2)} + \vec{P}_{K-1}\left(D_{21}^{(j-2)} - A_1^N A_0^{-1} D_{11}^{(j-2)}\right), \quad 2 \le j \le K-N+1, \tag{3.4}
$$

where $C_{11}(i-2)$ *and* $C_{21}(i-2)$ *are* $(N+1)\times(N+1)$ *sub-matrices of the* $2(N+1)\times 2(N+1)$ *product matrix* $C(i-2)$ *defined as*

$$
C(i) = \begin{cases} I_{2(N+1)}, & i = 0,\\ C(i-1)\begin{pmatrix} -A_1^{N-i-1}A_0^{-1} & I_{N+1}\\ -A_2^{N-i}A_0^{-1} & \mathbf{0}\end{pmatrix}, & i > 0\,. \end{cases} \tag{3.5}
$$

$D_{11}^{(j-2)}$ *and* $D_{21}^{(j-2)}$ *are* $(N+1)\times(N+1)$ *sub-matrices of the* $2(N+1)\times 2(N+1)$ *power matrix* $D^{(j-2)}$ *defined as*

$$
D = \begin{pmatrix} -A_1^N A_0^{-1} & I_{N+1}\\ -A_2^N A_0^{-1} & \mathbf{0}\end{pmatrix}, \quad D^{(j)} = \begin{cases} I_{2(N+1)}, & j = 0,\\ D^{(j-1)}D, & j > 0\,. \end{cases} \tag{3.6}
$$

PROOF: The proof is similar to the proof of Theorem 2.1. ■

Again, one calculates the matrices $C(i)$ and $D^{(j)}$ and then obtains sequentially the probability vectors $\vec{P}_m$, $0 \le m \le K$ as functions of $\vec{P}_{N-1}$, $\vec{P}_N$, $\vec{P}_{K-1}$ and $\vec{P}_K$. Together with the normalization equation $\sum_{m=0}^{K}\vec{P}_m\cdot\vec{e} = 1$, all required probability vectors are obtained.

The mean total number of customers in Q_2, $\mathbb{E}[L_2]$, is given by

$$
\begin{aligned}
\mathbb{E}[L_2] = {}& \vec{P}_{N-1}\left(\sum_{i=1}^{N-2} iC_{21}(N-i-2) - A_1^{N-1}A_0^{-1}\sum_{i=1}^{N-2} iC_{11}(N-i-2)\right)\vec{e}\\
&- \vec{P}_N A_2^N A_0^{-1}\sum_{i=1}^{N-2} iC_{11}(N-i-2)\vec{e}\\
&+ \vec{P}_{K-1}\left(\sum_{j=N-1}^{K-2} jD_{21}^{(K-j-2)} - A_1^N A_0^{-1}\sum_{j=N-1}^{K-2} jD_{11}^{(K-j-2)} + (K-1)I_{N+1}\right)\vec{e}\\
&- \vec{P}_K\left(A_2^N A_0^{-1}\sum_{j=N-1}^{K-2} jD_{11}^{(K-j-2)} + KI_{N+1}\right)\vec{e}.
\end{aligned} \tag{3.7}
$$

We can now calculate $\mathbb{E}[L_1]$ by

$$\mathbb{E}[L_1] = \left(1 - \vec{P}_K \vec{e}\right) \lambda_2/\mu_2 + \sum_{m=0}^{N-1} \vec{P}_m \vec{Z}_m, \tag{3.8}$$

where $\vec{Z}_0 = (1, 2, \ldots, N)$, and for all $1 \leq m \leq N-1$, $\vec{Z}_m = (0, \ldots, 0, 1, 2, \ldots, N-m)$.

4. MODEL 3

In this model both queues operate as single-server systems where Q_1 is an $M(\lambda_1)/M(\mu_1 L_2)/1/N$ system, and Q_2 is an $M(\lambda_2)/M(\mu_2 L_1)/1/K$ queue. Note that the combined service rate in each queue depends on the queue length in the other queue (resembling file-sharing programs).

4.1. Balance Equations

With L_1, L_2, and P_{nm} as before, Figure 3 is the corresponding transition-rate diagram.

