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ON STRENGTHENED WEIGHTED CARLEMAN'S INEQUALITY

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In this paper we prove a new refinement of the weighted arithmetic-geometric mean inequality and apply this result in obtaining a sharpened version of the weighted Carleman's inequality.

1. INTRODUCTION

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers, such that the series $\sum\limits_{n=1}^{\infty} a_n$ converges. Then the well-known Carleman inequality

$$\sum_{n=1}^{\infty}\Bigl(\prod_{k=1}^{n} a_k\Bigr)^{1/n} < e\sum_{n=1}^{\infty} a_n \tag{1}$$

holds, unless $a_n = 0$ for all $n \in \mathbb{N}$. Moreover, the constant e on the right-hand side of (1) is the best possible, that is, it cannot be replaced by any smaller constant. The relation (1) was discovered by Carleman in his paper [**1**] on quasi-analytic functions, where he gave necessary and sufficient conditions for a function not to be quasi-analytic (see [**6**] and [**10**] for further details).

Since its publication in 1922, the inequality (1) has been generalised in various ways by many different authors and has found a wide range of applications. The study of Carleman's inequality is also covered by a rich literature. Here we just mention the recent review papers [**7, 8, 12**], which give a comprehensive history, several proofs and some new developments related to Carleman's inequality and its integral analogue – the so-called Pólya-Knopp's inequality, and also many further references.

A possible way of generalising Carleman's inequality is either to obtain its finite sections, that is, to restrict the infinite series on its both-hand sides to a finite number of terms, to enlarge the left-hand side of (1) by adding suitable non-negative terms, or to find refinements of (1) by decreasing the weight coefficients in the series on its right-hand side. Moreover, although the constant factor e is optimal, Carleman's inequality can be strengthened by finding some appropriate approximations of this constant, as it was shown in a recently published sequence of papers dealing with such improvements: [**2, 3, 4, 7, 8, 9, 12, 14, 16, 17, 18**].

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In particular, in [9] Kaijser, Persson, and Öberg proved the following theorem:

THEOREM A. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers and let $(x_n^*)_{n\in\mathbb{N}}$ be the non-increasing rearrangement of the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = n(1+1/n)^n a_n$, $n \in \mathbb{N}$. Then the inequalities*

$$\sum_{n=1}^{N}\Big(\prod_{k=1}^{n} a_k\Big)^{1/n} + \sum_{n=1}^{N}\frac{1}{n(n+1)}\sum_{k=1}^{\lfloor n/2\rfloor}\big(\sqrt{x_{n-k+1}^*}-\sqrt{x_k^*}\big)^2 \leqslant \sum_{n=1}^{N}\Big(1-\frac{n}{N+1}\Big)\Big(1+\frac{1}{n}\Big)^n a_n < e\sum_{n=1}^{N}\Big(1-\frac{n}{N+1}\Big)a_n \tag{2}$$

hold for all $N \in \mathbb{N}$.

The factor $1 - n/(N+1)$ in (2) means that the usual partial sum $\sum_{n=1}^{N} a_n$ on the right-hand side of Carleman's inequality has been replaced by the corresponding, but strictly smaller Césaro sum, that is, the partial sums of the original sequence have been arithmetically averaged. Furthermore, if the sequence $(a_n)_{n\in\mathbb{N}}$ from Theorem A is such that the series $\sum_{n=1}^{\infty} a_n$ is convergent, then by letting $N \to \infty$ in (2) we obtain the relations

$$\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n} a_k\Big)^{1/n} + \sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{k=1}^{\lfloor n/2\rfloor}\big(\sqrt{x_{n-k+1}^*}-\sqrt{x_k^*}\big)^2 < \sum_{n=1}^{\infty}\Big(1+\frac{1}{n}\Big)^n a_n < e\sum_{n=1}^{\infty} a_n, \tag{3}$$

which may be regarded as refinements of (1).

On the other hand, since the inequalities on the right-hand sides of the relations (2) and (3) were derived from the rough estimate $(1+1/n)^n < e$, it is evident that they can be improved by using sharper approximations of e. Guided by this idea, in the paper [17] Yang obtained the following series expansion.

