J. Aust. Math. Soc. **114** (2023), 1–49
doi:10.1017/S1446788722000064

REPRESENTING STRUCTURED SEMIGROUPS ON ÉTALE GROUPOID BUNDLES

TRISTAN BICE

(Received 14 October 2020; accepted 10 February 2022; first published online 13 April 2022)

Communicated by James East

Abstract

We examine a semigroup analogue of the Kumjian–Renault representation of C*-algebras with Cartan subalgebras on twisted groupoids. Specifically, we represent semigroups with distinguished normal subsemigroups as 'slice-sections' of groupoid bundles.

2020 *Mathematics subject classification*: primary 06F05; secondary 20M18, 20M25, 20M30, 22A22, 46L85, 54D80, 46L05, 47D03.

Keywords and phrases: Weyl/étale groupoid, bundle, inverse semigroup, coset, filter.

1. Introduction

1.1. Background. A cornerstone of groupoid C*-algebra theory is the Kumjian–Renault Weyl groupoid representation. Specifically, given a C*-algebra A with a Cartan subalgebra C, they showed how to construct a Fell line bundle over an étale groupoid on which to represent A as continuous sections, thus realising the abstract algebra A as a concrete twisted groupoid C*-algebra.

One key aspect of their construction is the use of normalisers of C to construct an appropriate groupoid of germs. Recently it has become apparent that this normaliser semigroup S already contains a significant amount of structural information. For example, if A is already known to be a groupoid C*-algebra then this groupoid can be recovered just from the semigroup structure of S and C (as the groupoid of ultrafilters with respect to the 'domination relation' on S; see [4]).

This begs the question whether one can similarly represent a given abstract semigroup S, together with some additional structure coming from appropriate

The author is supported by the GAČR project EXPRO 20-31529X and RVO: 67985840.

subsemigroups, as a semigroup of sections of some groupoid bundle. This is borne out in the present paper, where we construct universal representations of S on groupoid bundles formed from cosets and filters of S. The basic idea is to extend the relevant theory of inverse semigroups, replacing the idempotents with normal subsemigroups analogous to Cartan subalgebras.

1.2. Outline. As a gentle introduction to our more general results, in Section 2 we first consider simpler classes of semigroups. We start with semilattices and their filters in Section 2.1, moving on to inverse semigroups and their cosets in Section 2.2. Finally, we introduce *structured semigroups* in Section 2.3, a generalisation of inverse semigroups that encompasses our primary motivating examples, namely the normaliser semigroups of Cartan subalgebras of C*-algebras.

In Section 3 we introduce groupoid bundles and examine their slice-sections. We give further motivational comments at the end to indicate how we construct groupoid bundles on which to represent structured semigroups.

Next in Section 4 we discuss appropriate morphisms between étale groupoids and groupoid bundles. Specifically, we consider relational Zakrzewski morphisms between étale groupoids, which are needed to include both continuous functions and group homomorphisms (but in opposite directions; see Remark 4.7), as well as the filter-coset Zakrzewski morphism $\triangleleft$ appearing later in Section 12.1. Building on these, we consider Zakrzewski–Pierce morphisms between groupoid bundles, which we show in Theorem 4.10 correspond to semigroup homomorphisms of their slice-sections.

From Section 5 we make the standing assumption that (S,N,Z) is a structured semigroup. We then introduce the domination relation, a generalisation of the usual ordering on an inverse semigroup which also has analogous properties that we proceed to examine. Next, in Section 6, we introduce dual subsets as something of a substitute for inverses. Optimal results here require an additional diagonality assumption on N that we discuss in Section 6.1, which becomes more relevant again later in Section 12.

Continuing on the path of extending inverse semigroup theory, we introduce atlases and cosets in Section 7. In Section 8 we show that the cosets form an étale groupoid; see Theorem 8.4. Again optimal results require an additional assumption, namely that Z is 'symmetric', as discussed in Section 8.1. Under this assumption, we use cosets to construct a universal étale representation in Theorem 8.11.

In Sections 9 and 10 we show how to split up cosets into certain equivalence classes. The main result is Theorem 10.2, which says that these form an étale bundle. Next, we represent S on this coset bundle in Section 11. The main result here is Theorem 11.2, which says that this is a universal bundle representation, assuming symmetry.

Finally, in Section 12, we take a closer look at filters or, more accurately, directed cosets. These are shown to form an étale subgroupoid of the coset groupoid which instead yield Zakrzewski universal representations (again under symmetry), as shown in Theorems 12.15 and 12.18.

2. Motivation

2.1. Semilattices. To put our results in context, it is instructive to first consider semilattices, that is, commutative semigroups consisting entirely of idempotents. For example, the open subsets $\mathcal{O}(X)$ of any topological space X form a semilattice under the intersection operation $\cap$. Of course the same applies to any subfamily of $\mathcal{O}(X)$ that is closed under pairwise intersections, like the family of open intervals (a, b) on the real line $\mathbb{R}$.

A slightly less trivial observation is that, in fact, all semilattices are of this form, at least up to isomorphism. Put another way, every semilattice has a faithful spatial representation. Formally, a *spatial representation* of a semilattice S on a space X is a map $\theta : S \to \mathcal{O}(X)$ such that $\theta[S]$ covers X and, for all $a, b \in S$,

$$\theta(ab) = \theta(a) \cap \theta(b).$$

A representation is *faithful* if it is injective, that is, $\theta(a) = \theta(b)$ if and only if $a = b$.

To construct a faithful representation of any given semilattice S, we can consider its *filters*, that is, those subsets $F \subseteq S$ such that, for all $a, b \in S$,

$$a, b \in F \quad \Leftrightarrow \quad ab \in F. \tag{Filter}$$

These are just the down-directed up-sets with respect to the order on S defined by

$$a \leq b \quad \Leftrightarrow \quad a = ab$$

(as products in S are meets/infima with respect to this ordering). Let $\mathcal{F}(S)$ denote the *filter spectrum*, that is, the space of all nonempty filters with the basis $(\mathcal{F}_a)_{a \in S}$, where

$$\mathcal{F}_a = \{F \in \mathcal{F}(S) : a \in F\}.$$

Let $\mathcal{F}$ on its own denote the map $a \mapsto \mathcal{F}_a$.

PROPOSITION 2.1. *The map $\mathcal{F}$ is a faithful spatial representation of S.*

PROOF. We immediately note that $\mathcal{F}_{ab} = \mathcal{F}_a \cap \mathcal{F}_b$, for all $a, b \in S$, thanks to the defining property of filters, that is, $\mathcal{F}$ is a spatial representation. To see that $\mathcal{F}$ is also faithful, first note that, for each $a \in S$, there is a minimal filter containing a, namely

$$a^{\leq} = \{b \in S : a \leq b\}.$$

Thus, for any $a, b \in S$, $\mathcal{F}_a = \mathcal{F}_b$ would imply $a^{\leq} \in \mathcal{F}_a = \mathcal{F}_b$ and $b^{\leq} \in \mathcal{F}_b = \mathcal{F}_a$, that is, $b \in a^{\leq}$ and $a \in b^{\leq}$ and hence $a = ab = b$. □

We call $\mathcal{F}$ the *filter representation* of S. Note, however, that this is not the only faithful representation. Indeed, the proof above shows that the subrepresentation obtained by restricting to principal filters (of the form $a^{\leq}$) is still faithful. What distinguishes the filter representation is its universality.

Roughly speaking, this means that the filter representation contains all possible spatial representations. To be more precise, first note that any continuous function

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

$\phi : Y \to X$ between topological spaces X and Y defines a semilattice homomorphism $\overline{\phi} : \mathcal{O}(X) \to \mathcal{O}(Y)$ (in the opposite direction) by taking preimages, that is,

$$\overline{\phi}(O) = \phi^{-1}[O].$$

In particular, if $\theta : S \to \mathcal{O}(X)$ is a spatial representation and $\phi : Y \to X$ is a continuous map on another space Y, then we immediately obtain a spatial representation $\overline{\phi} \circ \theta$ on Y by simply composing θ with $\overline{\phi}$, that is, for all $a \in S$,

$$\overline{\phi} \circ \theta(a) = \phi^{-1}[\theta(a)].$$

We call a spatial representation $\mu : S \to \mathcal{O}(X)$ *universal* if every spatial representation $\theta : S \to \mathcal{O}(Y)$ can be written uniquely as such a composition, that is, if there exists a unique continuous map $\phi : Y \to X$ such that $\theta = \overline{\phi} \circ \mu$.

PROPOSITION 2.2. *The filter representation $\mathcal{F}$ of any semilattice S is universal.*

PROOF. Take a spatial representation $\theta : S \to \mathcal{O}(X)$. For all $x \in X$, let

$$\phi(x) = \{a \in S : x \in \theta(a)\}.$$

Note that $a, b \in \phi(x)$ if and only if $x \in \theta(a) \cap \theta(b) = \theta(ab)$ if and only if $ab \in \phi(x)$, showing that $\phi(x)$ is a filter. Moreover, $\phi(x)$ is also nonempty, as $\theta[S]$ covers X, so $\phi(x) \in \mathcal{F}(S)$. Further note that $\phi(x) \in \mathcal{F}_a$ if and only if $a \in \phi(x)$ if and only if $x \in \theta(a)$, and hence

$$\phi^{-1}[\mathcal{F}_a] = \theta(a) \in \mathcal{O}(X).$$

Thus, ϕ is a continuous map from X to $\mathcal{F}(S)$ with $\theta = \overline{\phi} \circ \mathcal{F}$.

For uniqueness, suppose that we had another map $\phi' : X \to \mathcal{F}(S)$ such that $\overline{\phi'} \circ \mathcal{F} = \theta$, that is, $\phi'^{-1}[\mathcal{F}_a] = \theta(a)$, for all $a \in S$. Then again we note that $x \in \theta(a)$ if and only if $\phi'(x) \in \mathcal{F}_a$ if and only if $a \in \phi'(x)$ and hence $\phi'(x) = \{a \in S : x \in \theta(a)\} = \phi(x)$, for all $x \in X$. □

REMARK 2.3. The filter representation has various other subrepresentations that are of interest, particularly if one wants to represent semilattices on nicer topological spaces, for example with better separation properties.

For example, one could restrict to irreducible filters, which are always plentiful enough to retain faithfulness; see [8, Corollary 9]. The irreducible subrepresentation can even be extended to a categorical duality (see [9]) generalising the classic Stone duality between distributive lattices and spectral spaces (see [32]). The filter representation can also be extended to a different but analogous duality between semilattices and 'HMS spaces'; see [16].

One could also restrict to ultrafilters, that is, maximal proper filters, as in Wallman's work; see [34]. The space of ultrafilters is necessarily T_1, although it is only faithful for special kinds of semilattices, for example, separative semilattices with 0. We also briefly examine ultrafilters in more general semigroups later in Theorem 12.8. Indeed, the present paper could be viewed as a prelude to our subsequent work in [2, 3]

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

where ultrafilters play a much greater role in noncommutative extensions of the classic Gelfand duality (see [13, 14]).

2.2. Inverse semigroups. Our goal in the present paper is to extend Propositions 2.1 and 2.2 to noncommutative semigroups. As a first step in this direction, let us consider inverse semigroups S (see [20]), that is, such that each $a \in S$ has a unique generalised inverse a^{-1}, meaning that

$$aa^{-1}a = a \quad \text{and} \quad a^{-1}aa^{-1} = a^{-1}.$$

So semilattices are inverse semigroups where $a^{-1} = a$. Also the idempotents

$$\mathsf{E}(S) = \{e \in S : ee = e\}$$

of any inverse semigroup S form a semilattice. Consequently, inverse semigroups constitute a natural noncommutative generalisation of semilattices.

Topological spaces also have a natural noncommutative generalisation, namely étale groupoids; see [26, 29]. Let us digress for a moment to recall all the relevant definitions. First, a *groupoid* is a category where all the morphisms are isomorphisms. Following standard practice, we generally forget about the objects and just consider groupoids as collections of morphisms on which we have an inverse operation and a partially defined product. The *units* of a groupoid G are given by

$$G^0 = \{g \in G : gg = g\}.$$

We denote the *source* and *range* units of any $g \in G$ by

$$\mathsf{s}(g) = g^{-1}g \quad \text{and} \quad \mathsf{r}(g) = gg^{-1}.$$

Note that gh is defined if and only if $(g, h) \in G^2$, where

$$G^2 = \{(g, h) \in G \times G : \mathsf{s}(g) = \mathsf{r}(h)\}.$$

As usual, we call $B \subseteq G$ a *bisection* or *slice* of a groupoid G if the source and range $\mathsf{s}, \mathsf{r} : G \to G^0$ are injective on B or, equivalently, if $BB^{-1} \cup B^{-1}B \subseteq G^0$, where $B^{-1} = \{b^{-1} : b \in B\}$ and $AB = \{ab : a \in A \text{ and } b \in B\}$.

A *topological groupoid* is a groupoid G with a topology which makes the inverse $g \mapsto g^{-1}$ and product $(g, h) \mapsto gh$ continuous (on G^2). A topological groupoid is *étale* if the source s (and hence the range r and product too) is also an open map, that is, s maps open subsets of G to open subsets of G (which, in particular, implies that the units $G^0 = \mathsf{s}[G]$ form an open subset). Equivalently, a topology makes a groupoid étale precisely when the open bisections

$$\mathcal{B}(G) = \{B \in \mathcal{O}(G) : BB^{-1} \cup B^{-1}B \subseteq G^0\}$$

form both a basis for the topology of G and an inverse semigroup under pointwise products $(A, B) \mapsto AB$ and inverses $B \mapsto B^{-1}$; see [28, Theorem 5.18] and [5, Proposition 6.6]. This characterisation suggests that étale groupoids are natural structures on which to represent inverse semigroups.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

Accordingly, we call a map $\theta : S \to \mathcal{B}(G)$ an *étale representation* of an inverse semigroup S if $\theta[S]$ covers the étale groupoid G and, for all $a, b \in S$,

$$\theta(ab) = \theta(a)\theta(b)$$

(note that this is consistent with our previous notion of a spatial representation, as $AB = A \cap B$ for any $A, B \subseteq G^0$, in particular for any $A, B \in \mathcal{O}(X)$ when we consider a space X as an étale groupoid with the trivial product $xx = x$, for all $x \in X$). To obtain a faithful étale representation of an inverse semigroup S we can again consider filters with respect to the canonical ordering on S given by

$$a \leq b \quad \Leftrightarrow \quad a = aa^{-1}b.$$

As before, we let $\mathcal{F}(S)$ denote the nonempty filters in S, that is, the down-directed up-sets. These form an étale groupoid with inverse $F \mapsto F^{-1}$ and product

$$E \cdot F = (EF)^{\leq} \quad \text{when } (E^{-1}E)^{\leq} = (FF^{-1})^{\leq},$$

where $T^{\leq} = \{s \geq t : t \in T\}$ and the topology again has basis $(\mathcal{F}_a)_{a \in S}$, where $\mathcal{F}_a = \{F \in \mathcal{F}(S) : a \in F\}$; see [1, Propositions 2.7 and 2.38, Example 2.12 and Theorem 2.32], which extend results from [22, 23]. We call $\mathcal{F}(S)$ with this étale groupoid structure the *filter groupoid* of S. Again we let $\mathcal{F}$ on its own denote the map $a \mapsto \mathcal{F}_a$.

PROPOSITION 2.4. *The map $\mathcal{F}$ is a faithful étale representation of S.*

PROOF. For any $a, b \in S$, we immediately see that $\mathcal{F}_a \cdot \mathcal{F}_b \subseteq \mathcal{F}_{ab}$. Conversely, for any $F \in \mathcal{F}_{ab}$, set $E = (Fb^{-1})^{\leq} \in \mathcal{F}_a$, as $a \geq abb^{-1} \in Fb^{-1}$. For any $s, t \in S$ and $e \in \mathsf{E}(S)$, we see that $set \leq st$ and hence

$$(FF^{-1})^{\leq} \subseteq (Fb^{-1}bF^{-1})^{\leq} \subseteq (EE^{-1})^{\leq}.$$

For the reverse inclusion, first note that $FF^{-1}F \subseteq F^{\leq} = F$: for any $f, g, h \in F$, we have $i \in F$ with $i \leq f, g, h$, as F is a filter, and hence $i = ii^{-1}i \leq fg^{-1}h$. Thus,

$$Eb = (Fb^{-1})^{\leq}b \subseteq (Fb^{-1}b)^{\leq} \subseteq F^{\leq} = F.$$

As above, it then follows that $(EE^{-1})^{\leq} \subseteq (Ebb^{-1}E^{-1})^{\leq} \subseteq (FF^{-1})^{\leq}$. Thus, $E^{-1} \cdot F$ is defined and

$$F = E \cdot E^{-1} \cdot F \in \mathcal{F}_a \cdot \mathcal{F}_{a^{-1}} \cdot \mathcal{F}_{ab} \subseteq \mathcal{F}_a \cdot \mathcal{F}_{a^{-1}ab} \subseteq \mathcal{F}_a \cdot \mathcal{F}_b.$$

Thus, $\mathcal{F}_{ab} = \mathcal{F}_a \cdot \mathcal{F}_b$, showing that $a \mapsto \mathcal{F}_a$ is indeed an étale representation. As before, we see that $\mathcal{F}$ is faithful by considering principal filters. □

Of course, this is still not the only faithful étale representation and one would expect the filter representation again to be distinguished by its universality. However, universality fails, at least with respect to the obvious morphisms.

Specifically, we call a map $\phi : G \to H$ between étale groupoids an *étale morphism* if ϕ is a continuous star-bijective functor. Here *star-bijectivity* means that ϕ maps each star Ge, for $e \in G^0$, bijectively on the corresponding star $H\phi(e)$ (see [7], [15, Ch. 13];

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

star-bijective functors are also called *covering functors* in [21]). As noted below in Section 4, star-bijectivity ensures that preimages of slices are slices and that preimages respect products, that is, for all $A, B \in \mathcal{B}(G)$,

$$\phi^{-1}[AB] = \phi^{-1}[A]\phi^{-1}[B].$$

Thus, any étale morphism $\phi : H \to G$ again defines a semigroup homomorphism $\overline{\phi} : \mathcal{B}(G) \to \mathcal{B}(H)$ (in the opposite direction) between their open slice semigroups by just taking preimages. Composing $\overline{\phi}$ with any étale representation $\theta : S \to \mathcal{B}(G)$ on G then yields an étale representation $\overline{\phi} \circ \theta : S \to \mathcal{B}(H)$ on H.