Again, by algebraic manipulations on the (omitted) balance equations we arrive at

$$\lambda_2 \sum_{m=0}^{K-1} P_{\bullet m} = \mu_2 \sum_{m=0}^{K-1} P_{\bullet m+1} \mathbb{E}[L_1 | L_2 = m+1]. \tag{4.1}$$

Therefore, $\lambda_2 (1 - P_{\bullet K}) = \mu_2 (\mathbb{E}[L_1] - P_{\bullet 0}\mathbb{E}[L_1 | L_2 = 0]) = \mu_2 \left(\mathbb{E}[L_1] - \sum_{n=1}^{N} n P_{n0}\right)$, meaning that

$$\mathbb{E}[L_1] = (1 - P_{\bullet K})\lambda_2/\mu_2 + \sum_{n=1}^{N} n P_{n0}. \tag{4.2}$$

Furthermore,

$$\lambda_1(1 - P_{N\bullet}) = \mu_1 \mathbb{E}[L_2] - \mu_1 \sum_{m=1}^{K} m P_{0m}.$$

That is,

$$\mathbb{E}[L_2] = (1 - P_{N\bullet})\lambda_1/\mu_1 + \sum_{m=1}^{K} m P_{0m}. \tag{4.3}$$

We note that Eq. (4.2) is identical to Eq. (2.7) and exhibits that the effective arrival rate in Q_2, $\lambda_2(1 - P_{\bullet K})$, is smaller than the potential mean service rare there, being $\mu_2 \mathbb{E}[L_1]$. The same applies to Eq. (4.3). This coincides with the finite-buffer single-server $M(\lambda)/M(\mu)/1/K$ queue, where the following relation always holds $\lambda_{eff} \equiv \lambda(1 - P_K) < \mu$.

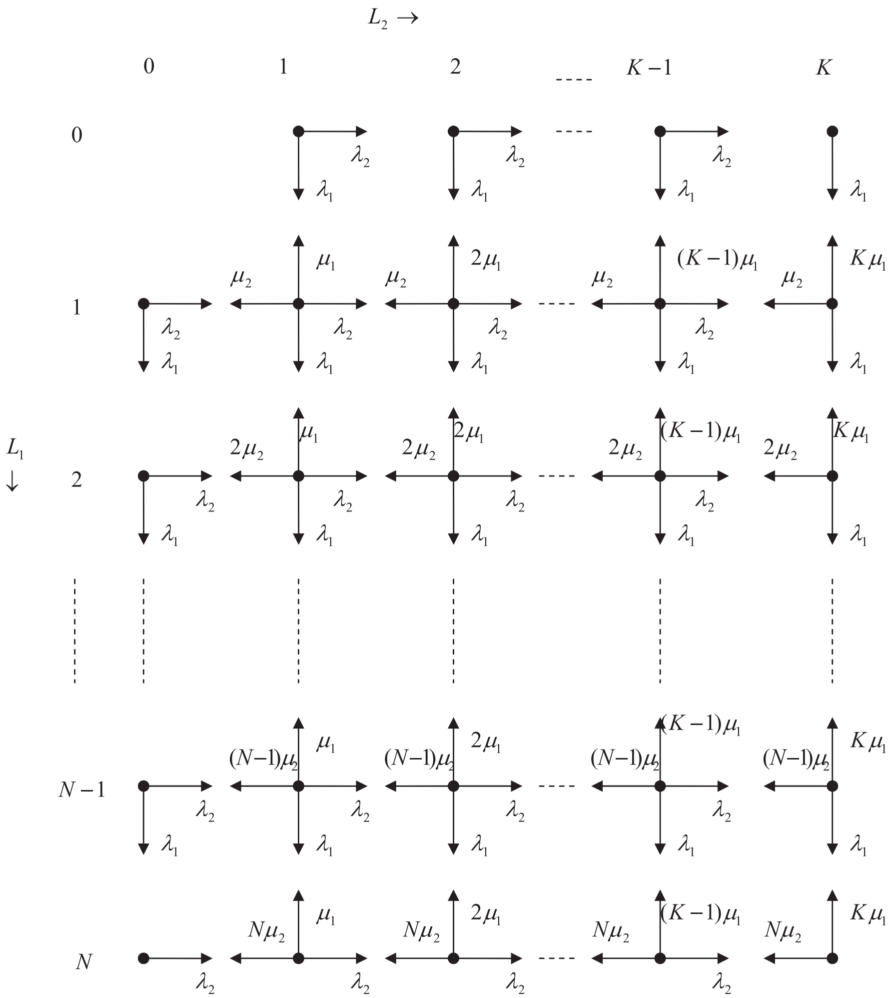

FIGURE 3. Transition rate diagram of (L_1, L_2) for Model 3.