LEMMA A. *If $x > 0$, then*

$$\Big(1+\frac{1}{x}\Big)^x = e\Big[1-\sum_{k=1}^{\infty}\frac{\alpha_k}{(1+x)^k}\Big], \tag{4}$$

where all the coefficients α_k are positive and given by the recurrence formula

$$\alpha_1 = \frac{1}{2},\quad \alpha_{k+1} = \frac{1}{k+1}\Big(\frac{1}{k+2}-\sum_{i=1}^{k}\frac{\alpha_i}{k+2-i}\Big),\ k\in\mathbb{N}.$$

In particular, $\alpha_2 = 1/24$, $\alpha_3 = 1/48$, $\alpha_4 = 73/5760$, and $\alpha_5 = 11/1280$.

The same result was proved also by Gyllenberg and Yan in [4], as an answer to an earlier question raised by Yang in [16].

If we denote

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{\alpha_k}{(1+x)^k}, \quad x > 0, \tag{5}$$

where the coefficients α_k are defined as in Lemma A, then by combining (2), (4) and (5) we have

$$\begin{aligned}\sum_{n=1}^{N} \Big(\prod_{k=1}^{n} a_k\Big)^{1/n} + \sum_{n=1}^{N} \frac{1}{n(n+1)} \sum_{k=1}^{\lfloor n/2 \rfloor} \big(\sqrt{x_{n-k+1}^*} - \sqrt{x_k^*}\,\big)^2 \\ \leqslant e \sum_{n=1}^{N} \Big(1 - \frac{n}{N+1}\Big)\big[1 - \varphi(n)\big] a_n < e \sum_{n=1}^{N} \Big(1 - \frac{n}{N+1}\Big) a_n\end{aligned} \tag{6}$$

and, under the assumption that the series $\sum_{n=1}^{\infty} a_n$ is convergent, by taking $\lim_{N\to\infty}$ also a corresponding result to (3).

A weighted version of (1) was noted already by Hardy in [5, p. 156]. (However, he said that it was G. Pólya who pointed out this fact to him.) A much more general weighted Carleman's inequality was recently proved by Yang in [17]. His result show

Theorem B. *Let $(a_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be sequences of real numbers, such that $a_n \geqslant 0$, $0 < w_{n+1} \leqslant w_n$, $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} w_n a_n < \infty$. Then*

$$\sum_{n=1}^{\infty} w_{n+1} \Big(\prod_{k=1}^{n} a_k^{w_k}\Big)^{1/W_n} \leqslant \sum_{n=1}^{\infty} \Big(1 + \frac{w_n}{W_n}\Big)^{W_n/w_n} w_n a_n = e \sum_{n=1}^{\infty} \Big[1 - \varphi\Big(\frac{W_n}{w_n}\Big)\Big] w_n a_n, \tag{7}$$

where φ is defined by (5) *and $W_n = \sum_{k=1}^{n} w_k$, $n \in \mathbb{N}$.*

Our aim in this paper is to state, prove and discuss a generalisation of Theorem A to the case with weights, which also may be seen as a sharpening of Theorem B (see Section 3). As a crucial step of the proof of our main result, and also of independent interest, in Section 2 we shall prove a new refinement of the weighted arithmetic-geometric mean inequality.

2. A refinement of the weighted arithmetic-geometric mean inequality

Before presenting our idea and results, we need to introduce some notation. For a given sequence $\underline{a} = (a_n)_{n\in\mathbb{N}}$ of non-negative real numbers and a sequence $\underline{w} = (w_n)_{n\in\mathbb{N}}$ of positive real numbers, by $A_n(\underline{a};\underline{w})$ and $G_n(\underline{a};\underline{w})$ we denote respectively the weighted arithmetic and geometric mean of $\underline{a}$ with the weights $\underline{w}$, that is,

$$A_n(\underline{a};\underline{w}) = \frac{1}{W_n} \sum_{k=1}^{n} w_k a_k, \quad G_n(\underline{a};\underline{w}) = \Big(\prod_{k=1}^{n} a_k^{w_k}\Big)^{1/W_n},$$