Just like before, we call an étale representation $\mu : S \to \mathcal{B}(G)$ *universal* if every étale representation $\theta : S \to \mathcal{B}(H)$ can be written uniquely as such a composition, that is, there exists a unique étale morphism $\phi : H \to G$ such that $\theta = \overline{\phi} \circ \mu$. To see why the proof of Proposition 2.2 no longer works note that, while it is still possible to define $\phi(g) = \{a \in S : g \in \theta(a)\}$, this may not be a filter anymore. To construct universal representations here we must consider a slightly broader inverse semigroup generalisation of filters in semilattices.

Specifically, we call $C \subseteq S$ a *coset* if, for all $a, b \in C$ and $c \in S$,

$$c \in C \quad \Leftrightarrow \quad ab^{-1}c \in C. \tag{Coset}$$

Equivalently, $C \subseteq S$ is a coset when $C = C^{\leq} = CC^{-1}C$; see [20, §1.4, Proposition 26]. The nonempty cosets $\mathcal{C}(S)$ again form an étale groupoid with inverse $C \mapsto C^{-1}$, product $B \cdot C = (BC)^{\leq}$ and subbasis $(\mathcal{C}_a)_{a \in S}$, where

$$\mathcal{C}_a = \{C \in \mathcal{C}(S) : a \in C\};$$

again see [1, Example 2.12 and Theorem 2.32]. We call $\mathcal{C}(S)$ with this étale groupoid structure the *coset groupoid* of S and again let $\mathcal{C}$ denote the map $a \mapsto \mathcal{C}_a$. As in the proof of Proposition 2.4, we see that $\mathcal{C}$ is a faithful étale representation of S which we call the *coset representation*.

PROPOSITION 2.5. *The coset representation of any inverse semigroup S is universal.*

PROOF. Take any étale representation $\theta : S \to \mathcal{B}(G)$. For $g \in G$, define

$$\phi(g) = \{a \in S : g \in \theta(a)\}.$$

If $a, b, c \in \phi(g)$ then $g = gg^{-1}g \in \theta(a)\theta(b)^{-1}\theta(c) = \theta(ab^{-1}c)$, which means that $ab^{-1}c \in \phi(g)$. On the other hand, if $a, b, ab^{-1}c \in \phi(g)$ then

$$g = gg^{-1}g \in \theta(b)\theta(a)^{-1}\theta(ab^{-1}c) = \theta(b)\theta(a)^{-1}\theta(a)\theta(b)^{-1}\theta(c) \subseteq G^0\theta(c) \subseteq \theta(c).$$

So $\phi(g)$ is a coset, which is also nonempty, as $(\theta(a))_{a \in S}$ covers G, that is, $\phi(g) \in \mathcal{C}(S)$.

Next, note that $\phi(\mathsf{s}(g)) = \mathsf{s}(\phi(g))$. Indeed, we immediately see that

$$\mathsf{s}(\phi(g)) = \phi(g)^{-1} \cdot \phi(g) = (\phi(g)^{-1}\phi(g))^{\leq} \subseteq \phi(g^{-1}g) = \phi(\mathsf{s}(g)).$$

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

Conversely, for any $a \in \phi(\mathsf{s}(g))$, taking $b \in \phi(g)$, we see that $a \geq ab^{-1}b \in \phi(g^{-1})\phi(g)$ and hence $a \in (\phi(g^{-1})\phi(g))^{\leq} = \phi(g^{-1}) \cdot \phi(g) = \phi(\mathsf{s}(g))$, that is, $\phi(\mathsf{s}(g)) \subseteq \mathsf{s}(\phi(g))$. Likewise, $\phi(\mathsf{r}(g)) = \mathsf{r}(\phi(g))$ so $\mathsf{s}(g) = \mathsf{r}(h)$ implies that $\mathsf{s}(\phi(g)) = \mathsf{r}(\phi(h))$, so $\phi(g) \cdot \phi(h)$ is a valid product. Again $\phi(g) \cdot \phi(h) \subseteq \phi(gh)$ is immediate, while the reverse inclusion follows as above. Thus, ϕ is a functor.

To show that ϕ is star-bijective, take any $i \in G^0$. If $\mathsf{s}(g) = \mathsf{s}(h) = i$ and $\phi(g) = \phi(h)$ then, for any $a \in \phi(g) = \phi(h)$, we see that $g, h \in \theta(a)$ and hence $g = h$, as $\theta(a)$ is a slice, showing that ϕ is star-injective. On the other hand, for any $C \in \mathcal{C}(S)$ with $\mathsf{s}(C) = \phi(i)$, taking $c \in C$, we see that $c^{-1}c \in C^{-1}C \subseteq \mathsf{s}(C) = \phi(i)$ and hence $i \in \theta(c^{-1}c) = \theta(c)^{-1}\theta(c) = \mathsf{s}[\theta(c)]$, as $\theta(c)$ is a slice. Thus, we have $g \in \theta(c)$ with $\mathsf{s}(g) = i$ and hence

$$\phi(g) = \phi(g) \cdot \phi(i) = (c\phi(i))^{\leq} = C \cdot \mathsf{s}(C) = C,$$

by [1, Theorem 2.32], proving that ϕ is also star-surjective.

For continuity note that, as before, $\phi(g) \in \mathcal{C}_a$ if and only if $a \in \phi(g)$ if and only if $g \in \theta(a)$, so

$$\phi^{-1}[\mathcal{C}_a] = \theta(a) \in \mathcal{O}(G).$$

Thus, ϕ is an étale morphism from G to $\mathcal{C}(S)$ with $\theta = \overline{\phi} \circ \mathcal{C}$. Uniqueness follows by the same argument as before. □

As hinted at above, there is actually an alternative approach here: instead of broadening the class of filters, we can broaden the class of étale morphisms. Specifically, we can consider more general relational Zakrzewski morphisms between étale groupoids; see Section 4 below. Indeed, there is a Zakrzewski morphism between filters and cosets of any inverse semigroup, which can be used to show that the filter representation is universal with respect to Zakrzewski morphisms (see Theorems 12.14 and 12.15 below for more general structured semigroup results).

2.3. Structured semigroups. The only problem with inverse semigroups is that they are not general enough, not if we want to apply the theory to semigroups commonly arising in other fields. Our primary motivating examples here come from operator algebras, where Cartan subalgebras of C*-algebras have received increasing attention in recent years; see [27]. A key role in their theory is played by the *normaliser semigroup* S of a Cartan subalgebra $C \subseteq A$ given by

$$S = \{a \in A : aCa^* + a^*Ca \subseteq C\}. \qquad \text{(Normalisers)}$$

Note that if A is a projectionless C*-algebra, like the well-known Jiang–Su algebra (which does indeed have Cartan subalgebras; see [11]), then S cannot contain any nontrivial idempotents and thus certainly cannot be an inverse semigroup. Even when A does have a large supply of projections, for example, when A has real rank zero, there is still no guarantee that they will lie in S.

However, the Cartan subalgebra C here is a subsemigroup of S which still behaves much like the idempotents in an inverse semigroup. For one thing, C is commutative,

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

and its self-adjoint part $C_{\mathrm{sa}} = \{a \in C : a = a^*\}$ also has a natural lattice structure. This exposes the possibility that inverse semigroup techniques could still be applied to Cartan pairs if we could somehow extend the relevant theory to *-semigroups S with a sufficiently well-behaved *-subsemigroup C. This is precisely what we did in our previous paper [1], where we examined 'Weyl *-semigroups' and their coset and filter groupoids.

In the present paper we have our eye on applications even further afield (see [2, 3]), namely to algebraic analogues of C*-algebras known as Steinberg algebras (see [10, 31]) as well as to noncommutative Cartan subalgebras of C*-algebras (see [12, 19]). This forces us to consider more general semigroups S without an involution *, with more general noncommutative subsemigroups C. But we can always replace C with its centre

$$\mathsf{Z}(C) = \{z \in C : \text{ for all } c \in C\ (cz = zc)\},$$

or some fixed subsemigroup Z, whenever commutativity is vital.

This naturally leads us to the concept of a 'structured semigroup'. Formally this a triple (S, N, Z) where S is a semigroup with distinguished subsemigroups N and $Z \subseteq \mathsf{Z}(N)$ satisfying certain weak normality conditions. To describe these conditions, first recall that $N \subseteq S$ is said to be *normal* if, for all $s \in S$,

$$sN = Ns. \tag{Normal}$$

Let us call $Z \subseteq S$ *binormal* if, for all $a, b \in S$,

$$ab, ba \in Z \quad\Rightarrow\quad aZb \cup bZa \subseteq Z. \tag{Binormal}$$

If N is another subset, we call it Z-*trinormal* if, for all $a, b \in S$ and $n \in N$,

$$abn = n = nab \quad \text{and} \quad ab, ba \in Z \quad\Rightarrow\quad bna \in N. \tag{Z-Trinormal}$$

Note that if $Z \subseteq \mathsf{Z}(N)$, as in the structured semigroups considered below, then we always have $abn = nab$ whenever $ab \in Z$ and $n \in N$. So, in this case, the left side of (Z-Trinormal) could be replaced with just $abn = n$.

DEFINITION 2.6. We call (S, N, Z) a *structured semigroup* if:

(1) S is a semigroup;
(2) N is a Z-trinormal subsemigroup of S;
(3) $Z \subseteq \mathsf{Z}(N)$ is a binormal subsemigroup of S.

Note that every normal subsemigroup Z is binormal, as $ab \in Z$ then implies $aZb = abZ \subseteq ZZ \subseteq Z$. However, the converse can fail; for example, Cartan subalgebras are always binormal in their normaliser semigroups, even though they are not necessarily normal (see [4, Example 7.3]). Similarly, we also immediately see that any binormal N is N-trinormal and hence Z-trinormal, for any $Z \subseteq N$. In particular, for any commutative binormal subsemigroup C of a semigroup S, we get a structured semigroup by simply taking $N = Z = C$. This applies to any Cartan subalgebra C in its normaliser semigroup S, as well as to the idempotents in any inverse semigroup, thanks to the following observation.

PROPOSITION 2.7. *The idempotents* $\mathsf{E}(S)$ *of any inverse semigroup* S *are normal.*

PROOF. For any $s \in S$ and $e \in \mathsf{E}(S)$, note that $ses^{-1}ses^{-1} = ss^{-1}sees^{-1} = ses^{-1}$, that is, $ses^{-1} \in \mathsf{E}(S)$. Moreover, $se = ss^{-1}se = ses^{-1}s$, showing that $s\mathsf{E}(S) \subseteq \mathsf{E}(S)s$, while the reverse inclusion follows by a dual argument. □

However, inverse semigroups can also form structured semigroups in other ways, that is, there may be other natural choices for N and Z; see, for example, Proposition 3.6 below.

The next question is how structured semigroups should be represented. As before, we could consider étale representations, that is, semigroup homomorphisms $\theta : S \to \mathcal{B}(G)$, for some étale groupoid G. As N is meant to be a kind of diagonal of S, it would be natural to further require that θ maps N onto open subsets of the diagonal of G, that is, its unit space G^0, as we do in Definition 8.9 below. However, these are the idempotents of $\mathcal{B}(G)$, so if θ were also faithful then this would imply that N also consists entirely of idempotents. So again we see that these kinds of étale representations are not general enough, not if we want structured semigroups with few idempotents to potentially have faithful representations.

To fix this, we again take inspiration from operator algebra theory, specifically Kumjian and Renault's Weyl groupoid construction from a Cartan pair (A, C); see [17, 27]. This results in the C*-algebra A being represented as continuous sections of a saturated Fell line bundle $\pi : F \to G$ over a locally compact étale groupoid G. Furthermore, the sections coming from the normaliser semigroup S are supported on open slices of G. Restricting to these supports yields a representation of S as partial sections defined on slices taking values in the groupoid $F^\times$ of invertible elements. This suggests that we should likewise represent more general structured semigroups as partial sections of groupoid bundles. Accordingly, we start the paper proper with a discussion of these bundles and their slice-sections.

3. Slice-sections

We assume that the reader is familiar with the basics of étale groupoids, as briefly outlined in the previous section. We refer the reader to [26, 29] for further background, the key difference in our work being that our étale groupoids (and even their unit spaces) can be highly non-Hausdorff.

Recall that an *isocofibration* $\pi : C \to D$ between categories C and D is a functor that is injective on objects/units or, equivalently, such that cd is defined in C whenever $\pi(c)\pi(d)$ is defined in D. Also a map $\pi : X \to Y$ on a topological space X is *locally injective* if every point in X has a neighbourhood on which π is injective or, equivalently, such that the open subsets of X on which π is injective cover X.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

DEFINITION 3.1. We call $\pi : F \to G$ a *groupoid bundle* if:

(1) F is a topological groupoid;
(2) G is an étale groupoid; and
(3) π is an open continuous isocofibration.

An *étale bundle* is a groupoid bundle $\pi : F \to G$ that is also locally injective.

If $\pi : F \to G$ is an étale bundle then, in particular, π is an open continuous and locally injective map, that is, a local homeomorphism. In this case, the source map s on F is also a local homeomorphism, which means that F is also an étale groupoid.

PROPOSITION 3.2. *If $\pi : F \to G$ is an étale bundle then F is an étale groupoid.*

PROOF. Assume that $\pi : F \to G$ is an étale bundle. First we show that F^0 is open in F. To see this, take any $f \in F^0$. As π is locally injective, f has an open neighbourhood O on which π is injective. We claim that

$$f \in O \cap \mathsf{s}^{-1}[O] \cap \pi^{-1}[G^0] \subseteq F^0.$$

To start with, $f \in F^0$ implies $\mathsf{s}(f) = f \in O$ and $\pi(f) \in G^0$, as π is a functor, showing that $f \in O \cap \mathsf{s}^{-1}[O] \cap \pi^{-1}[G^0]$. On the other hand, for any $e \in O \cap \mathsf{s}^{-1}[O] \cap \pi^{-1}[G^0]$, we see that $e, \mathsf{s}(e) \in O$ and $\pi(e) = \mathsf{s}(\pi(e)) = \pi(\mathsf{s}(e))$ and hence $e = \mathsf{s}(e) \in F^0$, as π is injective on O, proving the claim. As s and π are continuous, and O and G^0 are open, it follows that $O \cap \mathsf{s}^{-1}[O] \cap \pi^{-1}[G^0]$ is also open. As f was arbitrary, this shows that F^0 is open.

As π is injective on units, for any $O \subseteq F$,

$$\mathsf{s}[O] = F^0 \cap \pi^{-1}[\mathsf{s}[\pi[O]]].$$

If O is open then this shows that $\mathsf{s}[O]$ is also open, that is, s is an open map on F and hence F is an étale groupoid. □

The simplest examples of groupoid bundles come from the obvious structure on the Cartesian product of an étale groupoid and a topological group.

EXAMPLE 3.3 (trivial bundles). If G is an étale groupoid and T is a topological group then $G \times T$ is a topological groupoid in the product topology where

$$(g, t)(h, u) = (gh, tu),$$

when gh is defined. The projection $\pi : G \times T \to G$ is then a groupoid bundle. If T is discrete then π is also locally injective and hence a local homeomorphism.

More interesting 'twisted' bundles can be obtained by modifying the structure above via a given 2-cocycle, just like in [26] and [29, Section 5.1].

EXAMPLE 3.4 (twisted bundles). Again suppose that G is an étale groupoid and T is a topological group. Furthermore, suppose that $\sigma : G^2 \mapsto T$ is a continuous 2-cocycle,

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

that is,

$$\sigma(\mathsf{r}(g), g) = 1 = \sigma(g, \mathsf{s}(g)),$$

for all $g \in G$, and, for all $t \in T$ and $g, h, i \in G$ such that ghi is defined,

$$\sigma(g, h)t\sigma(gh, i) = \sigma(g, hi)t\sigma(h, i). \tag{3-1}$$

We claim that $G \times T$ is a topological groupoid in the product topology, where this time

$$(g, t)(h, u) = (gh, t\sigma(g, h)u),$$

when gh is defined. Indeed, we verify associativity by noting that

$$[(g, t)(h, u)](i, v) = (gh, t\sigma(g, h)u)(i, v) = (ghi, t\sigma(g, h)u\sigma(gh, i)v),$$
$$(g, t)[(h, u)(i, v)] = (g, t)(hi, u\sigma(h, i)v) = (ghi, t\sigma(g, hi)u\sigma(h, i)v),$$

where the last two expressions are equal by (3-1). Likewise, for all $g \in G$,

$$\sigma(g, g^{-1})t = \sigma(g, g^{-1})t\sigma(\mathsf{r}(g), g) = \sigma(g, \mathsf{s}(g))t\sigma(g^{-1}, g) = t\sigma(g^{-1}, g).$$

It follows that

$$\begin{aligned}(g^{-1}, \sigma(g, g^{-1})^{-1}t^{-1})(g, t) &= (\mathsf{s}(g), \sigma(g, g^{-1})^{-1}t^{-1}\sigma(g^{-1}, g)t)\\ &= (\mathsf{s}(g), \sigma(g, g^{-1})^{-1}t^{-1}t\sigma(g, g^{-1}))\\ &= (\mathsf{s}(g), 1)\end{aligned}$$

and, similarly, $(g, t)(g^{-1}, \sigma(g, g^{-1})^{-1}t^{-1}) = (\mathsf{r}(g), 1)$, that is,

$$(g, t)^{-1} = (g^{-1}, \sigma(g, g^{-1})^{-1}t^{-1}).$$

As the cocycle is continuous, so is the product and inverse on $G \times T$. So again $G \times T$ is a topological groupoid and $\pi : G \times T \to G$ is a groupoid bundle.

Our primary interest in groupoid bundles is that they form natural structures on which to represent semigroups as 'slice-sections'.

DEFINITION 3.5. A *slice-section* of a groupoid bundle $\pi : F \to G$ is a continuous map a where $\mathrm{dom}(a)$ is an open slice of G on which $\pi \circ a$ is the identity.

So if a is a slice-section then $\mathrm{ran}(a) \subseteq F$ and $\pi(a(g)) = g$, for $g \in \mathrm{dom}(a)$.