4.2. Deriving $(P_{nm})_{0\le n\le N,\ 0\le m\le K}$

The corresponding generator for the finite non-homogeneous QBD process is

$$Q = \begin{pmatrix} A_1^0 & A_0^0 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ A_2^1 & A_1^1 & A_0 & \mathbf{0} & \cdots & \vdots \\ \mathbf{0} & A_2 & A_1^2 & A_0 & \mathbf{0} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & A_1^{K-1} & A_0 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & A_2 & A_1^K \end{pmatrix},$$

where, A_0^0, A_0, A_1^0, A_2^1, and A_2 are the same as in Section 2.2, while A_1^1, A_1^2, …, A_1^K are a bit different and are given by

https://doi.org/10.1017/S0269964816000139 Published online by Cambridge University Press

For all $1 \le m \le K$,

$$(A_1^m)_{ij} = \begin{cases} -(\lambda_1 + \lambda_2 + m\mu_1 + i\mu_2) & j = i = 0, 1, \ldots, N-1 \\ -(\lambda_2 + m\mu_1 + N\mu_2) & j = i = N \\ \lambda_1 & j = i+1 \ , \ i = 0, 1, \ldots, N-1 \\ m\mu_1 & j = i-1 \ , \ i = 1, \ldots, N \\ 0 & \text{otherwise.} \end{cases}$$

The steady state probability vectors satisfy

$$\begin{aligned} &\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0} \\ &\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2^2 = \vec{0} \\ &\vec{P}_1 A_0 + \vec{P}_2 A_1^2 + \vec{P}_3 A_2^3 = \vec{0} \\ &\vdots \\ &\vec{P}_{N-2} A_0 + \vec{P}_{N-1} A_1^{N-1} + \vec{P}_N A_2^N = \vec{0} \\ &\vec{P}_{N-1} A_0 + \vec{P}_N A_1^N + \vec{P}_{N+1} A_2^N = \vec{0} \\ &\vdots \\ &\vec{P}_{K-2} A_0 + \vec{P}_{K-1} A_1^N + \vec{P}_K A_2^N = \vec{0} \\ &\vec{P}_{K-1} A_0 + \vec{P}_K A_1 = \vec{0}. \end{aligned} \tag{4.4}$$

THEOREM 4.1: *The following equation holds:*

$$\vec{P}_{K-i} = -\vec{P}_K A_2 A_0^{-1} C_{11}(i-2) + \vec{P}_{K-1}\left(C_{21}(i-2) - A_1^{K-1} A_0^{-1} C_{11}(i-2)\right), \quad 2 \le i \le K-1, \tag{4.5}$$

where $C_{11}(i-2)$ *and* $C_{21}(i-2)$ *are* $(N+1) \times (N+1)$ *sub-matrices of the* $2(N+1) \times 2(N+1)$ *product matrix* $C(i-2)$ *defined as in Theorem 2.1*

$$C(i) = \begin{cases} I_{2(N+1)}, & i = 0, \\ C(i-1)\begin{pmatrix} -A_1^{N-i-1} A_0^{-1} & I_{N+1} \\ -A_2 A_0^{-1} & \boldsymbol{0} \end{pmatrix}, & i > 0\ . \end{cases} \tag{4.6}$$

PROOF: The proof is similar to the proof of the second part of Theorem 2.1. ∎

The mean total number of customers in Q_2, $\mathbb{E}[L_2]$, is given by

$$\begin{aligned} \mathbb{E}[L_2] = {} & \vec{P}_{K-1}\left(\sum_{i=1}^{K-2} iC_{21}(K-i-2) - A_1^{K-1} A_0^{-1} \sum_{i=1}^{K-2} iC_{11}(K-i-2) - (K-1)I_{N+1}\right)\vec{e} \\ & - \vec{P}_K\left(A_2 A_0^{-1} \sum_{i=1}^{K-2} iC_{11}(K-i-2) + KI_{N+1}\right)\vec{e}. \end{aligned}$$