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where $W_n = \sum_{k=1}^{n} w_k$, $n \in \mathbb{N}$. We also put $W_0 = 0$. Further, if S_n is the symmetric group of degree n, then for arbitrary $\pi \in S_n$ and $k \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ let

$$w_{\pi,k} = w_{\pi(2k-1)} + w_{\pi(2k)},$$
$$A_{\pi,k}(\underline{a}; \underline{w}) = \frac{1}{w_{\pi,k}} \left[w_{\pi(2k-1)}a_{\pi(2k-1)} + w_{\pi(2k)}a_{\pi(2k)}\right],$$

and

$$G_{\pi,k}(\underline{a}; \underline{w}) = \left[a_{\pi(2k-1)}^{w_{\pi(2k-1)}} \cdot a_{\pi(2k)}^{w_{\pi(2k)}}\right]^{1/w_{\pi,k}}.$$

Especially, observe that if all weights in the sequence $\underline{w}$ are equal, their value $w = w_1 = w_2 = \ldots$ will not affect the value of the related means, that is, $A_n(\underline{a}; \underline{w}) = A_n(\underline{a}; \underline{1})$ and $G_n(\underline{a}; \underline{w}) = G_n(\underline{a}; \underline{1})$, where $\underline{1} = (1, 1, \ldots)$. In this case we can without loss of generality assume that $w = 1$ and denote

$$A_n(\underline{a}) = A_n(\underline{a}; \underline{1}) = \frac{1}{n} \sum_{k=1}^{n} a_k, \quad G_n(\underline{a}) = G_n(\underline{a}; \underline{1}) = \left(\prod_{k=1}^{n} a_k\right)^{1/n}, \quad n \in \mathbb{N}.$$

Now, we can state our first result. The following lemma gives a new generalisation and refinement of the weighted arithmetic-geometric mean inequality:

LEMMA 1. *Let $\underline{a} = (a_n)_{n \in \mathbb{N}}$ be a sequence of non-negative and $\underline{w} = (w_n)_{n \in \mathbb{N}}$ of positive real numbers. Then the inequality*

$$A_n(\underline{a}; \underline{w}) - G_n(\underline{a}; \underline{w}) \geqslant \frac{1}{W_n} \max_{\pi \in S_n} \sum_{k=1}^{\lfloor n/2 \rfloor} w_{\pi,k}\left[A_{\pi,k}(\underline{a}; \underline{w}) - G_{\pi,k}(\underline{a}; \underline{w})\right] \tag{8}$$

holds for all $n \in \mathbb{N}$.

PROOF: First, suppose that the number n is odd. For an arbitrary permutation $\pi \in S_n$ then we have

$$\begin{aligned}
A_n(\underline{a}; \underline{w}) &= \frac{1}{W_n} \sum_{k=1}^{n} w_{\pi(k)}a_{\pi(k)} \\
&= \frac{1}{W_n} \left\{ \sum_{k=1}^{\lfloor n/2 \rfloor} \left[w_{\pi(2k-1)}a_{\pi(2k-1)} + w_{\pi(2k)}a_{\pi(2k)}\right] + w_{\pi(n)}a_{\pi(n)} \right\} \\
&= \frac{1}{W_n} \left[\sum_{k=1}^{\lfloor n/2 \rfloor} w_{\pi,k}A_{\pi,k}(\underline{a}; \underline{w}) + w_{\pi(n)}a_{\pi(n)} \right] \\
&= \frac{1}{W_n} \sum_{k=1}^{\lfloor n/2 \rfloor} w_{\pi,k}\left[A_{\pi,k}(\underline{a}; \underline{w}) - G_{\pi,k}(\underline{a}; \underline{w})\right] \\
&\qquad + \frac{1}{W_n} \left[w_{\pi(n)}a_{\pi(n)} + \sum_{k=1}^{\lfloor n/2 \rfloor} w_{\pi,k}G_{\pi,k}(\underline{a}; \underline{w}) \right].
\end{aligned} \tag{9}$$