PROPOSITION 3.6. *The slice-sections $\mathcal{S}(\pi)$ of a groupoid bundle $\pi : F \to G$ form an inverse semigroup where, for all $a, b \in S$, the product is given by*

$$ab(gh) = a(g)b(h) \quad \textit{when } g \in \mathrm{dom}(a),\ h \in \mathrm{dom}(b) \textit{ and } (g, h) \in G^2$$

(*so* $\mathrm{dom}(ab) = \mathrm{dom}(a)\mathrm{dom}(b)$). *Moreover, $\mathcal{S}(\pi)$ has normal subsemigroups*

$$\mathcal{N}(\pi) = \{n \in \mathcal{S}(\pi) : \mathrm{dom}(n) \subseteq G^0\},$$
$$\mathsf{E}(\mathcal{S}(\pi)) = \{z \in \mathcal{S}(\pi) : \mathrm{ran}(z) \subseteq F^0\} \subseteq \mathsf{Z}(\mathcal{N}(\pi)).$$

In particular, $(S, N, Z) = (\mathcal{S}(\pi), \mathcal{N}(\pi), \mathsf{E}(\mathcal{S}(\pi)))$ is a structured semigroup.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

PROOF. If gh is defined in G, for some $g, h \in G$, then $a(g)b(h)$ is also defined in G, as $\pi(a(g)) = g$, $\pi(b(h)) = h$ and π is an isocofibration. Also, for any $i \in G$, we have at most one pair $g \in \mathrm{dom}(a)$ and $h \in \mathrm{dom}(b)$ with $gh = i$, as these domains are slices. Thus, the above product yields a well-defined function ab on the slice $\mathrm{dom}(ab) = \mathrm{dom}(a)\mathrm{dom}(b)$. Furthermore, for any $g, h \in H$ such that gh is defined,

$$\pi(ab(gh)) = \pi(a(g)b(h)) = \pi(a(g))\pi(b(h)) = gh,$$

as π is a functor, so again $\pi \circ ab$ is the identity on $\mathrm{dom}(ab)$. As the product in F is continuous, the function ab is also continuous and hence a slice-section. As the inverse map in F is continuous, every slice-section a has a unique inverse slice-section $a^{-1}(g) = a(g^{-1})^{-1}$, for all $g \in \mathrm{dom}(a)^{-1}$. Thus, the slice-sections do indeed form an inverse semigroup $S = \mathcal{S}(\pi)$.

Certainly the diagonal $N = \mathcal{N}(\pi)$ is a subsemigroup. For any $a \in S$ and $n \in N$,

$$\mathrm{dom}(ana^{-1}) \subseteq \mathrm{dom}(a)\mathrm{dom}(n)\mathrm{dom}(a)^{-1} \subseteq \mathrm{dom}(a)\mathrm{dom}(a)^{-1} \subseteq G^0,$$

as $\mathrm{dom}(n) \subseteq G^0$ and $\mathrm{dom}(a)$ is a slice. Thus $an = ana^{-1}a \in Na$, showing that $aN \subseteq Na$, while the reverse inclusion follows by a dual argument, that is, $aN = Na$, showing that N is normal. As the only idempotents in a groupoid are units, the idempotent sections are precisely those whose range consists of units, that is,

$$Z = \mathsf{E}(S) = \{z \in S : \mathrm{ran}(z) \subseteq F^0\},$$

which is normal by Proposition 2.7. As units commute with isotropy elements of any groupoid, it follows that $nz = zn$, for all $n \in N$ and $z \in Z$, that is, $Z \subseteq \mathsf{Z}(N)$. □

Note here that N will be strictly bigger than $Z = \mathsf{E}(S)$, as long as there are at least some slice-sections of π taking nonunit values. So in this case, the above structured semigroup is different from the canonical one mentioned after Definition 2.6 formed by simply taking $N = Z = \mathsf{E}(S)$.

We can also consider certain inverse subsemigroups of slice-sections.

EXAMPLE 3.7. If G is an étale groupoid with Hausdorff unit space G^0 then pointwise products KL of compact subsets $K, L \subseteq G$ are again compact. If $\pi : F \to G$ is a groupoid bundle then it follows that the slice-sections defined on (open) compact slices form an inverse subsemigroup $S \subseteq \mathcal{S}(\pi)$. Here it is natural to restrict to ample groupoids (where G^0 and hence G has a basis of compact open subsets) so these compact-slice-sections can distinguish points of G.

If we have a trivial ample groupoid bundle $\pi : G \times (\mathbb{F} \setminus \{0\}) \to G$, where $\mathbb{F}$ is a discrete field, then any compact-slice-section s can be extended to the entirety of G with 0 values outside the original domain. The linear span of these then forms an algebra A of functions from G to $\mathbb{F}$ commonly known as the 'Steinberg algebra' over G. The compact-slice-sections of nontrivial locally injective ample groupoid bundles $\pi : F \to G$ could thus be considered as 'twisted Steinberg semigroups'.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

Concrete (sub)semigroups of slice-sections arising in other contexts often only have 'local inverses'.

DEFINITION 3.8. Given a groupoid bundle $\pi : F \to G$ and semigroup $S \subseteq \mathcal{S}(\pi)$, let

$$S_g = \{a \in S : g \in \mathrm{dom}(a)\}. \tag{3-2}$$

We call S a *local-inverse semigroup* if, for all $g \in G$ and $a \in S_g$, we have $a' \in S_{g^{-1}}$ and $b \in S_g$ with $a'(h^{-1}) = a(h)^{-1}$, for all $h \in \mathrm{dom}(b)$.

So the values of a' above are the inverses of the values of a on some neighbourhood of g which is also the domain of some $b \in S$. In particular, $\mathrm{dom}(b) \subseteq \mathrm{dom}(a)$.

EXAMPLE 3.9. As in [18], we can consider a Fell bundle $\pi : F \to G$ over a locally compact étale groupoid G with Hausdorff unit space G^0. This is not quite a groupoid bundle: even with line bundles, where the fibres are one-dimensional, each fibre still has a 0. However, we can simply restrict π to the invertible elements $F^\times$. The slice-sections a defined on open slices such that $||a||$ vanishes at infinity again form a subsemigroup $S \subseteq \mathcal{S}(\pi|_{F^\times})$. Again we can extend any such slice-section to the entirety of G with 0 values outside the original domain. The linear span of these again forms an algebra, this time with a natural reduced algebra norm. Taking the completion then yields a C*-algebra A. Thus, S again forms a natural semigroup from which to generate A.

Note that the inverse of a function vanishing at infinity may not vanish at infinity and could even be unbounded, that is, $a \in S$ does not imply $a^{-1} \in S$, so S is not an inverse semigroup. However, if $a \in S_g$ then we have an open neighbourhood $O \subseteq \mathrm{dom}(a)$ of g on which a^{-1} is bounded. We can then define a function $a' \in S_{g^{-1}}$ which vanishes at infinity but coincides with a^{-1} on O^{-1}. Taking any $b \in S_g$ with $\mathrm{dom}(b) \subseteq O$ then witnesses the fact that S is a local-inverse semigroup.

Similarly, the sections of π with range in F^0 rarely vanish at infinity (only if their domain is also compact). Thus, it is more natural to allow scalar multiples of units as well, that is, instead of taking Z to be the idempotent slice-sections as in Proposition 3.6, we can take $Z = S \cap \mathcal{Z}(\pi)$, where $\mathcal{Z}(\pi)$ denotes the *central-diagonal*

$$\mathcal{Z}(\pi) = \{z \in \mathcal{S}(\pi) : \mathrm{ran}(z) \subseteq \mathbb{C}F^0\}.$$

As the name suggests, $\mathcal{Z}(\pi)$ is a central subsemigroup of the diagonal $\mathcal{N}(\pi)$ and hence $Z = S \cap \mathcal{Z}(\pi)$ is also a central subsemigroup of $N = S \cap \mathcal{N}(\pi)$. Moreover, note that we can always choose the a' in the previous paragraph so that $aa', a'a \in Z$. Also, unlike in the previous examples, N and Z here may not be normal (for example, in [4, Example 7.3] mentioned above) but they will still be binormal and hence (S, N, Z) will still form a structured semigroup.

Another possibility is to leave π as is; even though F is only a category, not a groupoid, the slice-sections $\mathcal{S}(\pi)$ still form a semigroup with a subsemigroup S of functions vanishing at infinity. Again taking $N = S \cap \mathcal{N}(\pi)$ and $Z = S \cap \mathcal{Z}(\pi)$, we see that N may not even be binormal now; however, it will still be Z-trinormal, that is, (S, N, Z) will still form a structured semigroup.

Our main goal is to obtain a kind of converse to Proposition 3.6. More precisely, we want to show how to represent a structured semigroup as a subsemigroup of $\mathcal{S}(\pi)$, for some étale bundle $\pi : F \to G$, which is universal for an appropriate class of representations on even more general groupoid bundles.

To get some hint as to how we might do this, take an étale bundle $\pi : F \to G$ and consider the inverse semigroup of all slice-sections $S = \mathcal{S}(\pi)$. The sets of the form S_g, for $g \in G$, form 'cosets' in an appropriate sense, generalising the usual cosets one considers in inverse semigroups; see (Coset) above. Indeed, these general cosets can still be defined via a *domination* relation $<$ on S generalising the usual order on an inverse semigroup (basically $<$ corresponds to inclusion of domains; see Proposition 5.2 below). Moreover, for any $g, h \in G$ such that gh is defined, the products $S_g S_h$ of slice-sections in S_g and S_h are $<$-coinitial in the family of slice-sections S_{gh}. In this way, the product structure of S encodes that of G.

For each $f \in F$ and $g \in G$ with $\pi(f) = g$, we can further consider the subfamily $S^f \subseteq S_g$ of slice-sections taking the value f at g

$$S^f = \{a \in S : a(g) = f\}.$$

These form equivalence classes in S_g modulo a relation $\sim_{S_g}$ again determined by the structured semigroup. Moreover, for any $e, f \in F$ such that ef is defined, S^{ef} coincides with the equivalence class containing $S^e S^f$, again showing how the product structure of F is encoded by that of S.

The goal is thus to examine the domination relation $<$ in structured semigroups and the cosets and equivalence classes it gives rise to, in order to construct étale bundles on which to obtain universal slice-section representations. For universality, however, we first need to consider morphisms between groupoid bundles which correspond to semigroup homomorphisms of the associated slice-sections.

4. Morphisms

Before considering groupoid bundles, we first need an appropriate notion of morphism for étale groupoids. One option would be to consider the (functional) étale morphisms from Section 2.2, which would be fine for Section 11 where we consider general cosets. However, when we restrict our attention to directed cosets in Section 12, we need more general (relational) Zakrzewski morphisms; see Theorem 12.14 below.

First let us consider any $\phi \subseteq G \times H$ as a relation 'from H to G', where $g\ \phi\ h$ means $(g, h) \in \phi$. The flip of ϕ is denoted by $\phi^{-1} \subseteq H \times G$, that is, $h\ \phi^{-1}\ g$ if and only if $g\ \phi\ h$. The *image* of $H' \subseteq H$ and *preimage* of $G' \subseteq G$ are denoted by

$$\phi[H'] = \{g \in G : \text{ there exists } h \in H'\ (g\ \phi\ h)\}, \tag{Image}$$

$$\phi^{-1}[G'] = \{h \in H : \text{ there exists } g \in G'\ (g\ \phi\ h)\}. \tag{Preimage}$$

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

In particular, we define the domain and range of $\phi \subseteq G \times H$ by

$$\mathrm{dom}(\phi) = \phi^{-1}[G] \subseteq H \quad \text{and} \quad \mathrm{ran}(\phi) = \phi[H] \subseteq G.$$

The composition of $\phi \subseteq G \times H$ and $\psi \subseteq H \times I$ is the relation $\phi \circ \psi \subseteq G \times I$ where

$$g\ (\phi \circ \psi)\ i \quad \Leftrightarrow \quad \text{there exists } h \in H\ (g\ \phi\ h\ \psi\ i).$$

DEFINITION 4.1. If G and H are groupoids, $\phi \subseteq G \times H$ is *functorial* if

$$\begin{aligned} g\ \phi\ h \quad &\Rightarrow \quad g^{-1}\ \phi\ h^{-1}, \quad \text{and} \\ g\ \phi\ h,\ g'\ \phi\ h' \text{ and } \mathsf{s}(h) = \mathsf{r}(h') \quad &\Rightarrow \quad \mathsf{s}(g) = \mathsf{r}(g') \text{ and } gg'\ \phi\ hh'. \end{aligned}$$

In particular, note that $\mathrm{dom}(\phi)$ is a subgroupoid when ϕ is functorial.

As usual, we call $\phi \subseteq G \times H$ a *function* when the image $\phi\{h\} = \phi[\{h\}]$ of every singleton h in $\mathrm{dom}(\phi)$ is again a singleton in $\mathrm{ran}(\phi)$, in which case

$$g\ \phi\ h \quad \Leftrightarrow \quad g = \phi(h).$$

So a functorial function $\phi \subseteq G \times H$ is just a functor on $\mathrm{dom}(\phi)$ in the usual sense. We also get functions when we restrict functorial relations to slices in the range.

PROPOSITION 4.2. *If $\phi \subseteq G \times H$ is functorial then*

$$B \subseteq G \textit{ is a slice} \quad \Rightarrow \quad \phi \cap (B \times H) \textit{ is a function}. \tag{4-1}$$

PROOF. It suffices to show that

$$f, g\ \phi\ h \quad \Rightarrow \quad \mathsf{s}(f) = \mathsf{s}(g).$$

To see this, note that $f, g\ \phi\ h$ implies $g^{-1}\ \phi\ h^{-1}$ by functoriality. As $\mathsf{s}(h) = \mathsf{r}(h^{-1})$, functoriality again yields $\mathsf{s}(f) = \mathsf{r}(g^{-1}) = \mathsf{s}(g)$. □

We extend the terminology in [7, 15] from functions to relations.

DEFINITION 4.3. Assume that G and H are groupoids and $\phi \subseteq G \times H$ is functorial. We call ϕ *star-injective* (*surjective*, *bijective*) if, whenever $h \in H$ and $\mathsf{r}(g)\ \phi\ h \in H^0$, there is at most (at least, precisely) one $i \in H$ such that $\mathsf{r}(i) = h$ and $g\ \phi\ i$.

More symbolically, these definitions can be summarised as follows:

$$\mathsf{r}(g)\ \phi\ h \in H^0 \quad \Rightarrow \quad |\phi^{-1}\{g\} \cap \mathsf{r}^{-1}\{h\}| \le 1, \tag{Star-Injective}$$

$$\mathsf{r}(g)\ \phi\ h \in H^0 \quad \Rightarrow \quad |\phi^{-1}\{g\} \cap \mathsf{r}^{-1}\{h\}| \ge 1, \tag{Star-Surjective}$$

$$\mathsf{r}(g)\ \phi\ h \in H^0 \quad \Rightarrow \quad |\phi^{-1}\{g\} \cap \mathsf{r}^{-1}\{h\}| = 1. \tag{Star-Bijective}$$

These have several equivalent characterisations; for example, ϕ is star-injective if and only if its kernel is discrete, that is, unit-preimages are units. More important for us is the fact that star-injectivity means ϕ^{-1} respects slices.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

PROPOSITION 4.4. *The functorial* $\phi \subseteq G\times H$ *is star-injective precisely when* $\phi^{-1}[G^0]\subseteq H^0$ *or, equivalently, when*

$$B \text{ is a slice of } G \quad \Rightarrow \quad \phi^{-1}[B] \text{ is a slice of } H. \tag{4-2}$$

PROOF. If ϕ is not star-injective then we have $h\in H^0$ and distinct $i,j\in H$ with $\mathsf{r}(h)=\mathsf{r}(i)=h$ and $g\ \phi\ i,j$. Then $G^0\ni \mathsf{s}(g)=g^{-1}g\ \phi\ i^{-1}j\notin H^0$ by functoriality.

Conversely, if we have $G^0\ni g\ \phi\ h\notin H^0$ then $g=\mathsf{r}(g)\ \phi\ \mathsf{r}(h)\neq h$, even though $\mathsf{r}(\mathsf{r}(h))=\mathsf{r}(h)$, showing that ϕ is not star-injective.

Now if $\phi^{-1}[G^0]\subseteq H^0$ then, for any slice $B\subseteq G^0$, functoriality yields

$$\phi^{-1}[B](\phi^{-1}[B])^{-1}=\phi^{-1}[B]\phi^{-1}[B^{-1}]\subseteq\phi^{-1}[BB^{-1}]\subseteq\phi^{-1}[G^0]\subseteq H^0.$$

Likewise, $(\phi^{-1}[B])^{-1}\phi^{-1}[B]\subseteq H^0$, showing that $\phi^{-1}[B]$ is a slice.

Conversely, if $\phi^{-1}\{g\}$ is a slice then, in particular, $i,j\in\phi^{-1}\{g\}\cap\mathsf{r}^{-1}\{h\}$ implies $i=j$. So if this holds for all $g\in G$ then ϕ is star-injective. □

On the other hand, star-surjectivity means ϕ^{-1} respects products.

PROPOSITION 4.5. *The functorial* $\phi\subseteq G\times H$ *is star-surjective if and only if, for all* $A,B\subseteq G$,

$$\phi^{-1}[AB]=\phi^{-1}[A]\phi^{-1}[B]. \tag{4-3}$$

PROOF. Assume that $\phi\subseteq G\times H$ is functorial and star-surjective. If $A,B\subseteq G$ then $\phi^{-1}[A]\phi^{-1}[B]\subseteq\phi^{-1}[AB]$ by the functoriality of ϕ. Conversely, take $a\in A$, $b\in B$ and $h\in H$ with $ab\ \phi\ h$. By functoriality, $\mathsf{r}(a)=\mathsf{r}(ab)\ \phi\ \mathsf{r}(h)$. Star-surjectivity then yields $i\in\phi^{-1}\{a\}$ with $\mathsf{r}(h)=\mathsf{r}(i)$. Again functoriality yields $b=a^{-1}(ab)\ \phi\ i^{-1}h$ and hence $h=i(i^{-1}h)\in\phi^{-1}[A]\phi^{-1}[B]$, showing that $\phi^{-1}[AB]\subseteq\phi^{-1}[A]\phi^{-1}[B]$ too.