Equation (4.2) can now be expressed as

$$\mathbb{E}[L_1] = \left(1 - \vec{P}_K \vec{e}\right)\lambda_2/\mu_2 + \vec{P}_0 \vec{Z}_0. \tag{4.7}$$

5. NUMERICAL EXAMPLES

We denote by Lq_i, $i = 1, 2$, the number of waiting customers in Q_i, and by $P_{\text{loss}}(i)$ the proportion of arriving customers that are blocked (and lost) in Q_i because the latter is at its full capacity. Tables 1–6 exhibit numerical results for a set of performance measures $\mathbb{E}[L_i]$, $\mathbb{E}[Lq_i]$, $\mathbb{E}[W_i]$, $P_{\text{loss}}(i)$, $i = 1, 2$, and $\text{Cov}\,(L_1, L_2)$ for different values of λ_1, λ_2, μ_1, μ_2, N, and K. Tables 1 and 4 relate to Model 1, Tables 2 and 5 relate to Model 2, while Tables 3 and 6 relate to Model 3.

Tables 1–3 are constructed as follows: the first numerical row in each table presents the values of the performance measures for a set of "Basic parameters" $\lambda_1 = 5$, $\lambda_2 = 5$, $\mu_1 = 8$, and $\mu_2 = 8$. The second, third, fourth, and fifth rows give, respectively, the values of the measures when in each row only one of the basic parameters is changed. Row six presents the results for another set of "Basic parameters" $\lambda_1 = 10$, $\lambda_2 = 10$, $\mu_1 = 5$, and $\mu_2 = 5$, and each of the following rows gives results when only one of the latter basic parameters is changed. Tables 4–6 are constructed in the same manner, where the first set of "Basic parameters" is $\lambda_1 = 5$, $\lambda_2 = 5$, $\mu_1 = 10$, and $\mu_2 = 10$, while the second set is $\lambda_1 = 10$, $\lambda_2 = 10$, $\mu_1 = 5$, and $\mu_2 = 5$.

5.1. Discussion

(1) When comparing the two classical models: (i) the multi-server $M(\lambda)/M(\mu)/s$ system (s identical servers, each with service rate μ, where the traffic intensity is $\rho = \frac{\lambda}{s\mu}$) with (ii) the single-server $M(\lambda)/M(s\mu)/1$ system (with same intensity $\rho = \frac{\lambda}{s\mu}$) the mean number of customers in the single-server system is always smaller than in the multi-server system, while the mean number of waiting customers in the single-server system is always greater than the one in the multi-server system. In the current study this relation does not always hold. Comparing the operation modes of Q_1 in Models 1 and 2 (see Tables 1 versus 2 and Tables 4 versus 5) it is seen that $\mathbb{E}[L_1]$ in Model 1 is not always smaller than in Model 2, while $\mathbb{E}[Lq_1]$ in Model 1 is not always greater than in Model 2. The same observation holds regarding Q_2 when comparing Models 1 and 3 (see Tables 1 versus 3 and Tables 4 versus 6). This result occurs since changing the service method of one of the queues effects both queues.

(2) When increasing λ_i, $i = 1, 2$, in all three models, $\mathbb{E}[L_i]$ is increased, while $\mathbb{E}[L_j]$ $j \neq i$ is decreased. This happens since higher level of L_i results in a higher number of servers attending Q_j, or a higher service rate at Q_j.

(3) In all three models, when increasing μ_i, $i = 1, 2$, $\mathbb{E}[L_i]$ is decreased, while $\mathbb{E}[L_j]$ $j \neq i$ is increased. This follows since smaller L_i causes a decrease in the number of available servers, or in the service rate, at Q_j.

(4) There is a negative correlation between L_1 and L_2 in all models: when one is decreased the other is increased, and vice versa.