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Since $w_{\pi(n)} + \sum_{k=1}^{\lfloor n/2 \rfloor} w_{\pi,k} = W_n$, applying the weighted arithmetic-geometric mean inequality to the sum of $\lfloor n/2 \rfloor + 1$ terms in the last row of (9) we obtain that $A_n(\underline{a};\underline{w})$ is not less than

$$\begin{aligned}\frac{1}{W_n}\sum_{k=1}^{\lfloor n/2 \rfloor} w_{\pi,k}\big[A_{\pi,k}(\underline{a};\underline{w}) - G_{\pi,k}(\underline{a};\underline{w})\big] + \Big[a_{\pi(n)}^{w_{\pi(n)}} \prod_{k=1}^{\lfloor n/2 \rfloor} G_{\pi,k}(\underline{a};\underline{w})^{w_{\pi,k}}\Big]^{1/W_n} \\ = \frac{1}{W_n}\sum_{k=1}^{\lfloor n/2 \rfloor} w_{\pi,k}\big[A_{\pi,k}(\underline{a};\underline{w}) - G_{\pi,k}(\underline{a};\underline{w})\big] + \Big[\prod_{k=1}^{n} a_{\pi(k)}^{w_{\pi(k)}}\Big]^{1/W_n} \\ = \frac{1}{W_n}\sum_{k=1}^{\lfloor n/2 \rfloor} w_{\pi,k}\big[A_{\pi,k}(\underline{a};\underline{w}) - G_{\pi,k}(\underline{a};\underline{w})\big] + G_n(\underline{a};\underline{w}).\end{aligned}$$

The inequality (8) now follows by taking the maximum over all $\pi \in S_n$. If the number n is even, the proof is similar or even simpler. $\square$

REMARK 1. Note that in the case when all weights are equal we have $w_{\pi,k} = 2$ and $W_n = n$, so the right-hand side of the relation (8) becomes

$$\begin{aligned}\frac{1}{n}\max_{\pi\in S_n}\sum_{k=1}^{\lfloor n/2 \rfloor}\Big[a_{\pi(2k-1)} + a_{\pi(2k)} - 2\sqrt{a_{\pi(2k-1)}a_{\pi(2k)}}\,\Big] \\ = \frac{1}{n}\max_{\pi\in S_n}\sum_{k=1}^{\lfloor n/2 \rfloor}\Big[\sqrt{a_{\pi(2k-1)}} - \sqrt{a_{\pi(2k)}}\,\Big]^2 = \frac{1}{n}\sum_{k=1}^{\lfloor n/2 \rfloor}\Big[\sqrt{a_{n-k+1}^*} - \sqrt{a_k^*}\,\Big]^2,\end{aligned}$$

where $(a_n^*)_{n\in\mathbb{N}}$ denotes the non-increasing rearrangement of the sequence $\underline{a}$. Hence, we obtained the relation

$$A_n(\underline{a}) - G_n(\underline{a}) \geqslant \frac{1}{n}\sum_{k=1}^{\lfloor n/2 \rfloor}\Big(\sqrt{a_{n-k+1}^*} - \sqrt{a_k^*}\,\Big)^2,$$

which was proved previously by Kaijser, Persson and Öberg in the paper [**9**]. Therefore, our Lemma 1 may be seen as a weighted generalisation of their result.

REMARK 2. It is evident that the inequality (8) is also an improvement of the relation

$$A_n(\underline{a};\underline{w}) - G_n(\underline{a};\underline{w}) \geqslant \frac{1}{W_n}\max_{1\leqslant i,j\leqslant n}(w_i + w_j)\Big[\frac{w_i a_i + w_j a_j}{w_i + w_j} - (a_i^{w_i}a_j^{w_j})^{1/(w_i+w_j)}\Big],$$

known in the literature as *Popoviciu's inequality* (see for example [**11**]).