On the other hand, if (4-3) holds and $\mathsf{r}(g)\ \phi\ h\in H^0$ then

$$h\in\phi^{-1}\{\mathsf{r}(g)\}=\phi^{-1}\{gg^{-1}\}=\phi^{-1}\{g\}\phi^{-1}\{g^{-1}\}=\phi^{-1}\{g\}(\phi^{-1}\{g\})^{-1}.$$

Thus, we have $i\in\phi^{-1}\{g\}$ and $j\in\phi^{-1}\{g^{-1}\}$ with $h=ij$. As $h\in H^0$, this means that $h=\mathsf{r}(h)=\mathsf{r}(ij)=\mathsf{r}(i)$, showing that ϕ is star-surjective. □

In fact, the proofs show that it suffices to consider singleton A and B in (4-2) and (4-3). We also note that ϕ is star-surjective if and only if $\mathrm{ran}(\phi)$ is an ideal of G; see (Ideal) below.

Extending the usual notion for functions, when G and H are topological spaces and $\phi\subseteq G\times H$, we call ϕ *continuous* if $\phi^{-1}[O]$ is open, for all open $O\subseteq G$.

DEFINITION 4.6. A *Zakrzewski morphism* $\phi\subseteq G\times H$ between étale groupoids G and H is a continuous star-bijective functorial relation.

The idea of considering relational morphisms between groupoids comes from [35] and was further studied in [30]. As in Section 2.2, a functional Zakrzewski morphism will be called an *étale morphism*.

REMARK 4.7. Zakrzewski morphisms include both continuous functions and group homomorphisms but in opposite directions. Indeed, if G and H are étale groupoids with $G = G^0$ and $H = H^0$, that is, if G and H are just topological spaces under the trivial product defined on the diagonal, a Zakrzewski morphism $\phi \subseteq G \times H$ is just a continuous function from an open subset of H to G. At the other extreme, if G^0 and H^0 are singletons, that is, if G and H are discrete groups, then nonempty $\phi \subseteq G \times H$ is a Zakrzewski morphism if and only if $\phi^{-1} \subseteq H \times G$ is a group homomorphism from G to H.

Zakrzewski morphisms can also be seen as generalisations of groupoid bundles. Indeed, if F and G are étale groupoids then a function $\pi : F \to G$ is a groupoid bundle if and only if $\pi^{-1} \subseteq F \times G$ is an open Zakrzewski morphism, that is, a Zakrzewski morphism such that $\pi^{-1}[O]$ is open, for all open $O \subseteq G$.

Now suppose we have a groupoid bundle $\pi : F \to G$ and a Zakrzewski morphism $\phi \subseteq G \times H$. Consider the subspace of $F \times H$ given by

$$\phi^\pi F = \{(f, h) \in F \times H : \pi(f) \ \phi \ h\}.$$

Note that $\phi^\pi F$ is a topological groupoid under the product

$$(f, h)(f', h') = (ff', hh') \quad \text{when } hh' \text{ is defined.}$$

Let $\pi_\phi : \phi^\pi F \to H$ denote the projection onto H, that is, $\pi_\phi(f, h) = h$.

PROPOSITION 4.8. *If $\pi : F \to G$ is a groupoid bundle and $\phi \subseteq G \times H$ is a Zakrzewski morphism then $\pi_\phi : \phi^\pi F \to H$ is also a groupoid bundle.*

PROOF. The definition of the topology and product on $\phi^\pi F$ ensures that π_ϕ is a continuous isocofibration. It only remains to show that π_ϕ is also an open map. To see this, take open $O \subseteq F$ and $N \subseteq H$ and note that

$$\begin{aligned}\pi_\phi[(O \times N) \cap \phi^\pi F] &= \{h \in N : \text{ there exists } f \in O \ (\pi(f) \ \phi \ h)\} \\ &= N \cap \phi^{-1}[\pi[O]],\end{aligned}$$

which is open because ϕ is continuous and π is an open map. □

This $\pi_\phi : \phi^\pi F \to H$ is the *pullback bundle* of π induced by ϕ.

DEFINITION 4.9. Let $\pi : F \to G$ and $\pi' : F' \to G'$ be groupoid bundles. If

(1) $\phi \subseteq G \times G'$ is a Zakrzewski morphism,
(2) $\tau : \phi^\pi F \to F'$ is a continuous functor and
(3) $\pi_\phi = \pi' \circ \tau$ (that is, $\pi'(\tau(f, g')) = g'$, for all $(f, g') \in \phi^\pi F$)

then we call the pair (ϕ, τ) a *Zakrzewski–Pierce morphism* from π to π'. If ϕ is also a function then (ϕ, τ) is a *Pierce morphism*.

Similar morphisms for ring bundles are considered in [25, Definition 6.1], hence the name (for analogous C*-bundle morphisms, see [33, Definition 4.3]). Like in

[25, Lemma 6.2], Zakrzewski–Pierce morphisms of groupoid bundles naturally yield semigroup homomorphisms of their slice-sections.

THEOREM 4.10. *If (ϕ, τ) is a Zakrzewski–Pierce morphism from $\pi : F \to G$ to $\pi' : F' \to G'$, we have a semigroup homomorphism $\tau/\phi : \mathcal{S}(\pi) \to \mathcal{S}(\pi')$ given by*

$$\frac{\tau}{\phi}(a) = \tau \circ (((a \circ \phi) \times \mathrm{id}) \circ \delta),$$

where $\delta(g') = (g', g')$ and $\mathrm{id}(g') = g'$.

PROOF. Assume that a is a slice-section of π. In particular, $\mathrm{dom}(a)$ is an open slice, as is

$$\mathrm{dom}\Big(\frac{\tau}{\phi}(a)\Big) = \mathrm{dom}(a \circ \phi) = \phi^{-1}[\mathrm{dom}(a)],$$

because ϕ is continuous and star-injective; see (4-2). As a and τ are also continuous, so is $(\tau/\phi)(a)$. Moreover, $\phi \cap (\mathrm{dom}(a) \times G')$ and hence $a \circ \phi$ is function, by (4-1). Thus, $(\tau/\phi)(a)$ is a function too such that, for all $g' \in \mathrm{dom}((\tau/\phi)(a))$,

$$\frac{\tau}{\phi}(a)(g') = \tau((a \circ \phi)(g'), g')$$

(which is defined because $((a \circ \phi)(g'), g') \in \phi^{\pi}$, as a is a section of π). It follows that $\pi'((\tau/\phi)(a)(g')) = \pi'(\tau((a \circ \phi)(g'), g')) = \pi_{\phi}((a \circ \phi)(g'), g') = g'$. This all shows that $(\tau/\phi)(a)$ is a slice-section of π', that is, $(\tau/\phi)(a) \in \mathcal{S}(\pi')$.

Now, given another slice-section $b \in \mathcal{S}(\pi)$, note that

$$\mathrm{dom}\Big(\frac{\tau}{\phi}(ab)\Big) = \phi^{-1}[\mathrm{dom}(ab)] = \phi^{-1}[\mathrm{dom}(a)\mathrm{dom}(b)], \quad \text{and}$$

$$\mathrm{dom}\Big(\frac{\tau}{\phi}(a)\frac{\tau}{\phi}(b)\Big) = \mathrm{dom}\Big(\frac{\tau}{\phi}(a)\Big)\mathrm{dom}\Big(\frac{\tau}{\phi}(b)\Big) = \phi^{-1}[\mathrm{dom}(a)]\phi^{-1}[\mathrm{dom}(b)],$$

so $\mathrm{dom}((\tau/\phi)(ab)) = \mathrm{dom}((\tau/\phi)(a)(\tau/\phi)(b))$, by the star-surjectivity of ϕ. Moreover, for any $g' \in \mathrm{dom}((\tau/\phi)(a))$ and $h' \in \mathrm{dom}((\tau/\phi)(b))$ such that $g'h'$ is defined, note that

$$\begin{aligned}
\frac{\tau}{\phi}(ab)(g'h') &= \tau((ab \circ \phi)(g'h'), g'h') \\
&= \tau((a \circ \phi)(g')(b \circ \phi)(h'), g'h') \\
&= \tau(((a \circ \phi)(g'), g')((b \circ \phi)(h'), h')) \\
&= \tau((a \circ \phi)(g'), g')\tau((b \circ \phi)(h'), h') \\
&= \frac{\tau}{\phi}(a)(g')\frac{\tau}{\phi}(b)(h') \\
&= \Big(\frac{\tau}{\phi}(a)\frac{\tau}{\phi}(b)\Big)(g'h').
\end{aligned}$$

This shows that $(\tau/\phi)(ab) = (\tau/\phi)(a)(\tau/\phi)(b)$, that is, (τ/ϕ) is semigroup homomorphism. □

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

5. Domination

We now make the following standing assumption throughout.

$$(S, N, Z) \text{ is a structured semigroup (see Definition 2.6).}$$

DEFINITION 5.1. We define relations on S as follows:

$$\begin{aligned} a <_s b \quad &\Leftrightarrow \quad asb = a = bsa,\ as, sa \in N \quad \text{and} \quad bs, sb \in Z; \\ a < b \quad &\Leftrightarrow \quad \text{there exists } s \in S\ (a <_s b). \end{aligned}$$

When $a <_s b$, we say that b *dominates* a *via* s.

To get some intuition for domination, we imagine that S is a subsemigroup of the slice-sections $\mathcal{S}(\pi)$ of some groupoid bundle $\pi : F \to G$. As long as N lies in the diagonal $\mathcal{N}(\pi)$, $a <_s b$ means that s is an inverse of b on the domain of a.

PROPOSITION 5.2. *If $\pi : F \to G$ is a groupoid bundle and $a, s, b \in \mathcal{S}(\pi)$,*

$$a = asb \quad \textit{and} \quad as \in \mathcal{N}(\pi) \quad \Leftrightarrow \quad \textit{for all } g \in \mathrm{dom}(a)\ (s(g^{-1}) = b(g)^{-1})$$

(implicit in $s(g^{-1}) = b(g)^{-1}$ is that $g^{-1} \in \mathrm{dom}(s)$ and $g \in \mathrm{dom}(b)$).

PROOF. If $a = asb$ and $g \in \mathrm{dom}(a)$ then $a(g) = asb(g) = a(h)s(i)b(j)$, for some $h, i, j \in G$ with $g = hij$. Thus, $\mathsf{r}(g) = \mathsf{r}(hij) = \mathsf{r}(h)$ so $g = h$, as $\mathrm{dom}(a)$ is a slice. If $as \in \mathcal{N}(\pi)$ too then $gi = hi \in \mathrm{dom}(as) \subseteq G^0$ and hence $i = g^{-1}$. Thus, $g = hij = gg^{-1}j = j$ so $a(g) = a(g)s(g^{-1})b(g)$ and hence $s(g^{-1})b(g) \in F^0$, that is, $s(g^{-1}) = b(g)^{-1}$. This proves $\Rightarrow$.

Now suppose that $s(g^{-1}) = b(g)^{-1}$, for all $g \in \mathrm{dom}(a)$. In particular, $\mathrm{dom}(a) \subseteq \mathrm{dom}(s)^{-1}$ and hence $\mathrm{dom}(as) = \mathrm{dom}(a)\mathrm{dom}(s) \subseteq \mathrm{dom}(s)^{-1}\mathrm{dom}(s) \subseteq G^0$ so $as \in \mathcal{N}(\pi)$. Also

$$asb(g) = a(g)s(g^{-1})b(g) = a(g)b(g)^{-1}b(g) = a(g),$$

for all $g \in \mathrm{dom}(a)$. In particular, $\mathsf{s}[\mathrm{dom}(a)] \subseteq \mathrm{dom}(sb)$ and hence

$$\mathrm{dom}(asb) = \mathrm{dom}(a)\mathrm{dom}(sb) = \mathrm{dom}(a)$$

which, together with the above computation, yields $a = asb$, proving $\Leftarrow$. □

In particular, if $\pi : F \to G$ is a groupoid bundle and (S, N, Z) is a structured semigroup with $S \subseteq \mathcal{S}(\pi)$ and $N \subseteq \mathcal{N}(\pi)$ then it follows from the above that

$$a < b \quad \Rightarrow \quad \mathrm{dom}(a) \subseteq \mathrm{dom}(b).$$

On the other hand, if $S = \mathcal{S}(\pi)$, $\mathcal{N}(\pi) \subseteq N$ and $\mathsf{E}(S) \subseteq Z$ then the converse holds. If we instead consider functions vanishing at infinity as in Example 3.9 then $<$ is instead equivalent to compact containment $\Subset$ of domains, that is, $a < b$ means $\mathrm{dom}(a) \subseteq K \subseteq \mathrm{dom}(b)$, for some compact $K \subseteq G$ (see [4, Proposition 4.4]).

On inverse semigroups, $<$ is just the usual ordering.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

PROPOSITION 5.3. *If S is an inverse semigroup and $N = Z = \mathsf{E}(S)$ then*

$$a < b \quad \Leftrightarrow \quad a \in Nb \quad \Leftrightarrow \quad a <_{b^{-1}} b. \tag{5-1}$$

PROOF. If $a < b$ then we have $s \in S$ with $a = asb \in Nb$. Conversely, if $a = nb$, for some $n \in N$, then $ab^{-1} = nbb^{-1} \in N$ and $ab^{-1}b = nbb^{-1}b = nb = a$. Likewise $b^{-1}a = b^{-1}nb \in N$ and $bb^{-1}a = bb^{-1}nb = nbb^{-1}b = nb = a$. As $bb^{-1}, b^{-1}b \in N$, this shows that $a <_{b^{-1}} b$. □

Our primary goal in this section is to show that the domination relation $<$ on general structured semigroups still has many of the same properties as the usual order relation on inverse semigroups.

First we note that $<$ could have been defined in a more one-sided way. Note that here and elsewhere we make use of the fact that a result proved from S immediately yields a dual result for the opposite semigroup S^{op} (where $a \cdot^{\mathrm{op}} b = ba$).

PROPOSITION 5.4. *For any $a, b, b' \in S$,*

$$\begin{aligned} a <_s b \quad &\Leftrightarrow \quad asb = a,\ as \in N \quad \text{and} \quad bs, sb \in Z, \\ &\Leftrightarrow \quad bsa = a,\ sa \in N \quad \text{and} \quad bs, sb \in Z. \end{aligned}$$

PROOF. If $asb = a$, $as \in N$ and $bs, sb \in Z$ then $as \in N$ commutes with $bs \in Z$. Thus, $bsa = bsasb = asbsb = a$ and $bsas = asbs = as \in N$ so $sa = sasb \in N$, as N is Z-trinormal. Thus, $a <_s b$, proving the first $\Leftrightarrow$, while the second follows dually. □

From now on we use the above characterisations of $<$.

PROPOSITION 5.5. *For all $a, b', b, c', c \in S$,*

$$a <_{b'} b <_{c'} c \quad \Rightarrow \quad a <_{c'} c. \tag{Transitivity}$$

PROOF. If $a <_{b'} b <_{c'} c$ then we see that $ac' = ab'bc' \in NN \subseteq N$ and $ac'c = ab'bc'c = ab'b = a$. □

We can also switch the subscript with the right argument as follows.

PROPOSITION 5.6. *For all $a, b, c, c' \in S$,*

$$ab \in N \quad \text{and} \quad b <_{c'} c \quad \Rightarrow \quad abc' <_c c'. \tag{Switch}$$

PROOF. If $ab \in N$ and $b <_{c'} c$ then we see that $abc'c \in NZ \subseteq N$ and $abc'cc' = abc'$. □

Next we show that $<$ preserves the product.

PROPOSITION 5.7. *For any $a, b', b, c, d', d \in S$,*

$$a <_{b'} b \quad \text{and} \quad c <_{d'} d \quad \Rightarrow \quad ac <_{d'b'} bd. \tag{Multiplicativity}$$

PROOF. If $a <_{b'} b$ and $c <_{d'} d$ then $b'bb'acd' = b'acd' \in NN \subseteq N$ so $acd'b' = bb'acd'b' \in N$, as N is Z-trinormal. Also $acd'b'bd = ab'bcd'd = ac$, as $b'b \in Z$

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

commutes with $cd' \in N$. Moreover, as Z is binormal,

$$d'b'bd \in d'Zd \subseteq Z \supseteq bZb' \ni bdd'b'. \qquad \square$$

Next we note multiplying by elements of N on the left does not affect $<$.

PROPOSITION 5.8. *For any $a, b', b \in S$ and $n \in N$,*

$$a <_{b'} b \quad \Rightarrow \quad an, na <_{b'} b. \tag{N-Invariance}$$

PROOF. If $a <_{b'} b$ then $nab' \in NN \subseteq N$ and $nab'b = na$ and hence $na <_{b'} b$, while $an <_{b'} b$ follows by duality. $\square$

We also have a similar result for elements of Z.

PROPOSITION 5.9. *For any $a, b, b' \in S$ and $z \in Z$,*

$$az = a <_{b'} b \quad \Rightarrow \quad a <_{b'} bz \quad \text{and} \quad a <_{zb'} b. \tag{Z-Invariance}$$

PROOF. If $az = a <_{b'} b$ then $b'bz, bzb' \in Z$ and $ab'bz = az$, that is, $a <_{b'} bz$. Also $azb' = ab' \in N$ and $azb'b = a$, that is, $a <_{zb'} b$. $\square$

We can even split up pairs whose products lie in Z.

PROPOSITION 5.10. *For any $a, b, b', c, c' \in S$ with $cc', c'c \in Z$,*

$$acc' = a <_{b'} b \quad \Rightarrow \quad ac <_{c'b'} bc. \tag{Z-Splitting}$$

PROOF. If $acc' = a <_{b'} b$ and $cc' \in Z$ then $c'b'bc, bcc'b' \in Z$, $acc'b'bc = ac$ and $acc'b' = ab'bcc'b' \in NZ \subseteq N$, that is, $ac <_{c'b'} bc$. $\square$

For use in the next section, we denote the up-closure of $A \subseteq S$ by

$$A^{<} = \{b > a : a \in A\}.$$

6. Duals

DEFINITION 6.1. The *dual* of $A \subseteq S$ is defined by

$$A^* = \{s \in S : A \ni a <_s b\}.$$

So $s \in A^*$ precisely when some $a \in A$ is dominated by another element via s. In particular, if $A \subseteq B$ then $A^* \subseteq B^*$, a simple fact we often use below. In inverse semigroups, duals are just up-closures of inverses.