6. SUMMARY

In this paper, we broaden the scope of analytic investigation of 2-queue models where customers of each queue act as servers for the other queue. We consider cases where both queues are finite and analyze three models, distinguished by the way in which customers of each queue serve the customers of the opposite queue. Specifically, in Model 1 we assume that

Table 1. Numerical results for $N = 2$ and $K = 4$ for Model 1

	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$\mathbb{E}[Lq_1]$	$\mathbb{E}[Lq_2]$	$\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	$\mathrm{Cov}\,(L_1, L_2)$	$P_{\mathrm{loss}}(1)$	$P_{\mathrm{loss}}(2)$
Basic parameters	1.0007	1.9312	0.6041	1.2219	0.3154	0.5234	−0.9195	0.3654	0.2620
$\lambda_1 = 15$	1.6830	0.8249	1.2315	0.3786	0.4660	0.1726	−0.3859	0.7592	0.0442
$\lambda_2 = 15$	0.6086	3.4627	0.0687	2.4908	0.1409	0.7750	−0.3223	0.1363	0.7021
$\mu_1 = 18$	0.3855	3.1350	0.1320	2.2029	0.0845	1.4785	−0.4239	0.0875	0.5759
$\mu_2 = 18$	1.5590	0.6264	1.3702	0.3129	1.0317	0.1324	−0.5992	0.6978	0.0535
Basic parameters	1.2960	2.8201	0.2368	1.9054	0.2447	0.4984	−0.3559	0.4704	0.4341
$\lambda_1 = 20$	1.6252	2.4546	0.3732	1.5843	0.2596	0.3584	−0.2313	0.6870	0.3151
$\lambda_2 = 20$	1.2163	3.5464	0.0402	2.5575	0.2068	0.5935	−0.1520	0.4119	0.7012
$\mu_1 = 1$	1.8534	2.1765	0.5069	1.3486	1.3765	0.2878	−0.1194	0.8653	0.2438
$\mu_2 = 1$	1.2008	3.8603	0.0021	2.8606	0.2003	3.2164	−0.0317	0.4006	0.8800

Table 2. Numerical results for $N = 2$ and $K = 4$ for Model 2

	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$\mathbb{E}[Lq_1]$	$\mathbb{E}[Lq_2]$	$\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	$\mathrm{Cov}\,(L_1, L_2)$	$P_{\mathrm{loss}}(1)$	$P_{\mathrm{loss}}(2)$
Basic parameters	0.8855	2.1538	0.4432	1.71147	0.2502	0.6086	−0.7774	0.2922	0.2922
$\lambda_1 = 15$	1.5865	1.0761	0.9975	0.4871	0.3367	0.2284	−0.3996	0.6859	0.0577
$\lambda_2 = 15$	0.5946	3.4975	0.0493	2.9522	0.1363	0.8018	−0.2767	0.1276	0.7092
$\mu_1 = 18$	0.3550	3.1942	0.0968	2.9359	0.0764	1.5460	−0.3346	0.0703	0.5868
$\mu_2 = 18$	1.3989	0.8539	1.1414	0.5964	0.6791	0.1842	−0.6801	0.5880	0.0730
Basic parameters	1.2695	2.9262	0.1704	1.8272	0.2310	0.5325	−0.2918	0.4505	0.4505
$\lambda_1 = 20$	1.6017	2.6081	0.2714	1.2778	0.2408	0.3921	−0.1846	0.6674	0.3348
$\lambda_2 = 20$	1.2125	3.5634	0.0308	2.3817	0.2052	0.6031	−0.1394	0.4091	0.7046
$\mu_1 = 1$	1.8401	2.3733	0.3719	0.9051	1.2533	0.3233	−0.0988	0.8532	0.2659
$\mu_2 = 1$	1.2007	3.8608	0.0018	2.6619	0.2003	3.2202	−0.0313	0.4005	0.8801