3. A strengthened Carleman's inequality

Our main result in this paper reads:

Theorem 1. *Let $\underline{a} = (a_n)_{n\in\mathbb{N}}$ and $\underline{w} = (w_n)_{n\in\mathbb{N}}$ respectively be a sequence of non-negative and a sequence of positive real numbers, and let the sequence $\widetilde{\underline{a}} = (\widetilde{a}_n)_{n\in\mathbb{N}}$ be defined by $\widetilde{a}_n = W_n(1 + w_{n+1}/W_n)^{W_n/w_n}a_n$, $n \in \mathbb{N}$. Then the inequality*

$$(10)\quad \sum_{n=1}^{N} w_{n+1}G_n(\underline{a};\underline{w}) + \sum_{n=1}^{N} \frac{w_{n+1}}{W_nW_{n+1}}\, r_n \leqslant \sum_{n=1}^{N}\Big(1 - \frac{W_n}{W_{N+1}}\Big)\Big(1 + \frac{w_{n+1}}{W_n}\Big)^{W_n/w_n} w_na_n,$$

where

$$r_n = \max_{\pi\in S_n} \sum_{k=1}^{\lfloor n/2\rfloor} w_{\pi,k}\big[A_{\pi,k}(\widetilde{\underline{a}};\underline{w}) - G_{\pi,k}(\widetilde{\underline{a}};\underline{w})\big] \geqslant 0,$$

holds for all $N \in \mathbb{N}$.

Proof: Let $n \in \{1, 2, \ldots, N\}$. Since

$$\begin{aligned} G_n(\widetilde{\underline{a}};\underline{w}) &= \Big[\prod_{k=1}^{n} W_k^{w_k}\Big(1 + \frac{w_{k+1}}{W_k}\Big)^{W_k} a_k^{w_k}\Big]^{1/W_n} \\ &= \Big[\prod_{k=1}^{n} W_k^{w_k}\Big(\frac{W_{k+1}}{W_k}\Big)^{W_k} a_k^{w_k}\Big]^{1/W_n} = W_{n+1}G_n(\underline{a};\underline{w}), \end{aligned}$$

Lemma 1, applied to the sequence $\widetilde{\underline{a}}$ instead of $\underline{a}$, yields

$$W_{n+1}G_n(\underline{a};\underline{w}) + \frac{r_n}{W_n} \leqslant \frac{1}{W_n}\sum_{k=1}^{n} w_kW_k\Big(1 + \frac{w_{k+1}}{W_k}\Big)^{W_k/w_k} a_k.$$

By multiplying this relation by w_{n+1}/W_{n+1}, taking the sum $\sum_{n=1}^{N}$, and then reversing the order of the summation we get

$$\begin{aligned} &\sum_{n=1}^{N} w_{n+1}G_n(\underline{a};\underline{w}) + \sum_{n=1}^{N}\frac{w_{n+1}}{W_nW_{n+1}}\, r_n \\ &\qquad \leqslant \sum_{n=1}^{N}\frac{w_{n+1}}{W_nW_{n+1}}\sum_{k=1}^{n} w_kW_k\Big(1 + \frac{w_{k+1}}{W_k}\Big)^{W_k/w_k} a_k \\ &\qquad = \sum_{k=1}^{N} w_kW_k\Big(1 + \frac{w_{k+1}}{W_k}\Big)^{\frac{W_k}{w_k}} a_k\sum_{n=k}^{N}\frac{w_{n+1}}{W_nW_{n+1}} \\ &\qquad = \sum_{k=1}^{N} w_kW_k\Big(1 + \frac{w_{k+1}}{W_k}\Big)^{W_k/w_k} a_k\sum_{n=k}^{N}\Big(\frac{1}{W_n} - \frac{1}{W_{n+1}}\Big) \\ &\qquad = \sum_{k=1}^{N} W_k\Big(\frac{1}{W_k} - \frac{1}{W_{N+1}}\Big)\Big(1 + \frac{w_{k+1}}{W_k}\Big)^{W_k/w_k} w_ka_k \\ &\qquad = \sum_{k=1}^{N}\Big(1 - \frac{W_k}{W_{N+1}}\Big)\Big(1 + \frac{w_{k+1}}{W_k}\Big)^{W_k/w_k} w_ka_k, \end{aligned}$$

so (10) is proved. ∎

REMARK 3. Observe that the first inequality in the relation (2) is only a special case of the inequality (10), obtained when all the weights in $\underline{w}$ are equal. Hence, considering also Remark 1, we see that Theorem 1 is a weighted generalisation of Theorem A.