PROPOSITION 6.2. *If S is an inverse semigroup and $N = Z = \mathsf{E}(S)$ then*

$$A^* = A^{-1<}.$$

PROOF. If $s \in A^*$ then we have $a \in A$ with $a <_s b$ and hence $a <_{b^{-1}} b$, by (5-1), so

$$aa^{-1} = ab^{-1}ba^{-1} = ab^{-1}(ab^{-1})^{-1} = ab^{-1} = asbb^{-1} = bb^{-1}as = as.$$

Then $a^{-1} = a^{-1}aa^{-1} = a^{-1}as \in Ns$ so $a^{-1} < s$, showing that $A^* \subseteq A^{-1<}$.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

Conversely, if $s \in A^{-1<}$ then we have $a \in A$ with $a^{-1} < s$. Again (5-1) yields $a^{-1} <_{s^{-1}} s$. Taking inverses yields $a <_s s^{-1}$ so $s \in A^*$, showing that $A^{-1<} \subseteq A^*$. □

Again we want to show that duals in structured semigroups still behave much like up-closures of inverses. For example, from (Multiplicativity), we see that the * operation is a kind of antimorphism, that is, for all $A, B \subseteq S$.

$$A^*B^* \subseteq (BA)^*. \tag{Antimorphism}$$

Next we note some relations between the * and $^<$ operations.

PROPOSITION 6.3. *For any $A \subseteq S$,*

$$A^{<*} \subseteq A^{*<}, \tag{6-1}$$
$$A^{<<} \subseteq A^{**}. \tag{6-2}$$

Consequently, $A^* \neq \emptyset$ whenever $\emptyset \neq A \subseteq A^<$.

PROOF.

(6-1) If $c' \in A^{<*}$ then we have $a, b, b', c \in S$ with $A \ni a <_{b'} b <_{c'} c$. Then $a <_{b'bc'} c$ by (Transitivity) and (Z-Invariance), so $A^* \ni b'bc' <_c c' \in A^{*<}$ by (Switch).

(6-2) If $c \in A^{<<}$ then we have $a, b', b, c' \in S$ with $A \ni a <_{b'} b <_{c'} c$. Then again $A^* \ni b'bc' <_c c'$ and hence $c \in A^{**}$.

For the last statement, note that if $A^* = \emptyset$ and $A \subseteq A^<$ then (6-2) would yield $A \subseteq A^{<<} \subseteq A^{**} \subseteq \emptyset^* = \emptyset$. □

If A is closed under 'triple products', these inclusions become equalities.

PROPOSITION 6.4. *If $AA^*A \subseteq A$ then*

$$A^{<*} = A^{*<} \subseteq A^*. \tag{6-3}$$
$$A^{<<} = A^{**}. \tag{6-4}$$

PROOF.

(6-3) Take $a' \in A^{*<}$ so we have $a, b, b', c \in S$ with $A \ni c <_{b'} b$ and $b' <_a a'$. As $ab'ba' \in Z$, $ba'ab' = bb' \in Z$ and $bb'c = c$, (Z-Splitting) and (Switch) yield

$$ab'c <_{b'ba'} ab'b <_{a'} a.$$

Then (N-Invariance) yields $cb'ab'c < ab'b <_{a'} a$ and (Transitivity) yields $cb'ab'c <_{a'} a$. Thus $a' \in A^{<*} \cap A^*$, as

$$cb'ab'c \in AA^*A^{**}A^*A \subseteq A(AA^*A)^*A \subseteq AA^*A \subseteq A,$$

showing that $A^{*<} \subseteq A^{<*} \cap A^*$ and hence $A^{<*} = A^{*<} \subseteq A^*$, by (6-1).

(6-4) If we take $a \in A^{**}$ then we likewise have $a', b, b', c \in S$ with $A \ni c <_{b'} b$ and $b' <_a a'$, which again yields $cb'ab'c < ab'b < a$ and hence $a \in A^{<<}$. Combined with (6-2), this shows that $A^{<<} = A^{**}$. □

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

6.1. Diagonality. We can improve some results when N is 'diagonal'.

DEFINITION 6.5. We call $D \subseteq S$ *diagonal* if, for all $a, d, b \in S$,

$$ad, d, db \in D \quad \Rightarrow \quad adb \in D. \tag{Diagonal}$$

EXAMPLE 6.6. If S is a semigroup of slice-sections of a groupoid bundle $\pi : F \to G$ then the canonical diagonal $N = \{n \in S : \mathrm{dom}(n) \subseteq G^0\}$ is indeed diagonal in S. To see this just note that if $an, n, nb \in N$ then $anb \in N$ because

$$\mathrm{dom}(anb) = \mathrm{dom}(a)\mathrm{dom}(n)\mathrm{dom}(n)\mathrm{dom}(b) = \mathrm{dom}(an)\mathrm{dom}(nb) \subseteq G^0.$$

EXAMPLE 6.7. Any subsemigroup $I \subseteq S$ of idempotents (for example, all idempotents in an inverse semigroup S) is diagonal, as $ai, i, ib \in I$ implies $aib = aiib \in II \subseteq I$.

More generally, any regular subsemigroup $R \subseteq S$ (that is, for every $r \in R$ we have $r' \in R$ with $rr'r = r$) is diagonal, as $ar, r, rb \in R$ yields $r' \in R$ with

$$arb = arr'rb \in RRR \subseteq R.$$

EXAMPLE 6.8. The range of any 'conditional expectation' Φ satisfies (Diagonal): if $R = \mathrm{ran}(\Phi)$ for an idempotent map Φ on S such that, for all $a, b \in S$,

$$\Phi(\Phi(a)b) = \Phi(a)\Phi(b) = \Phi(a\Phi(b)),$$

then $ar, r, rb \in R$ implies $\Phi(arb) = ar\Phi(b) = a\Phi(rb) = arb$ so $arb \in R$.

Diagonality allows us to exchange the subscript and right argument of $<$ in a slightly more general situation than in (Switch).

PROPOSITION 6.9. *If N is diagonal then, for all $a, b, c, d, d' \in S$,*

$$ab, bc \in N \quad \textit{and} \quad b <_{d'} d \quad \Rightarrow \quad abc <_d d'. \tag{Exchange}$$

PROOF. If $b <_{d'} d$ and $ab, bc \in N$ then $abcdd' = add'bc = abc$ and (Diagonal) yields $abcd = add'bcd \in N$, as $ab = add'b, d'b, d'bcd \in N$, that is, $abc <_d d'$. □

This yields is a kind of transitivity, namely,

$$a <_{b'} b \quad \text{and} \quad b' <_c c' \quad \Rightarrow \quad a <_{c'} c. \tag{*-Transitivity}$$

Indeed, if $a <_{b'} b$ and $b' <_c c'$ then (Exchange) yields $a = ab'b <_{c'} c$.

Then (6-3) and (6-4) hold even without the triple product assumption.

PROPOSITION 6.10. *If N is diagonal then, for any $C \subseteq S$,*

$$C^{*<} \subseteq C^*, \quad C^{**} = C^{<<} \quad \textit{and} \quad C^{***} \subseteq C^*. \tag{6-5}$$

PROOF. If $a' \in C^{*<}$ then we have $a, b, b', c \in S$ with $C \ni c <_{b'} b$ and $b' <_a a'$. By (*-Transitivity), $c <_{a'} a$ and hence $a' \in C^*$, showing that $C^{*<} \subseteq C^*$.

If $a \in C^{**}$ then we have $a', b, b', c \in S$ with $C \ni c <_{b'} b$ and $b' <_a a'$. By (Exchange), $c <_{b'} bb'b <_{a'} a$ and hence $a \in C^{<<}$, showing $C^{**} \subseteq C^{<<}$. The reverse inclusion was already proved in (6-2).

Now it follows that $C^{***} \subseteq C^{*<<} \subseteq C^{*<} \subseteq C^*$. □

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

7. Atlases

DEFINITION 7.1. We call $A \subseteq S$ an *atlas* and $C \subseteq S$ a *coset* if

$$AA^*A \subseteq A \subseteq A^<, \tag{Atlas}$$

$$CC^*C \subseteq C = C^<. \tag{Coset}$$

So an atlas is a triple-product-closed round subset, and a coset is also an up-set. These generalise the same notions for inverse semigroups from [20, Section 1.4 before Proposition 26] (as the order there is reflexive, all subsets are trivially round).

REMARK 7.2. When S is a group with identity e and $N = Z = \{e\}$, a coset C of S is precisely a coset in the usual sense, specifically a left coset of the subgroup $C^{-1}C$ or a right coset of the subgroup CC^{-1}. In particular, each singleton $\{a\}$ is a coset of the trivial subgroup $\{e\}$.

More generally, if S is an inverse semigroup and $N = Z = \mathsf{E}(S)$ then every principal filter $a^\leq$ is a coset. If S is the normaliser semigroup of a Cartan subalgebra C of some C*-algebra and $N = Z = C$ then again we have principal filters $a^<$, for each $a \in S$, which are also cosets, by Proposition 12.3 below. We also have principal filters in the more general structured C*-algebras considered in [2], thanks to [2, Proposition 6.11] and [3, Lemmas 5.2 and 5.3].

However, general structured semigroups can have few cosets, for example, S itself is the only nonempty coset in Example 11.4 below. Indeed, an important consequence of one of our results is that we have faithful bundle representations precisely when there are enough cosets to distinguish the elements of S via their corresponding equivalence relations, at least when Z is symmetric; see Corollary 11.3 below.

First we show that atlases generate cosets.

PROPOSITION 7.3. *If A is an atlas, A^* and $A^< = A^{**}(\supseteq A)$ are cosets.*

PROOF. If A is an atlas, $A^*A^{**}A^* \subseteq (AA^*A)^* \subseteq A^*$ by (Antimorphism). Also $A^* \subseteq A^{<*} = A^{*<} \subseteq A^*$ by (6-3), that is, $A^* = A^{*<}$. This shows that A^* is a coset and hence $A^{**} = A^{<<} = A^<$ is also a coset, by (Transitivity), $A \subseteq A^<$ and (6-4). □

PROPOSITION 7.4. *If $A, B \subseteq S$ are atlases and $c \in S$ then*

$$A \subseteq B \quad \text{and} \quad A^* = B^* \quad \Rightarrow \quad (Ac)^* = (Bc)^*. \tag{7-1}$$

PROOF. Assume that $A, B \subseteq S$ are atlases with $A \subseteq B$ and $A^* = B^*$. If $d \in (Bc)^*$, then we have $b \in B$ and $d' \in S$ with $bc <_d d'$. As $A \subseteq A^<$ and $B \subseteq B^<$, we can take $n \in AA^* \cap N$, $z \in A^*A \cap N$ and $b' \in B^*$ with $b'b \in Z$. As Z is binormal, $nbzb' \in NZ \subseteq N$ and hence (N-Invariance) yields $nbzb'bc <_d d'$. Also

$$nbzb'b = nbb'bz \in AA^*BB^*BA^*A = AB^*BB^*BB^*A \subseteq AB^*A = AA^*A \subseteq A,$$

so $d \in (Ac)^*$, showing that $(Bc)^* \subseteq (Ac)^* \subseteq (Bc)^*$. □

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

DEFINITION 7.5. For any $A \subseteq S$ and $b \in S$ we define

$$\begin{aligned} A|b \quad &\Leftrightarrow \quad \text{there exists } b' \in S\ (bb', b'b \in Z \quad \text{and} \quad \text{there exists } a \in A\ (a = abb')), \\ b|A \quad &\Leftrightarrow \quad \text{there exists } b' \in S\ (bb', b'b \in Z \quad \text{and} \quad \text{there exists } a \in A\ (a = b'ba)). \end{aligned}$$

We read $A|b$ and $b|A$ as saying b *acts on* A (from the right and left, respectively). Indeed, when b acts on A, the product is another atlas; see Proposition 7.7 below.

PROPOSITION 7.6. *If $A, B \subseteq S$ are atlases satisfying $A^*A \subseteq (BB^*)^<$ then their product AB is also an atlas. Consequently, $(AB)^*$ is a coset and, for all $a \in A$,*

$$a|B \quad \textit{and} \quad (aB)^* = (AB)^*. \tag{7-2}$$

PROOF. If $A^*A \subseteq (BB^*)^<$ then (Multiplicativity) and the fact B is an atlas yields $A^*AB \subseteq (BB^*)^< B^< \subseteq (BB^*B)^< \subseteq B^<$ and hence

$$(AB)^*A \subseteq (AB)^*A^{**} \subseteq (A^*AB)^* \subseteq B^{<*} = B^{*<} \subseteq B^*.$$

Thus, $AB(AB)^*AB \subseteq ABB^*B \subseteq AB \subseteq A^<B^< \subseteq (AB)^<$ by (Multiplicativity), so AB is an atlas.

Take $a', c \in S$ with $A \ni c <_{a'} a$. Then $a'c \in A^*A \subseteq (BB^*)^<$ so, taking any $b \in B$, $a'cb \in (BB^*)^<B \subseteq B^<$. Taking $e, d \in S$ with $B \ni e <_d a'cb$, we see that $a'ae = a'aa'cbde = a'cbde = e$ so $a|B$. For any other $f \in A$, (N-Invariance) yields

$$(fB)^* \subseteq (aa'fB)^* \subseteq (aA^*AB)^* \subseteq (aB^<)^* = (aB)^*$$

by (7-1) (applied to S^{op} with a, $B^<$ and B in place of c, A and B, respectively). This shows that $(AB)^* \subseteq (aB)^* \subseteq (AB)^*$. □

Note that we can take $B = A^*$ above, as $A^*A \subseteq A^{<*}A^< \subseteq (A^*A)^<$, that is,

$$A \text{ is an atlas} \quad \Rightarrow \quad AA^* \text{ is an atlas.}$$

In this case, the cosets $(AA^*)^*$ and $(AA^*)^<$ they generate are the same as $(AA^*)^* \subseteq (A^{**}A^*)^* \subseteq (AA^*)^{**} = (AA^*)^<$ and $(AA^*)^< \subseteq (A^{**}A^*)^< \subseteq (AA^*)^{*<} = (AA^*)^*$. Likewise, $(A^*A)^*$ and $(A^*A)^<$ coincide, and we denote these cosets by

$$\mathsf{s}(A) = (A^*A)^* = (A^*A)^< \quad \text{and} \quad \mathsf{r}(A) = (AA^*)^* = (AA^*)^<.$$

We soon see that these are the source and range in a groupoid of cosets.

PROPOSITION 7.7. *If $A|b$ and A is an atlas then Ab is an atlas with*

$$\mathsf{r}(Ab) = \mathsf{r}(A).$$

PROOF. Take $b' \in S$ with $bb', b'b \in Z$ and $a \in A$ with $a = abb'$. As $A \subseteq A^<$, we can also take $a' \in A^*$ with $aa', a'a \in Z$.

First, note that

$$b'A^* \subseteq (Ab)^*. \tag{7-3}$$

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

To see this, take any $c' \in A^*$, so we have $c, d \in S$ with $A \ni d <_{c'} c$. Then (N-Invariance) yields $da'a <_{c'} c$ and (Z-Splitting) yields $da'ab <_{b'c'} cb$. Thus $b'c' \in (Ab)^*$, as $da'a \in AA^*A \subseteq A$, showing that $b'A^* \subseteq (Ab)^*$.

Next we claim that

$$b(Ab)^* \subseteq A^*.$$

To see this, take $c' \in (Ab)^*$, so we have $c \in S$ and $d \in A$ with $db <_{c'} c$. Take $d' \in A^*$ with $dd', d'd \in Z$ and let $n = da'ad' \in N$ so (N-Invariance) yields $ndb <_{c'} c$. Also $ndbb' = da'ad'dbb' = da'abb'd'd = da'ad'd = nd$ and hence $ndbb'b = ndb$ so (Z-Splitting) yields $nd = ndbb' <_{bc'} cb'$. Thus $bc' \in A^*$, as $nd \in AA^*AA^*A \subseteq A$, showing that $b(Ab)^* \subseteq A^*$ and hence

$$Ab(Ab)^*Ab \subseteq AA^*Ab \subseteq Ab. \tag{7-4}$$

Next, note that $a'aa' = a'abb'a' = bb'a'aa'$, so b and $a'aa' \in A^*AA^* \subseteq A^*$ witness $b'|A^*$. Thus, (7-3) applied in S^{op} yields $A^{**}b \subseteq (b'A^*)^*$. Then (6-4) and (7-4) yield

$$Ab \subseteq A^{<<}b = A^{**}b \subseteq (b'A^*)^* \subseteq (Ab)^{**} = (Ab)^{<<} \subseteq (Ab)^<,$$

showing that Ab is an atlas.

Note that $Ab(Ab)^* \subseteq AA^* \subseteq A^{**}(Abb')^* \subseteq (Abb'A^*)^* \subseteq (Ab(Ab)^*)^* = \mathsf{r}(Ab)$, so taking duals yields $\mathsf{r}(Ab) \subseteq \mathsf{r}(A) \subseteq \mathsf{r}(Ab)^* = \mathsf{r}(Ab)$. □

PROPOSITION 7.8. *If $A \subseteq S$ is an atlas and $n \in N$ then*

$$A|n \quad \Leftrightarrow \quad n \in \mathsf{s}(A) \quad \Rightarrow \quad (An)^< = A^<. \tag{7-5}$$

PROOF. If $n \in \mathsf{s}(A)$ then $An \subseteq A^<(A^*A)^< \subseteq (AA^*A)^< \subseteq A^<$ and hence

$$A^< \subseteq (An)^< \subseteq A^{<<} \subseteq A^<.$$

Also, we have $m, n' \in S$ with $A^*A \ni m <_n n'$, so $nn', n'n \in Z$ and $mnn' = m$. Taking any $a \in A$, note that $am \in AA^*A \subseteq A$ and $amnn' = am$, showing $A|n$.

Conversely, if $A|n$, we have $n' \in S$ with $nn', n'n \in Z$ and $a \in A$ with $a = ann'$. Then $a'a = a'ann'$, for any $a' \in A^*$ with $a'a \in N$, so $a'a <_n n'$ and $n \in \mathsf{s}(A)$. □

8. Cosets

Our goal here is to show that the nonempty cosets

$$C(S) = \{C \subseteq S : CC^*C \subseteq C = C^< \neq \emptyset\}$$

form an étale groupoid. This generalises similar results for filters in inverse semigroups in [23, 24] (we have more to say about filters in Section 12).