TABLE 3. Numerical results for $N = 2$ and $K = 4$ for Model 3

	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$\mathbb{E}[Lq_1]$	$\mathbb{E}[Lq_2]$	$\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	Cov (L_1, L_2)	$P_{\text{loss}}(1)$	$P_{\text{loss}}(2)$
Basic parameters	0.4652	3.0496	0.1532	2.1684	0.1099	1.5591	−0.7902	0.1532	0.6088
$\lambda_1 = 15$	1.4586	1.2437	0.6459	0.7122	0.2746	0.2879	−0.8371	0.6459	0.1361
$\lambda_2 = 15$	0.2144	3.8303	0.0355	2.8372	0.0445	2.3689	−0.1288	0.0355	0.8922
$\mu_1 = 18$	0.1053	3.7839	0.0159	2.7949	0.0214	5.5153	−0.1047	0.0159	0.8628
$\mu_2 = 18$	1.2881	1.1815	0.5717	0.7427	0.6015	0.2876	−1.1638	0.5717	0.1784
Basic parameters	0.7573	3.327	0.2377	2.3946	0.0993	0.9911	−0.4507	0.2377	0.6617
$\lambda_1 = 20$	1.2749	2.8267	0.4995	1.9203	0.1273	0.5184	−0.4977	0.4995	0.4547
$\lambda_2 = 20$	0.6338	3.7766	0.1727	2.7812	0.0766	1.2086	−0.1622	0.1727	0.8438
$\mu_1 = 1$	1.7379	2.3033	0.7883	1.4616	0.8211	0.3234	−0.2746	0.7883	0.2878
$\mu_2 = 1$	0.5872	3.9340	0.1499	2.9341	0.0691	6.7025	−0.0351	0.1499	0.9413

TABLE 4. Numerical results for $N = 8$ and $K = 12$ for Model 1

	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$\mathbb{E}[Lq_1]$	$\mathbb{E}[Lq_2]$	$\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	Cov (L_1, L_2)	$P_{\text{loss}}(1)$	$P_{\text{loss}}(2)$
Basic parameters	7.7099	0.1490	7.6342	0.0725	10.1863	0.0298	−0.5851	0.8486	0.0017
$\lambda_1 = 1$	0.1070	11.5324	0.0070	10.5338	0.1070	11.4825	−0.1619	0.0003	0.7991
$\lambda_2 = 1$	0.7.9691	0.0130	7.9561	0.0002	61.4255	0.0130	−0.0022	0.9740	6.6×10^{-11}
$\mu_1 = 20$	0.9824	9.5185	0.7504	8.6146	0.2117	3.4664	−7.0224	0.0719	0.4508
$\mu_2 = 20$	7.9219	0.0331	7.8889	0012	23.9816	0.0066	−0.0043	0.9339	6.9×10^{-7}
Basic parameters	7.3169	0.8698	6.8480	0.5534	3.1207	0.0880	−2.9090	0.7655	0.0111
$\lambda_1 = 20$	7.8514	0.3959	7.4772	0.1321	4.1961	0.0396	−0.2689	0.9064	0.0003
$\lambda_2 = 20$	2.4671	9.8005	0.5739	8.8431	0.2606	0.9102	−5.0244	0.0534	0.4617
$\mu_1 = 1$	7.9574	0.3769	7.5892	0.1211	21.6121	0.0377	−0.0516	0.9632	0.0006
$\mu_2 = 1$	1.9984	11.7286	0.0001	10.7286	0.2000	5.8689	−0.1735	0.0009	0.8002

Table 5. Numerical results for $N = 8$ and $K = 12$ for Model 2

	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$\mathbb{E}[Lq_1]$	$\mathbb{E}[Lq_2]$	$\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	Cov (L_1, L_2)	$P_{\text{loss}}(1)$	$P_{\text{loss}}(2)$
Basic parameters	2.3645	4.4548	1.9111	4.0014	0.5216	0.9825	−7.7898	0.0933	0.0932
$\lambda_1 = 1$	0.1008	11.5488	0.0008	11.4448	0.1008	11.5448	−0.0914	3.3×10^{-7}	0.8000
$\lambda_2 = 1$	7.5506	0.1066	7.4506	0.0066	7.5506	0.1066	−0.1289	0.8000	2.6×10^{-8}
$\mu_1 = 20$	0.2685	10.5543	0.0186	10.3044	0.0537	4.2235	−0.3038	0.0004	0.5002
$\mu_2 = 20$	6.6046	0.3624	6.3548	0.1126	2.6438	0.0725	−0.8148	0.5004	0.0007
Basic parameters	3.4649	5.423	1.6799	3.6222	0.3882	0.6023	−6.4185	0.1075	0.0996
$\lambda_1 = 20$	6.6604	2.4147	4.6085	0.4239	0.6492	0.2459	−1.9659	0.4870	0.0046
$\lambda_2 = 20$	2.0565	10.5564	0.0638	8.5558	0.2064	1.0553	−1.3926	0.0036	0.4998
$\mu_1 = 1$	7.6409	2.6039	5.2114	0.6271	3.1451	0.2634	−0.4291	0.7570	0.0116
$\mu_2 = 1$	1.9983	11.7287	0.0001	9.7304	0.2000	5.8693	−0.1731	0.0009	0.8002