Theorem 1 is also a sharpening of Theorem B. In fact, we have:

COROLLARY 1. *Let $\underline{a} = (a_n)_{n\in\mathbb{N}}$ and $\underline{w} = (w_n)_{n\in\mathbb{N}}$ be sequences of real numbers, such that $a_n \geqslant 0$ and $0 < w_{n+1} \leqslant w_n$, $n \in \mathbb{N}$. If the sequence $\underline{r} = (r_n)_{n\in\mathbb{N}}$ is defined as in Theorem 1 and the function φ is given by (5), then the inequality*

$$\begin{aligned}\sum_{n=1}^{N} w_{n+1}G_n(\underline{a};\underline{w}) + \sum_{n=1}^{N}\frac{w_{n+1}}{W_nW_{n+1}}\, r_n &\leqslant \sum_{n=1}^{N}\Bigl(1-\frac{W_n}{W_{N+1}}\Bigr)\Bigl(1+\frac{w_n}{W_n}\Bigr)^{W_n/w_n} w_na_n\\ &= e\sum_{n=1}^{N}\Bigl(1-\frac{W_n}{W_{N+1}}\Bigr)\Bigl[1-\varphi\Bigl(\frac{W_n}{w_n}\Bigr)\Bigr]w_na_n \end{aligned}\tag{11}$$

holds for all $N \in \mathbb{N}$.

PROOF: The first relation in (17) follows directly from the fact that the sequence $\underline{w}$ is non-increasing, while the second one is a consequence of Lemma A. ∎

REMARK 4. Suppose that the sequences $\underline{a}$ and $\underline{w}$ from Corollary 1 are such that the series $\sum\limits_{n=1}^{\infty} w_na_n$ is convergent. Let $W = \sum\limits_{n=1}^{\infty} w_n$. Since $0 < W \leqslant \infty$, by taking $\lim\limits_{N\to\infty}$ in (10) and (17) we have

$$\begin{aligned}&\sum_{n=1}^{\infty} w_{n+1}G_n(\underline{a};\underline{w}) + \sum_{n=1}^{\infty}\frac{w_{n+1}}{W_nW_{n+1}}\, r_n\\ &\qquad\leqslant \sum_{n=1}^{\infty}\Bigl(1-\frac{W_n}{W}\Bigr)\Bigl(1+\frac{w_{n+1}}{W_n}\Bigr)^{W_n/w_n} w_na_n \leqslant \sum_{n=1}^{\infty}\Bigl(1-\frac{W_n}{W}\Bigr)\Bigl(1+\frac{w_n}{W_n}\Bigr)^{W_n/w_n} w_na_n\\ &\qquad= e\sum_{n=1}^{\infty}\Bigl(1-\frac{W_n}{W}\Bigr)\Bigl[1-\varphi\Bigl(\frac{W_n}{w_n}\Bigr)\Bigr]w_na_n \leqslant e\sum_{n=1}^{\infty}\Bigl[1-\varphi\Bigl(\frac{W_n}{w_n}\Bigr)\Bigr]w_na_n. \end{aligned}\tag{12}$$

Therefore, the inequalities in (12) give a refinement and improvement of Theorem B. Moreover, as a special case of the relation (12) we obtain the non-weighted results described in the Introduction.