THEOREM 8.1. *$C(S)$ is a groupoid with inverse $C \mapsto C^*$ and product*

$$B \cdot C = (BC)^< \quad \textit{when } \mathsf{s}(B) = \mathsf{r}(C).$$

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

PROOF. If $(B, C) \in \mathcal{C}^2 = \{(B, C) : B, C \in \mathcal{C}(S) \text{ and } \mathsf{s}(B) = \mathsf{r}(C)\}$ then $(BC)^<$ is a coset, by Propositions 7.3 and 7.6. Also, as $B \neq \emptyset \neq C$, it follows that $\emptyset \neq BC \subseteq B^<C^< \subseteq (BC)^<$, so the product is well defined on $\mathcal{C}(S)$. Moreover, if $c \in C$, Proposition 7.6 yields $B|c$ and $(BC)^< = (Bc)^<$, so Proposition 7.7 yields

$$\mathsf{r}((BC)^<) = \mathsf{r}((Bc)^<) = \mathsf{r}(Bc) = \mathsf{r}(B).$$

Thus, $(A, B) \in \mathcal{C}^2$ if and only if $(A, (BC)^<) \in \mathcal{C}^2$, in which case, for any $c \in C$,

$$(A(BC)^<)^< = (Abc)^< \subseteq (ABC)^< \subseteq (A(BC)^<)^<.$$

Likewise, for any $(A, B) \in \mathcal{C}^2$, we see that $\mathsf{s}((AB)^<) = \mathsf{s}(B)$, so $((AB)^<, C) \in \mathcal{C}^2$ if and only if $(B, C) \in \mathcal{C}^2$, in which case $((AB)^<C)^< = (ABC)^< = (A(BC)^<)^<$, showing that the product is associative.

Again by Proposition 7.3, as well as the last part of Proposition 6.3, if $C \in \mathcal{C}(S)$ then $C^* \in \mathcal{C}(S)$, so the involution is well defined on $\mathcal{C}(S)$. Also, for any $(B, C) \in \mathcal{C}^2$, $c \in C$ and $c' \in C^*$ with $cc' \in N$,

$$B \subseteq (Bcc')^< = (BCC^*)^< \subseteq (B(B^*B)^<)^< \subseteq B,$$

showing that $C \cdot C^*$ is a unit. Likewise, $(B^*BC)^< = C$ so $B^* \cdot B$ is a unit, showing that the involution takes each element to its inverse. Thus, $\mathcal{C}(S)$ is a groupoid. □

The units of this groupoid have a couple of simple characterisations.

PROPOSITION 8.2. *$C \in \mathcal{C}(S)$ is a unit if and only if $C \cap N \neq \emptyset$ if and only if $C \cap Z \neq \emptyset$.*

PROOF. If $C \in \mathcal{C}(S)$ is a unit then $C = (CC^*)^< \supseteq CC^*$. Taking any $b, b', c \in S$ with $C \ni c <_{b'} b$, we see that $bb' \in CC^* \cap Z \subseteq C \cap Z$. Conversely, if $n \in C \cap N$ then, for any B with $(B, C) \in \mathcal{C}^2$, Proposition 7.6 and (7-5) yield $B = (Bn)^< = (BC)^<$. Likewise, $B = (CB)^<$ for any B with $(C, B) \in \mathcal{C}^2$, so C is a unit in $\mathcal{C}(S)$. □

The following slices of $\mathcal{C}(S)$ will play an important role very soon.

PROPOSITION 8.3. *For every $a \in S$, we have a slice in $\mathcal{C}(S)$ given by*

$$\mathcal{C}_a = \{C \in \mathcal{C}(S) : a \in C\}.$$

PROOF. If $B, C \in \mathcal{C}_a$ and $\mathsf{s}(B) = \mathsf{s}(C)$ then, by (7-2),

$$B = B \cdot \mathsf{s}(B) = (a\mathsf{s}(B))^< = (a\mathsf{s}(C))^< = C \cdot \mathsf{s}(C) = C.$$

Likewise, $\mathsf{r}(B) = \mathsf{r}(C)$ implies $B = C$, so $\mathcal{C}_a$ is a slice. □

The canonical topology on $\mathcal{C}(S)$ is generated by the slices $(\mathcal{C}_a)_{a\in S}$. So a basis for this topology is given by $\mathcal{C}_F = \bigcap_{f\in F} \mathcal{C}_f$, for finite $F \subseteq S$.

THEOREM 8.4. *The coset groupoid is étale in the canonical topology.*

PROOF. If $C^* \in \mathcal{C}_b$ then $b \in C^*$, so we have $a, b' \in S$ with $C \ni a <_b b'$ and hence $C \in \mathcal{C}_a$ and $\mathcal{C}_a^* \subseteq \mathcal{C}_b$, showing that the involution $C \mapsto C^*$ is continuous on $\mathcal{C}(S)$. Similarly,

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

if $B \cdot C \in \mathcal{C}_a$ then $a \in B \cdot C$, so we have $b \in B$ and $c \in C$ with $bc < a$, that is, $B \in \mathcal{C}_b$, $C \in \mathcal{C}_c$ and $\mathcal{C}_b \cdot \mathcal{C}_c \subseteq \mathcal{C}_a$, showing that the product is also continuous.

To see that the source s is an open map on $\mathcal{C}(S)$, take any finite $F \subseteq S$ and $C \in \mathcal{C}_F$. Further, take finite $G \subseteq C$ with $F \subseteq G^<$ and fix some $a, b', b \in S$ with $G \ni a <_{b'} b \in F$. We claim that

$$\mathsf{s}(C) \in \mathcal{C}_{b'G} \subseteq \mathsf{s}[\mathcal{C}_F].$$

To see this, first note that $b' \in G^* \subseteq C^*$ and $G \subseteq C$ imply $b'G \subseteq C^*C \subseteq \mathsf{s}(C)$ and hence $\mathsf{s}(C) \in \mathcal{C}_{b'G}$. Next note that if $B \in \mathcal{C}_{b'G}$ then $b'a \in B \cap N$ so B is a unit, by Proposition 8.2, and $b'bb'a = b'a$ so $b|B$ and $(bB)^< \in \mathcal{C}_{bb'G} \subseteq \mathcal{C}_F$ (as $A \in \mathcal{C}_{bb'G}$ implies $bb'G \subseteq A$ and hence $F \subseteq G^< \subseteq (bb'G)^< \subseteq A^< \subseteq A$). Thus $B = \mathsf{s}(B) = \mathsf{s}(bB) \in \mathsf{s}[\mathcal{C}_F]$, showing that $\mathcal{C}_{b'G} \subseteq \mathsf{s}[\mathcal{C}_F]$. As C was arbitrary, this shows $\mathsf{s}[\mathcal{C}_F]$ is open, which, as F was arbitrary, shows that s is an open map. □

8.1. Symmetry. The structured semigroups we are interested in often satisfy a certain additional assumption, which also allows us to say more about cosets.

DEFINITION 8.5. We call $Y \subseteq S$ *symmetric* if, for all $a, b \in S$,

$$ab \in Y \quad \Rightarrow \quad baba \in Y. \tag{Symmetric}$$

For example, the idempotents $\mathsf{E}(S)$ in any semigroup S are always symmetric, as $ab \in \mathsf{E}(S)$ implies $b(ababab)a = baba \in \mathsf{E}(S)$. Likewise, one can verify that the slice-sections $\mathcal{S}(\pi|_{F^\times})$ of the invertible part of a Fell bundle π have a symmetric central-diagonal $\mathcal{Z}(\pi|_{F^\times})$ (see Example 3.9).

Symmetry yields more one-sided versions of $<$ than in Proposition 5.4.

PROPOSITION 8.6. *If Z is symmetric and $a, b \in S$ then*

$$asb = a,\ as \in N \quad \text{and} \quad (bs \in Z \quad \text{or}\ sb \in Z) \ \Rightarrow\ a <_s bsb \quad \text{and} \quad a <_{sbs} b.$$
$$bsa = a,\ sa \in N \quad \text{and} \quad (bs \in Z \quad \text{or}\ sb \in Z) \ \Rightarrow\ a <_s bsb \quad \text{and} \quad a <_{sbs} b.$$

PROOF. If $bs \in Z$ then $bsbs \in ZZ \subseteq Z$ and $sbsb \in Z$, by symmetry. If $sb \in Z$ then, likewise, $sbsb \in ZZ \subseteq Z$ and $bsbs \in Z$, by symmetry. If $asb = a$ and $as \in N$ too then $asbs \in NZ \subseteq N$ and $asbsb = asb = a$, so $a <_s bsb$ and $a <_{sbs} b$. This proves the first $\Rightarrow$ and the second follows by a dual argument. □

PROPOSITION 8.7. *If Z is symmetric then, for all $a, b \in S$ and $n \in N$,*

$$a <_{b'} nb \quad \Rightarrow \quad a <_{b'nbb'n} b. \tag{8-1}$$

If N is diagonal and Z is symmetric then, for all $a, b \in S$ and $n \in N$,

$$a <_{b'n} b \quad \Rightarrow \quad a <_{b'} nbb'nb.$$

PROOF. If $a <_{b'} nb$ then $ab'n \in NN \subseteq N$, $ab'nb = a$ and $b'nb \in Z$ so $a <_{b'nbb'n} b$, by Proposition 8.6 with $s = b'n$.

Now suppose that N is also diagonal and $a <_{b'n} b$. In particular, $bb'n \in Z$ so, by symmetry, $nbb'nbb' \in Z \subseteq N$. As $ab'n, n \in N$ too, diagonality yields

$ab' = ab'nbb'nbb' \in N$. As $ab'nb = a$ and $b'nb \in Z$, it follows that $a <_{b'} nbb'nb$, again by Proposition 8.6 but with $s = b'$ and nb replacing b. □

PROPOSITION 8.8. *If Z is symmetric, $A \subseteq S$ is an atlas and $a, b', b \in S$, then*

$$abb' = a \in A \quad \text{and} \quad bb' \in Z \quad \Rightarrow \quad A|b. \tag{8-2}$$

PROOF. If $bb' \in Z$ then $b'bb'b \in Z$ by symmetry. Also $bb'bb' \in ZZ \subseteq Z$ and $abb'bb' = abb' = a$, showing that $b'bb'$ witnesses $A|b$. □

Let us now extend our notion of étale representation from Section 2.2.

DEFINITION 8.9. An *étale representation* of the structured semigroup (S, N, Z) on an étale groupoid G is a semigroup homomorphism $\theta : S \to \mathcal{B}(G)$ to the open slices of G such that $\theta[S]$ covers G, $\theta[N] \subseteq \mathcal{O}(G^0)$ and, for all $g \in G$ and $a \in S$,

$$g \in \theta(a) \quad \Rightarrow \quad \text{there exists } b < a \ (g \in \theta(b)). \tag{LocallyRound}$$

Intuitively, (Locally Round) is saying we can always shrink open neighbourhoods arising from the representation. On a more technical level, we must restrict the representations in this way for the coset representation to be universal, as shown below. The following observation shows that we can at least rest assured that requiring (Locally Round) and $\theta[N] \subseteq \mathcal{O}(G^0)$ is still consistent with the original notion of an étale representation of an inverse semigroup introduced in Section 2.2.

PROPOSITION 8.10. *If S is an inverse semigroup and $N = Z = \mathsf{E}(S)$ then any semigroup homomorphism $\theta : S \to \mathcal{B}(G)$ is an étale representation.*

PROOF. As θ is semigroup homomorphism, it maps idempotents in S to idempotents in $\mathcal{B}(G)$, which are precisely the open subsets of the unit space G^0, that is, $\theta[\mathsf{E}(S)] \subseteq \mathcal{O}(G^0)$. Moreover, the domination relation $<$ here is just the usual order on the inverse semigroup S; see (5-1). In particular, $<$ is reflexive so (Locally Round) is immediately verified by taking $b = a$. □

As before in Section 2.2, we call an étale representation $\mu : S \to \mathcal{B}(G)$ *universal* if, for every étale representation $\theta : S \to \mathcal{B}(H)$, there exists a unique étale morphism $\phi : H \to G$ such that $\theta = \overline{\phi} \circ \mu$, where $\overline{\phi} : \mathcal{B}(G) \to \mathcal{B}(H)$ is the preimage map. Again as before, we let C denote the map $a \mapsto C_a$.

THEOREM 8.11. *If Z is symmetric, C is a universal étale representation.*

PROOF. First, we must show that C is a semigroup homomorphism. For $a, b \in S$, certainly $C_a \cdot C_b \subseteq C_{ab}$. Next, for $n \in N$, we have a partial converse, namely,

$$C_{an} \subseteq C_a. \tag{8-3}$$

Indeed, if $C \in C_{an}$ then we have $c \in C$ with $c < an$. This implies $c < a$, by (8-1), and hence $a \in C^< \subseteq C$, that is, $C \in C_a$, which proves (8-3).

More generally, if $C \in C_{ab}$ then we have $c, c' \in S$ with $C \ni c <_{c'} ab$. In particular, $abc' \in Z$ and $abc'c = c$ and hence $bc'|C$, by (8-2). By Proposition 7.7, $B = (bc'C)^<$ is

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

a coset with $\mathsf{s}(B) = \mathsf{s}(C)$. Also $B \in \mathcal{C}_{bc'c} \subseteq \mathcal{C}_b$, by (8-3), as $c'c \in N$. As $b \in B$, we have $b' \in B^*$ with $bb' \in Z \subseteq N$ and hence $C \cdot B^* \in \mathcal{C}_{ab} \cdot \mathcal{C}_{b'} \subseteq \mathcal{C}_{abb'} \subseteq \mathcal{C}_a$, again by (8-3). Thus, $C = C \cdot B^* \cdot B \in \mathcal{C}_a \cdot \mathcal{C}_b$. This shows that $\mathcal{C}_{ab} = \mathcal{C}_a \cdot \mathcal{C}_b$, for all $a, b \in S$, so $\mathcal{C}$ is indeed a semigroup homomorphism.

To see that $\mathcal{C}$ is even an étale representation, note first that $n \in N$ implies that $\mathcal{C}_n$ consists entirely of unit cosets, thanks to Proposition 8.2. Also any coset C must satisfy $C \subseteq C^<$ by definition, so if $C \in \mathcal{C}_a$ then we have $b \in C$ with $b < a$ and hence $C \in \mathcal{C}_b$, that is, (Locally Round) is also satisfied. As each $C \in \mathcal{C}(S)$ is nonempty, we have $a \in C$ and hence $C \in \mathcal{C}_a$, that is, $(\mathcal{C}_a)_{a \in S}$ covers $\mathcal{C}(S)$, as required.

To see that $\mathcal{C}$ is universal, take any other étale representation $\theta : S \to \mathcal{B}(G)$. For each $g \in G$, define $\phi(g) \subseteq S$ by

$$\phi(g) = \{a \in S : g \in \theta(a)\}.$$

We claim that $\phi(g) \in \mathcal{C}(S)$. Certainly $\phi(g) \neq \emptyset$, as $\theta[S]$ covers G. Also $\phi(g) \subseteq \phi(g)^<$, as θ is required to satisfy (Locally Round). Conversely, suppose that $\phi(g) \ni a < b$ so $g \in \theta(a) = \theta(asb) = \theta(a)\theta(s)\theta(b)$ and $\theta(a)\theta(s) = \theta(as) \in \theta[N] \subseteq \mathcal{O}(G^0)$. Arguing as in the proof of Proposition 5.2 (or considering the special case of Proposition 5.2 where $\pi : G \to G$ is the identity bundle and hence $\mathcal{B}(G) \approx \mathcal{S}(\pi)$), it follows that $g \in \theta(b)$ and hence $b \in \phi(g)$, showing that $\phi(g)^< \subseteq \phi(g)$. Finally, note that if $b \in \phi(g)^*$ then $a <_b c$, for some $a \in \phi(g)$, which means that $g \in \theta(a)$ and hence $g^{-1} \in \theta(b)$, again by Proposition 5.2. This shows that $\phi(g)^* \subseteq \phi(g^{-1})$ so, for any $a, c \in \phi(g)$ and $b \in \phi(g)^*$, we see that $g = gg^{-1}g \in \theta(a)\theta(b)\theta(c) = \theta(abc)$, that is, $abc \in \phi(g)$. Thus, we have shown that $\phi(g)$ is a nonempty coset, proving the claim.

Now $\phi(g)^* \subseteq \phi(g^{-1}) = \phi(g^{-1})^{**} \subseteq \phi(g)^*$, that is, $\phi(g)^* = \phi(g^{-1})$. Also

$$\mathsf{r}(\phi(g)) = \phi(g) \cdot \phi(g)^* = (\phi(g)\phi(g)^*)^< = (\phi(g)\phi(g^{-1}))^< \subseteq \phi(gg^{-1})^< = \phi(\mathsf{r}(g)).$$

As $\phi(\mathsf{r}(g))$ contains a unit coset and hence an element of N, $\phi(\mathsf{r}(g))$ is also a unit coset. Taking any $a \in \phi(g)$ and $a' \in \phi(g^{-1})$, we see that $\mathsf{r}(g) = gg^{-1} \in \theta(a)\theta(a') = \theta(aa')$. Thus, $aa' \in \phi(\mathsf{r}(g))$ and then (6-4) and (7-2) yield

$$\phi(\mathsf{r}(g)) = (\phi(\mathsf{r}(g))aa')^< \subseteq (\phi(\mathsf{r}(g))\phi(g)\phi(g^{-1}))^< \subseteq (\phi(\mathsf{r}(g)g)\phi(g^{-1}))^< = \mathsf{r}(\phi(g)).$$

Likewise, $\mathsf{s}(\phi(g)) = \phi(\mathsf{s}(g))$. If we have another $h \in G$ with $\mathsf{s}(g) = \mathsf{r}(h)$ then

$$\mathsf{s}(\phi(g)) = \phi(\mathsf{s}(g)) = \phi(\mathsf{r}(h)) = \mathsf{r}(\phi(h)),$$

so $\phi(g) \cdot \phi(h)$ is defined and

$$\phi(g) \cdot \phi(h) = (\phi(g)\phi(h))^< \subseteq \phi(gh)^< = \phi(gh).$$

Likewise, $\phi(gh) \cdot \phi(h^{-1}) \subseteq \phi(ghh^{-1}) = \phi(g)$ and hence

$$\phi(gh) = \phi(gh) \cdot \phi(h^{-1}) \cdot \phi(h) \subseteq \phi(g) \cdot \phi(h).$$

This shows that $\phi(gh) = \phi(g) \cdot \phi(h)$ and hence ϕ is a functor.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

Now suppose that $x \in G^0$, $C \in \mathcal{C}(S)$ and $\mathsf{r}(C) = \phi(x)$. Taking any $c \in C$ and $c' \in C^*$, we see that $cc' \in CC^* \subseteq \mathsf{r}(C) = \phi(x)$, so $x \in \theta(cc') = \theta(c)\theta(c')$ and hence $x = \mathsf{r}(g)$, for some $g \in \theta(c) \cap \theta(c')^{-1}$. Then (6-4) and (7-2) again yield

$$C = \mathsf{r}(C) \cdot C = (\mathsf{r}(C)c)^{<} = (\phi(x)c)^{<} = (\phi(x)\phi(g))^{<} = \phi(g).$$

This shows that ϕ is star-surjective. To see that ϕ is also star-injective, just note that $\mathsf{r}(g) = \mathsf{r}(h)$ and $\phi(g) = \phi(h)$, or even just $\phi(g) \cap \phi(h) \neq \emptyset$, implies $g = h$, as $a \in \phi(g) \cap \phi(h)$ implies that g and h are in the same slice $\theta(a)$.