Table 6. Numerical results for $N = 8$ and $K = 12$ for Model 3

	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$\mathbb{E}[Lq_1]$	$\mathbb{E}[Lq_2]$	$\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	Cov (L_1, L_2)	$P_{\text{loss}}(1)$	$P_{\text{loss}}(2)$
Basic parameters	0.0688	11.8531	0.0233	10.856	0.0138	25.8674	−0.2932	0.0027	0.9083
$\lambda_1 = 1$	0.0085	11.9811	0.0001	10.9811	0.0085	141.2160	−0.0008	5.1×10^{-9}	0.9830
$\lambda_2 = 1$	7.9680	0.0144	6.9682	0.0014	61.1161	0.0143	−0.0131	0.9739	0.0001
$\mu_1 = 20$	0.0215	11.9527	0.0005	10.9527	0.0043	55.6901	−0.0012	8.4×10^{-8}	0.9571
$\mu_2 = 20$	7.7282	0.3204	6.7517	0.2646	17.3796	0.0654	−2.2110	0.9110	0.0201
Basic parameters	0.3221	11.6846	0.1387	10.6958	0.0326	9.9648	−1.3459	0.0126	0.8827
$\lambda_1 = 20$	6.3539	2.6691	5.4875	2.2549	1.1394	0.3144	−13.5063	0.7212	0.1512
$\lambda_2 = 20$	0.2051	11.9370	0.0361	10.9370	0.0205	11.6628	−0.0400	0.0001	0.9488
$\mu_1 = 1$	7.9498	0.3863	6.9500	0.1298	20.6973	0.0387	−0.1161	0.9616	0.0009
$\mu_2 = 1$	0.2010	11.9788	0.0338	10,9788	0.0201	59.5821	−0.0062	6×10^{-7}	0.9800

each of the customers present in Q_2 individually act as a server for the customers in Q_1, with service time, specific for Q_1, exponentially distributed with mean $1/\mu_1$. That is, the number of available servers in Q_1 changes dynamically according to the number of customers in Q_2. Therefore, Q_1 operates as an $M(\lambda_1)/M(\mu_1)/L_2/N$ system. In addition, we assume that all customers of Q_1 join hands together to form a single-server, serving the customers in Q_2 with changing service rate $\mu_2 L_1$. That is, Q_2 operates as an $M(\lambda_2)/M(\mu_2 L_1)/1/K$ system. In Model 2 we analyzed the case where both queues are multi-server systems, where Q_1 operates as an $M(\lambda_1)/M(\mu_1)/L_2/N$ and Q_2 as an $M(\lambda_2)/M(\mu_2)/L_1/K$. In Model 3 we considered the case where both queues have the same mode of operation, but, in contrast to Model 2, each queue is a single-server system in which Q_1 is an $M(\lambda_1)/M(\mu_1 L_2)/1/N$ system, and Q_2 an $M(\lambda_2)/M(\mu_2 L_1)/1/K$ system. We derive the steady state probability vectors of the system's state for each model, and calculate the mean total number of customers in the queues. The derivation of the probability vectors uses the special 3-diagonal structure of the generator matrix of the QBD process to reduce the required computational effort.

In Model 2 we show that the carried loads of both queues are equal, while in Model 3 the effective arrival rate is smaller than the realized service rate for both queues. Numerical results, presented in Section 5, further exhibit the differences between the models while insights are discussed.

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