Instead of using the series expansion (4) or its finite sections (see, for example [**13, 15, 16, 18**]), the factor $(1+1/x)^x$ can be estimated by different types of approximations. In particular, in the paper [**14**] Yan and Sun proved that

$$\Bigl(1+\frac{1}{x}\Bigr)^x < e\Bigl(1+\frac{1}{x+1/5}\Bigr)^{-1/2}, \quad x \geqslant 1,$$

and applied this relation to obtain the Carleman type inequality

$$\sum_{n=1}^{\infty}\Bigl(\prod_{k=1}^{n} a_k\Bigr)^{1/n} < e\sum_{n=1}^{\infty}\Bigl(1+\frac{1}{n+1/5}\Bigr)^{-1/2} a_n, \tag{13}$$

for a sequence $\underline{a} = (a_n)_{n\in\mathbb{N}}$ of non-negative real numbers such that the series $\sum_{n=1}^{\infty} a_n$ is convergent. In [**9**], the relation (13) was sharpened as follows:

$$\sum_{n=1}^{N}\Bigl(\prod_{k=1}^{n} a_k\Bigr)^{1/n} + \sum_{n=1}^{N}\frac{1}{n(n+1)}\sum_{k=1}^{\lfloor n/2\rfloor}\bigl(\sqrt{x^*_{n-k+1}} - \sqrt{x^*_k}\,\bigr)^2$$
$$\leqslant e\sum_{n=1}^{N}\Bigl(1-\frac{n}{N+1}\Bigr)\Bigl(1+\frac{1}{n+c^*}\Bigr)^{-1/2} a_n$$
$$< e\sum_{n=1}^{N}\Bigl(1-\frac{n}{N+1}\Bigr)\Bigl(1+\frac{1}{n+1/5}\Bigr)^{-1/2} a_n, \tag{14}$$

where the sequence $(x^*_n)_{n\in\mathbb{N}}$ is defined as in Theorem A and

$$c^* = \frac{8-e^2}{e^2-4} \approx 0.1802696. \tag{15}$$

It was also shown that the constant c^* from (15) is the best possible for the relation

$$\Bigl(1+\frac{1}{n}\Bigr)^n \leqslant e\Bigl(1+\frac{1}{n+c^*}\Bigr)^{-1/2} \tag{16}$$

to hold for all $n \in \mathbb{N}$, that is, it cannot be replaced by any smaller constant. Consequently, the relation (14) cannot be further improved by an approximation of the form (16).

Observing that

$$\inf\Bigl\{c \in \mathbb{R} : \Bigl(1+\frac{1}{x}\Bigr)^x \leqslant e\Bigl(1+\frac{1}{x+c}\Bigr)^{-1/2},\ x \geqslant 1\Bigr\}$$
$$= \sup_{x\geqslant 1}\Bigl[\frac{1}{e^2(1+1/x)^{-2x}-1} - x\Bigr] = c^*,$$

since the function under the sign of sup is decreasing on $[1,\infty)$, we obtain the following weighted generalisation of (14):

Corollary 2. *Suppose $\underline{a} = (a_n)_{n\in\mathbb{N}}$ and $\underline{w} = (w_n)_{n\in\mathbb{N}}$ are sequences of real numbers, such that $a_n \geqslant 0$ and $0 < w_{n+1} \leqslant w_n$, $n \in \mathbb{N}$. If the sequence $\underline{r} = (r_n)_{n\in\mathbb{N}}$ is defined as in Theorem 1 and the constant c^* is given by (15), then the inequality*

$$\sum_{n=1}^{N} w_{n+1}G_n(\underline{a};\underline{w}) + \sum_{n=1}^{N}\frac{w_{n+1}}{W_nW_{n+1}}\, r_n \leqslant \sum_{n=1}^{N}\Bigl(1-\frac{W_n}{W_{N+1}}\Bigr)\Bigl(1+\frac{w_n}{W_n}\Bigr)^{W_n/w_n} w_na_n$$
$$\leqslant e\sum_{n=1}^{N}\Bigl(1-\frac{W_n}{W_{N+1}}\Bigr)\Bigl(1+\frac{1}{\frac{W_n}{w_n}+c^*}\Bigr)^{-1/2} w_na_n \tag{17}$$

holds for all $N \in \mathbb{N}$ and the constant c^* is the best possible.

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