Now note that, for any $a \in S$, $\phi^{-1}[C_a] = \theta(a) \in \mathcal{B}(G)$ because

$$\phi(g) \in C_a \quad \Leftrightarrow \quad a \in \phi(g) \quad \Leftrightarrow \quad g \in \theta(a).$$

Thus, ϕ is an étale morphism from G to $\mathcal{C}(S)$ with $\theta = \overline{\phi} \circ C$.

Finally, for uniqueness, suppose that we had another map $\psi : G \to \mathcal{C}(S)$ such that $\overline{\psi} \circ C = \theta$, that is, $\psi^{-1}[C_a] = \theta(a)$, for all $a \in S$. Then again we see that $g \in \theta(a)$ if and only if $\psi(g) \in C_a$ if and only if $a \in \psi(g)$ and hence $\psi(g) = \{a \in S : g \in \theta(a)\} = \phi(g)$, for all $g \in G$. □

REMARK 8.12. If the subbasic slices of a groupoid are closed under pointwise products then the same is true of arbitrary open slices; see the proof of [6, Proposition 3.18]. In particular, this applies to $\mathcal{C}(S)$ if C is a semigroup homomorphism, as shown in the first part of the proof above. As $C \mapsto C^*$ is immediately seen to be continuous on $\mathcal{C}(S)$, as mentioned at the start of the proof of Theorem 8.4, the open slices of $\mathcal{C}(S)$ are also closed under pointwise inverses. It follows that $\mathcal{C}(S)$ is étale, again by [6, Proposition 3.18], which provides an alternative proof of Theorem 8.4 in the symmetric case.

While the coset representation C is universal, it can only be faithful when N consists entirely of idempotents. If we want faithful representations of more general structured semigroups, including those with few idempotents, then we need to consider finer representations on groupoid bundles. In particular, to construct such a bundle over the coset groupoid, we split up the cosets into equivalence classes with respect to certain relations that we now proceed to examine.

9. Equivalence

DEFINITION 9.1. For any $A \subseteq S$, we define a relation $\sim_A$ on $A^{<}$ by

$$a \sim_A b \quad \Leftrightarrow \quad \text{there exists } s, t \in A^* \ (sat = sbt).$$

Here A will usually be an atlas, in which case $A^{<}$ is a coset.

We think of $a \sim_A b$ as saying that a and b have the same 'germ' at A.

EXAMPLE 9.2. Suppose that $S = N = Z = C_0(X)$ are continuous functions from X to $\mathbb{C}$ vanishing at infinity, where X is a locally compact Hausdorff space. Then each $x \in X$

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

defines a coset (in fact an ultrafilter)

$$S_x = \{a \in S : a(x) \neq 0\}$$

(note that this is consistent with (3-2) when we identify each $a \in S$ with the restriction to its support $a|_{\mathrm{supp}(a)}$). Moreover, for all $a, b \in S_x$,

$$a \sim_{S_x} b \quad \Leftrightarrow \quad a(y) = b(y) \quad \text{for all } y \text{ in some neighbourhood } N \text{ of } x.$$

Indeed, if $a \sim_{S_x} b$, that is, $sat = sbt$, for some $s, t \in S_x^* = S_x$, then $a(x) = b(x)$, for all $x \in \mathrm{supp}(s) \cap \mathrm{supp}(t)$. Conversely, if $a(y) = b(y)$ for all y in a neighbourhood N of x then, taking any $s \in S_x$ with $\mathrm{supp}(s) \subseteq N$, we see that $sas = sbs$ so $a \sim_{S_x} b$.

Another example to think of would be inverse semigroups, where $a \sim_A b$ is just saying that a and b have a common lower bound in A. We prove something similar for more general semigroups in (9-4) below by considering the subsets

$$A^Z = \{z \in Z : \text{ there exists } a \in A\ (az = a)\},$$
$$^ZA = \{z \in Z : \text{ there exists } a \in A\ (za = a))\}.$$

First, however, we need to examine their basic properties.

PROPOSITION 9.3. *If A is an atlas then A^Z is a subsemigroup of Z and*

$$A^Z = A^{<Z} = {}^Z(A^*) = (A^Z)^Z = \mathsf{s}(A)^Z \subseteq \mathsf{s}(A), \tag{9-1}$$
$$A \ni a'' <_{a'} a \quad \Rightarrow \quad a'^ZAa \subseteq A^Z, \tag{9-2}$$
$$a \in A \cap S^> \quad \Rightarrow \quad {}^ZAa = aA^Z. \tag{9-3}$$

PROOF. If $y, z \in A^Z$ then we have $a, b \in A$ with $ay = a$ and $bz = b$. As A is an atlas, $A \subseteq A^<$, so we have $a', a'' \in S$ with $A \ni a'' <_{a'} a$. Then $ba'a \in AA^*A \subseteq A$ and, moreover, $ba'ayz = ba'az = bza'a = ba'a$. Thus, $yz \in A^Z$, which shows that A^Z is a subsemigroup of Z. A dual argument works for ^ZA.

(9-1) If $z \in A^{*Z}$ then we have $a, a', a'' \in S$ with $A \ni a'' <_{a'} a$ and $a'z = a'$ and hence $za'' = za''a'a = a''a'za = a''a'a = a''$, showing that $z \in {}^ZA$. Conversely, if $z \in {}^ZA$, then we have $a \in A$ with $za = a$. As $A \subseteq A^<$, we have $a', a'' \in S$ with $A \ni a'' <_{a'} a$ and hence $a'aa'z = a'zaa' = a'aa' \in A^*AA^* \subseteq A^*$, showing that $z \in A^{*Z}$. Thus, $^ZA = A^{*Z}$ and, likewise, $^Z(A^*) = A^Z$. Then (6-4) yields $A^{<Z} = A^{**Z} = {}^Z(A^*) = A^Z$.

Now if $z \in A^Z$ then we have $a \in A$ with $az = a$. Taking $a', a'' \in A$ with $A \ni a'' <_{a'} a$, we see that $A^Z \ni a'a = a'az$, so $z \in (A^Z)^Z$, and $a'a <_z z$, so $z \in (A^*A)^< = \mathsf{s}(A)$. This shows that $A^Z \subseteq \mathsf{s}(A)$ and $A^Z \subseteq (A^Z)^Z \subseteq \mathsf{s}(A)^Z$. Conversely, if $z \in \mathsf{s}(A)^Z$ then we have $b \in \mathsf{s}(A)$ with $bz = b$. Taking $a \in A$, note that $abz = ab \in A\mathsf{s}(A) \subseteq A^<$ so $z \in A^{<Z} = A^Z$, showing that $\mathsf{s}(A)^Z \subseteq A^Z$.

(9-2) If $A \ni a'' <_{a'} a$ and $zb = b \in A$ then we have $b', b'' \in S$ with $A \ni b'' <_{b'} b$ and hence $bb'a'' \in AA^*A \subseteq A$ and $bb'a''a'za = zbb'a''a'a = bb'a''$, that is, $bb'a''$ witnesses $a'za \in A^Z$.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

(9-3) If $A \ni a <_{a'} a''$ and $z \in {}^{Z}\!A$ then (9-2) yields $a'za'' \in A^{Z}$ and hence $za = zaa'a'' = aa'za'' \in aA^{Z}$. $\square$

Dually to (9-2), $A \ni a'' <_{a'} a$ also implies $aA^{Z}a' \subseteq {}^{Z}\!A$.

Proposition 9.4. *If $A \subseteq S$ is an atlas and $a, b \in A^{<}$ then*

$$a \sim_A b \quad \Leftrightarrow \quad {}^{Z}\!AaA^{Z} \cap {}^{Z}\!AbA^{Z} \neq \emptyset \tag{9-4}$$

$$\Leftrightarrow \quad \textit{there exists } s \in A^{*}\ (sa = sb \in A^{Z} \quad \textit{and} \quad as = bs \in {}^{Z}\!A). \tag{9-5}$$

Proof. Take $a, b \in A^{<}$. If $s \in A^{*}$ and $sa = sb$ or $as = bs$ then $sas = sbs$ so $a \sim_A b$. This proves the last $\Leftarrow$.

If $a \sim_A b$ then we have $s, t \in A^{*}$ with $sat = sbt$. Taking any $s', s'', t', t'' \in S$ with $A^{*} \ni s'' <_{s'} s$ and $A^{*} \ni t'' <_{t'} t$, we see that ${}^{Z}\!AaA^{Z} \ni s'satt' = s'sbtt' \in {}^{Z}\!AbA^{Z}$. This proves the first $\Rightarrow$.

Conversely, assume that ${}^{Z}\!AaA^{Z} \cap {}^{Z}\!AbA^{Z} \neq \emptyset$. First, we claim that $aA^{Z} \cap bA^{Z} \neq \emptyset$. To see this, take $w, x \in {}^{Z}\!A$ and $y, z \in A^{Z}$ with $way = xbz$. As $a \in A^{<}$, we have $a', a'' \in S$ with $A \ni a'' <_{a'} a$. Note that $waya'a = waa'ay = aa'way \in aa'{}^{Z}\!AaA^{Z} \subseteq aA^{Z}$ by Proposition 9.3. Likewise, as $b \in A^{<}$, we have $b', b'' \in S$ with $A \ni b'' <_{b'} b$ and then $xbzb'b \in bA^{Z}$. As $way = xbz$,

$$aA^{Z} \supseteq aA^{Z}b'b \ni waya'ab'b = xbzb'ba'a \in bA^{Z}a'a \subseteq bA^{Z}.$$

This proves the claim.

Next, we claim that we have $t \in A^{*}$ with $at = bt \in {}^{Z}\!A$ and $ta \in A^{Z}$. To see this, take $p, q \in A^{Z}$ with $ap = bq$. As $p \in A^{Z}$, we have $c \in A \subseteq A^{<}$ with $cp = c$ and $c', c'' \in S$ with $A \ni c'' <_{c'} c$. Likewise, we have $d, d', d'' \in S$ with $A \ni d'' <_{d'} d$ and $dq = d$. Let $z = c'cd'd = d'dc'c \in A^{Z} \cap A^{*}A$ by Proposition 9.3. Now just note that $az = ac'cd'd = ac'cpd'd = apc'cd'd = apz$ and, likewise,

$$bz = bd'dc'c = bd'dqc'c = bqd'dc'c = bqz = apz = az.$$

Taking the a' above, note that $t = za' \in A^{*}AA^{*} \subseteq A^{*}$, $ta = za'a \in A^{Z}A^{Z} \subseteq A^{Z}$ and $bza' = aza' \in aA^{Z}a' \subseteq {}^{Z}\!A$ by Proposition 9.3, proving the claim.

Likewise, we have $u \in A^{*}$ with $ua = ub \in A^{Z}$ and $au \in {}^{Z}\!A$. Again Proposition 9.3 yields $taub = taua \in A^{Z}A^{Z} \subseteq A^{Z}$ and $btau = atau \in A^{Z}A^{Z} \subseteq A^{Z}$, that is,

$$sa = sb \in A^{Z} \quad \text{and} \quad as = bs \in {}^{Z}\!A,$$

for $s = tau$. This completes the cycle of equivalences. $\square$

One could also write down other characterisations of $\sim_A$ that lie somewhere in between (9-4) and (9-5). One could also replace A^{*} with some coinitial subset A'. That is to say, if A is an atlas and $A' \subseteq A^{*} = (A')^{<}$ then, for example,

$$a \sim_A b \quad \Leftrightarrow \quad \text{there exists } s \in A'\ (as = bs). \tag{9-6}$$

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

Indeed, if we have $s \in A' \subseteq A^*$ with $as = bs$ then $aA^Z \cap bA^Z \neq \emptyset$ and so certainly ${}^ZAaA^Z \cap {}^ZAbA^Z \neq \emptyset$ and hence $a \sim_A b$, by (9-4). Conversely, if $a \sim_A b$ then we have $s \in A^*$ with $as = bs$ by (9-5). If $A^* = (A')^<$ too then we have $a', t \in S$ with $A' \ni a' <_t s$ and hence $aa' = asta' = bsta' = ba'$, proving (9-6).

We can now finally show that $\sim_A$ is an equivalence relation.

COROLLARY 9.5. *If $A \subseteq S$ is an atlas then $\sim_A$ is an equivalence relation.*

PROOF. We immediately see that $\sim_A$ is a reflexive and symmetric relation on $A^<$. On the other hand, if $a \sim_A b \sim_A c$ then we have $s, t \in A^*$ with $sa = sb$ and $bt = ct$ by (9-5). Thus, $sat = sbt = sct$, showing that $a \sim_A c$. So $\sim_A$ is also transitive and hence an equivalence relation on $A^<$. □

Also, multiples of ${}^ZA$ and A^Z do not change the equivalence class.

COROLLARY 9.6. *For any atlas $A \subseteq S$, $y \in {}^ZA$, $z \in A^Z$ and $a \in A^<$,*

$$ya \sim_A a \sim_A az. \tag{9-7}$$

PROOF. As $y^2 \in {}^ZA$, $z^2 \in A^Z$ and $y(ya)z^2 = y^2az^2 = y^2(az)z$, (9-4) yields (9-7). □

Moreover, these equivalence relations even respect products.

COROLLARY 9.7. *If $A, B \subseteq S$ are atlases with $\mathsf{s}(A) = \mathsf{r}(B)$ then*

$$a \sim_A c \quad \text{and} \quad b \sim_B d \quad \Rightarrow \quad ab \sim_{AB} cd. \tag{9-8}$$

PROOF. If $a \sim_A c$ and $b \sim_B d$ then (9-5) yields $s \in A^*$ and $t \in B^*$ with $sa = sc$ and $bt = dt$. Then $tsabts = tscdts$ so $ab \sim_{AB} cd$, because $ts \in B^*A^* \subseteq (AB)^*$ (as $\mathsf{s}(A) = \mathsf{r}(B)$, AB is an atlas; see Proposition 7.6). □

10. Bundles

In the present section we show that the equivalence classes of the previous section form an étale bundle over the coset groupoid. Firstly, if $A \subseteq S$ is an atlas and $a \in A^<$, we define $[a, A]$ to be the pair consisting of the $\sim_A$-equivalence class of a together with the coset $A^<$ generated by A, that is,

$$[a, A] = (a^{\sim_A}, A^<).$$

We denote the set of all such pairs by

$$\widetilde{\mathcal{C}}(S) = \{[c, C] : c \in C \in \mathcal{C}(S)\}.$$

PROPOSITION 10.1. *$\widetilde{\mathcal{C}}(S)$ is a groupoid under the product*

$$[a, A][b, B] = [ab, AB] \quad \textit{when } \mathsf{s}(A) = \mathsf{r}(B),$$

where $[c, C]^{-1} = [c', C^*]$, for any c' with $C \ni d <_{c'} c$.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

PROOF. By (9-8), the product is well defined. Also, as S is a semigroup and $C(S)$ is a groupoid, $\widetilde{C}(S)$ can be viewed as a semicategory with unit cosets as objects. To show that $\widetilde{C}(S)$ is a category we must find a unit $[z, U] \in \widetilde{C}(S)$ for every unit $U \in C(S)$.

We claim that $[z, U] \in \widetilde{C}(S)$ is a unit, for any $z \in U^Z = U^{*Z} = {}^Z U$, by (9-1). Indeed, if $C \in C(S)$ and $\mathsf{s}(C) = U$ then $C^Z = U^Z$ by (9-1). For any $c \in C$, this means that $cz \sim_C c$ so $[c, C][z, U] = [cz, CU] = [c, C]$. Likewise, if $\mathsf{r}(C) = U$ then ${}^Z C = {}^Z U$ so $[z, U][c, C] = [zc, UC] = [c, C]$, proving the claim. Thus, $\widetilde{C}(S)$ is a category.

Next note that, for any $c \in C$, we have $c', d \in S$ with $C \ni d <_{c'} c$ and then $[c', C^*][c, C] = [c'c, \mathsf{s}(C)]$ and $[c, C][c', C^*] = [cc', \mathsf{r}(C)]$. As $c'c \in C^Z = \mathsf{s}(C)^Z$ and $cc' \in {}^Z C = {}^Z\mathsf{r}(C)$, these are both units in $\widetilde{C}(S)$ by what we just proved and hence

$$[c', C^*] = [c, C]^{-1}. \tag{10-1}$$

This shows that every $[s, C] \in \widetilde{C}(S)$ is invertible so $\widetilde{C}(S)$ is a groupoid. □

Let $\rho_C : \widetilde{C}(S) \to C(S)$ denote the canonical projection onto $C(S)$, that is,

$$\rho_C([a, A]) = A^<.$$

The canonical topology on $\widetilde{C}(S)$ is generated by the sets

$$\widetilde{C}^s = \{[s, C] : s \in C \in C(S)\},$$
$$\widetilde{C}_s = \{[c, C] : c, s \in C \in C(S)\} = \rho_C^{-1}[C_s].$$

THEOREM 10.2. *$\rho_C : \widetilde{C}(S) \to C(S)$ is an étale bundle.*

PROOF. To see that the inverse map on $\widetilde{C}(S)$ is continuous, suppose that $[a, A]^{-1} \in \widetilde{C}^{b'}$. This means that $[a, A]^{-1} = [b', A^*]$ so, in particular, $b' \in A^*$. Taking $b, c \in S$ with $A \ni c <_{b'} b$, (10-1) yields $[a, A] = [b', A^*]^{-1} = [b, A]$, so $[a, A] \in \widetilde{C}^b \cap \widetilde{C}_c$. Moreover, for any other $[b, B] \in \widetilde{C}^b \cap \widetilde{C}_c$, we see that $B \ni c <_{b'} b$ and hence $[b, B]^{-1} = [b', B^*] \in \widetilde{C}^{b'}$, by (10-1). Likewise, as in the proof of Theorem 8.4, if $[a, A]^{-1} \in \widetilde{C}_{b'}$ then again $b' \in A^*$ so, taking $b, c \in S$ with $A \ni c <_{b'} b$, we see that $[a, A] \in \widetilde{C}_c$ and $(\widetilde{C}_c)^{-1} \subseteq \widetilde{C}_{b'}$. Thus, the inverse map is continuous.

For the product, suppose that $[a, A][b, B] \in \widetilde{C}^s$. As $[a, A][b, B] = [ab, AB]$, this means that $ab \sim_{AB} s$, so (9-6) yields $a' \in A^*$ and $b' \in B^*$ with $abb'a' = sb'a'$. Taking $a'', b'', c, d \in S$ with $A \ni c <_{a'} a''$ and $B \ni d <_{b'} b''$, we see that $[a, A] \in \widetilde{C}^a \cap \widetilde{C}_c$ and $[b, B] \in \widetilde{C}^b \cap \widetilde{C}_d$. Also, for any other $[a, C] \in \widetilde{C}^a \cap \widetilde{C}_c$ and $[b, D] \in \widetilde{C}^b \cap \widetilde{C}_d$ such that $[a, C][b, D]$ is defined, (9-6) again yields $ab \sim_{CD} s$, as $c \in C$ and $d \in D$, so $a' \in C^*$ and $b' \in D^*$, and hence $[a, C][b, D] = [ab, CD] \in \widetilde{C}^s$. Similarly, if $[a, A][b, B] \in \widetilde{C}_s$, then we have $c \in A$ and $d \in B$ with $cd < s$. Then $[a, A] \in \widetilde{C}_c$ and $[b, B] \in \widetilde{C}_d$. Moreover, for any other $[e, C] \in \widetilde{C}_c$ and $[f, D] \in \widetilde{C}_d$ such that $[e, C][f, D]$ is defined, $[e, C][f, D] = [ef, CD] \in \widetilde{C}_{cd} \subseteq \widetilde{C}_s$. This shows that the product is continuous.

So $\widetilde{C}(S)$ is a topological groupoid, and we already know that $C(S)$ is an étale groupoid, by Theorem 8.4. It only remains to prove that ρ_C is a locally injective continuous open isocofibration. Local injectivity is immediate from the fact that ρ_C is injective on $\widetilde{C}^s$, for all $s \in S$. Continuity is immediate from the fact that $\widetilde{C}_s = \rho_C^{-1}[C_s]$

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

is open, for all $s \in S$. The definition of the product on $C(S)$ also immediately shows that ρ_C is an isocofibration.

To see that ρ_C is also open, suppose that F and G and finite subsets of S and

$$[a, A] \in \widetilde{C}^F_G = \bigcap_{f \in F} \widetilde{C}^f \cap \bigcap_{g \in G} \widetilde{C}_g.$$

For all $f \in F$, this means that $a \sim_A f$, so we have $a'_f \in A^*$ with $aa'_f = fa'_f$. Then we can take $a_f, b_f \in S$ with $A \ni b_f <_{a'_f} a_f$. Note that

$$H = \{a\} \cup \{b_f : f \in F\} \cup F \cup G \subseteq A$$

and hence $\rho_C([a, A]) = A \in C_H$. Moreover, for all $B \in C_H$ and $f \in F$, note that $b_f \in B$, so $a'_f \in B^*$ and hence $a \sim_B f$. Thus, $[a, B] \in \widetilde{C}^F_G$, so $B = \rho_C([a, B]) \in \rho_C[\widetilde{C}^F_G]$, showing that $C_H \subseteq \rho_C[\widetilde{C}^F_G]$. This shows that ρ_C is also an open map. □

We refer to $\rho_C : \widetilde{C}(S) \to C(S)$ as the *coset bundle*.

11. Representations

Recall the notion of an étale representation from Definition 8.9.

DEFINITION 11.1. A *bundle representation* of the structured semigroup (S, N, Z) on a groupoid bundle $\pi : F \to G$ is a semigroup homomorphism $\theta : S \to \mathcal{S}(\pi)$ such that $a \mapsto \mathrm{dom}(\theta(a))$ is an étale representation.

If $\theta : S \to \mathcal{S}(\pi)$ is a semigroup homomorphism then so is $a \mapsto \mathrm{dom}(\theta(a))$. To verify that it is an étale representation, it thus suffices to check that these domains cover G, (Locally Round) holds and N gets mapped to $\mathcal{O}(G^0)$, that is, $\theta[N] \subseteq \mathcal{N}(\pi)$. Note that (Locally Round) here ensures that $\theta[S]$ is a local-inverse semigroup, as per Definition 3.8; see Proposition 5.2.

Again, we call a bundle representation $\mu : S \to \mathcal{S}(\pi)$ *universal* if, for every bundle representation $\theta : S \to \mathcal{S}(\pi')$, there exists a unique Pierce morphism (ϕ, τ) from π to π' such that $\theta = (\tau/\phi) \circ \mu$ for $\tau/\phi : \mathcal{S}(\pi) \to \mathcal{S}(\pi')$ defined as in Theorem 4.10.

THEOREM 11.2. *If Z is symmetric then $a \mapsto \widetilde{a}$ is a universal bundle representation where $\widetilde{a} \in \mathcal{S}(\rho_C)$ is such that, for all $C \in \mathrm{dom}(\widetilde{a}) = C_a$,*

$$\widetilde{a}(C) = [a, C].$$

PROOF. For any $a \in S$, $\mathrm{dom}(\widetilde{a}) = C_a$ is a slice by Proposition 8.3. To see that $\widetilde{a}$ is continuous, first note that $\widetilde{a}^{-1}[\widetilde{C}_s] = C_a \cap C_s$. On the other hand, if $\widetilde{a}(C) = [a, C] \in \widetilde{C}^s$ then $az = sz$, for some $z \in C^Z$. Take any $b \in C$ with $bz = b$. If $B \in C_a \cap C_b$ then $z \in B^Z$ and hence $\widetilde{a}(B) = [a, B] = [s, B] \in \widetilde{C}^s$. Thus, $C \in C_a \cap C_b \subseteq \widetilde{a}^{-1}[\widetilde{C}^s]$. We have shown that preimages of subbasic sets are open and hence $\widetilde{a}$ is continuous.

If Z is symmetric then $a \mapsto \mathrm{dom}(\widetilde{a}) = C_a$ is an étale representation, thanks to Theorem 8.11. In particular, for any $a, b \in S$, $\mathrm{dom}(\widetilde{ab}) = \mathrm{dom}(\widetilde{a}) \cdot \mathrm{dom}(\widetilde{b})$ and, for any

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

$A \in \mathrm{dom}(\widetilde{a})$ and $B \in \mathrm{dom}(\widetilde{b})$ with $\mathsf{s}(A) = \mathsf{r}(B)$,

$$\widetilde{ab}(A \cdot B) = [ab, AB] = [a, A][b, B] = \widetilde{a}(A)\widetilde{b}(B).$$

This shows that $\widetilde{ab} = \widetilde{a}\widetilde{b}$, which in turn shows that $a \mapsto \widetilde{a}$ is a semigroup homomorphism and hence a bundle representation.

To see that $a \mapsto \widetilde{a}$ is universal, take a bundle representation $\theta : S \to \mathcal{S}(\pi)$. By Theorem 8.11, we have a unique étale morphism $\phi : G \to C(S)$ such that $\mathrm{dom}(\theta(a)) = \phi^{-1}[C_a]$, for all $a \in S$. Indeed, the proof shows, for all $g \in G$, that

$$\phi(g) = \{a \in S : g \in \mathrm{dom}(\theta(a))\}.$$

We claim we have a continuous functor $\tau : \phi^{\rho_C}\widetilde{C}(S) \to F$ given by

$$\tau([a, \phi(g)], g) = \theta(a)(g).$$

To see that τ is well defined, take $a, b \in \phi(g)$ with $a \sim_{\phi(g)} b$. Then $as = bs$, for some $s \in \phi(g)^* = \phi(g^{-1})$, and hence

$$\theta(a)(g)\theta(s)(g^{-1}) = (\theta(a)\theta(s))(\mathsf{r}(g)) = \theta(as)(\mathsf{r}(g)) = \theta(bs)(\mathsf{r}(g)) = \theta(b)(g)\theta(s)(g^{-1}).$$

Multiplying by $\theta(s)(g^{-1})^{-1}$ yields $\theta(a)(g) = \theta(b)(g)$, as required. To see that τ is a functor, note that $([a, \phi(g)], g), ([b, \phi(h)], h) \in \phi^{\rho_C}\widetilde{C}(S)$ and $\mathsf{s}(g) = \mathsf{r}(h)$ implies

$$\begin{aligned}\tau(([a, \phi(g)], g)([b, \phi(h)], h)) = \tau([ab, \phi(gh)], gh) &= \theta(ab)(gh) = \theta(a)(g)\theta(b)(h)\\ &= \tau([a, \phi(g)], g)\tau([b, \phi(h)], h).\end{aligned}$$

For continuity, note that if $\tau([a, \phi(g)], g) = \theta(a)(g) \in O$, for some open $O \subseteq F$, then the continuity of $\theta(a)$ yields open $O' \subseteq G$ with $g \in O'$ and $\theta(a)[O'] \subseteq O$. Then $([a, \phi(g)], g) \in \widetilde{C}^a \times O'$ and, for any other $([a, \phi(h)], h) \in (\widetilde{C}^a \times O')$, we see that

$$\tau([a, \phi(h)], h) = \theta(a)(h) \in \theta(a)[O'] \subseteq O.$$

This shows that τ is continuous, completing the proof of the claim. Furthermore, noting that $\pi(\tau([a, \phi(g)], g)) = \pi(\theta(a)(g)) = g = \rho_{C\phi}([a, \phi(g)], g)$, that is, $\pi \circ \tau = \rho_{C\phi}$, we see that (ϕ, τ) is a Pierce morphism from ρ_C to π.

For all $g \in G$ and $a \in \phi(g)$, we see that

$$\theta(a)(g) = \tau([a, \phi(g)], g) = \tau(\widetilde{a}(\phi(g)), g) = \tau((\widetilde{a} \circ \phi)(g), g) = \frac{\tau}{\phi}(\widetilde{a})(g),$$

where $\tau/\phi : \mathcal{S}(\rho_C) \to \mathcal{S}(\pi)$ is the homomorphism from Theorem 4.10. Also

$$\mathrm{dom}\Big(\frac{\tau}{\phi}(\widetilde{a})\Big) = \phi^{-1}[\mathrm{dom}(\widetilde{a})] = \phi^{-1}[C_a] = \mathrm{dom}(\theta(a))$$

and hence $\theta(a) = (\tau/\phi)(\widetilde{a})$, for all $a \in S$, that is, $\theta = (\tau/\phi) \circ \widetilde{C}$, where $\widetilde{C}(a) = \widetilde{a}$.

To see that (ϕ, τ) is unique, take another Pierce morphism (ϕ', τ') such that $\theta = (\tau'/\phi') \circ \widetilde{C}$. It follows that $a \mapsto \mathrm{dom}(\theta(a)) = \mathrm{dom}((\tau'/\phi')(\widetilde{a})) = \phi'^{-1}[C_a]$ is an étale morphism and hence $\phi' = \phi$, by the uniqueness part of Theorem 8.11. Now, for any $([a, \phi(g)], g) \in \phi^{\rho c}\widetilde{C}(S)$, just note that

$$\tau'([a, \phi(g)], g) = \frac{\tau'}{\phi'}(\widetilde{a})(g) = \theta(a)(g) = \tau([a, \phi(g)], g).$$

Thus, $\tau' = \tau$, proving uniqueness and hence universality. □

It follows that cosets and their equivalence classes determine when structured semigroups have faithful bundle representations, at least under symmetry.

COROLLARY 11.3. *If Z is symmetric then the following assertions are equivalent.*

(1) *(S, N, Z) has a faithful bundle representation.*
(2) *The coset bundle representation of (S, N, Z) is faithful.*
(3) *For all distinct $a, b \in S$, we have $C \in \mathcal{C}(S)$ with $\{a, b\} \cap C \neq \emptyset$ and*

$$a \nsim_C b.$$

PROOF. Certainly (2) implies (1). Conversely, if θ is a faithful bundle representation then Theorem 11.2 yields a Pierce morphism (ϕ, τ) such that $\theta = (\tau/\phi) \circ \widetilde{C}$, where $\widetilde{C}$ is the coset bundle representation. As θ is injective, so is $\widetilde{C}$, showing that (1) implies (2).

More explicitly, (2) is saying that, for any distinct $a, b \in S$, we have $C \in \mathcal{C}(S)$ such that either only one of $\widetilde{a}$ and $\widetilde{b}$ is defined at C, so certainly $a \nsim_C b$, or they are both defined at C but have different values at C, that is,

$$[a, C] = \widetilde{a}(C) \neq \widetilde{b}(C) = [b, C],$$

which again means $a \nsim_C b$. We immediately see this is equivalent to (3). □

Unlike the situation with inverse semigroups in Proposition 2.4, it is indeed possible for a (symmetric) structured semigroup to have no faithful representations.

EXAMPLE 11.4. Take $S = N = Z = \{0, a\}$ where $a^2 = a0 = 0a = 00 = 0$. This is certainly a structured semigroup with symmetric Z. For any representation $a \mapsto \widehat{a}$ on a groupoid bundle $\pi : F \to G$, we would have $\mathrm{dom}(\widehat{a}) = \mathrm{dom}(\widehat{0}) \subseteq G^0$, as $a^2 = 0 \in N$, and $\mathrm{ran}(\widehat{0}) \subseteq F^0$, as $00 = 0$, and hence $\widehat{0} = \widehat{a0} = \widehat{a}\widehat{0} = \widehat{a}$, showing that the representation cannot be faithful.

REMARK 11.5. On the one hand, this might motivate one to look for even more general kinds of representations than those arising from slice-sections of groupoid bundles. On the other hand, one might look for some natural restrictions or additional structures to place on structured semigroups to ensure that they do have faithful bundle representations. In our subsequent work (see [2, 3]) we actually do a bit of both, considering semigroups that sit nicely within larger rings and their representations as sections of more general category bundles.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

12. Filters

As usual, we call $D \subseteq S$ *directed* if

$$a, b \in D \quad \Rightarrow \quad \text{there exists } d \in D\ (d < a, b). \tag{Directed}$$

A *filter* is a directed up-set $D = D^<$. So the nonempty directed cosets

$$\mathcal{D}(S) = \{D \in C(S) : D \text{ is directed}\}$$

are all filters. Directed cosets offer a number of advantages over arbitrary cosets. For example, one immediately notes that the canonical subbasis of $C(S)$ becomes a basis when restricted to $\mathcal{D}(S)$, namely $(\mathcal{D}_a)_{a\in S}$, where $\mathcal{D}_a = C_a \cap \mathcal{D}(S)$. Our goal here is to show that the directed coset subbundle is still a groupoid bundle and that the corresponding subrepresentation of S is 'Zakrzewski universal'.

REMARK 12.1. There will be plenty of directed cosets arising from principal filters in our primary motivating examples, as already noted in Remark 7.2. But in general, a structured semigroup S may have few directed cosets; again we see in Corollary 12.19 below that we have faithful bundle representations precisely when there are enough directed cosets to distinguish the elements of S via their corresponding equivalence relations, at least when Z is symmetric.

First we look at products. Recall the notation $c \mid D$ from Definition 7.5.

PROPOSITION 12.2. *For any $c \in S$ and $D \subseteq S$,*

$$c \mid D \quad \textit{and} \quad D \textit{ is directed} \quad \Rightarrow \quad cD \textit{ is directed.} \tag{12-1}$$

PROOF. If $c'cd = d \in D$ and D is directed then, for any $a, b \in D$, we have $e \in D$ with $e < a, b, d$. In particular, we have $d' \in S$ with $e <_{d'} d$ and hence $e = dd'e = c'cdd'e = c'ce$. Then (Z-Splitting) (applied to S^{op}) yields $ce < ca, cb$, showing that cD is directed. □

Likewise, duals respect directedness.

PROPOSITION 12.3. *If $D \subseteq S$ is directed then so is D^* and*

$$D^*DD^* \subseteq D^{*<}. \tag{12-2}$$

PROOF. Take $b', d' \in D^*$, so we have $a, b, c, d \in S$ with $D \ni a <_{b'} b$ and $D \ni c <_{d'} d$. As D is directed, we have $e, f \in D$ with $e <_{f'} f < a, c$ so (Transitivity) yields $e, f <_{b'} b$ and $e, f <_{d'} d$. Then (Switch) yields $b'fd' <_b b'$ and $b'fd' <_d d'$. Also $e <_{b'fd'} df'b$, as $eb'fd' \in NN \subseteq N$ and $eb'fd'df'b = eb'ff'b = ff'eb'b = e$. Thus, $b'fd' \in D^*$, showing that D^* is indeed directed.

Now suppose that $e \in D$ and $b', d' \in D^*$, so we again have $a, b, c, d \in S$ with $D \ni a <_{b'} b$ and $D \ni c <_{d'} d$. As D is directed, we have $f, g \in D$ with $g <_{f'} f < a, c, e$. We immediately verify that $g <_{b'fd'} df'b$ (for example, $gb'fd'df'b = gd'db'ff'b =$

$ff'gb'b = g$) so $b'fd' \in D^*$. Taking $e' \in S$ with $f <_{e'} e$, we also immediately verify that $b'fd' <_{de'b} b'ed'$ (for example, $b'fd'de'bb'ed' = b'fe'bb'ed' = b'bb'fe'ed' = b'fd'$) and hence $b'ed' \in D^{*<}$, showing that $D^*DD^* \subseteq D^{*<}$. □

When N is diagonal, directed cosets are precisely filters and, moreover, for any $(d^{\sim_D}, D) \in \widetilde{\mathcal{D}}(S)$, the second coordinate is just the up-closure of the first (where $\widetilde{\mathcal{D}}(S) = \rho_C^{-1}[\mathcal{D}(S)] = \{(d^{\sim_D}, D) \in \widetilde{C}(S) : D \in \mathcal{D}(S)\}$).

Proposition 12.4. *Any directed $D = D^{**} \subseteq S$ is a filter and a coset. Conversely, if N is diagonal then every filter is a coset and*

$$d \in D \in \mathcal{D}(S) \quad \Rightarrow \quad D = (d^{\sim_D})^<.$$

Proof. As D is directed and hence round, $D^< \subseteq D^{<<} \subseteq D^{**} = D$, that is, D is also an up-set and hence a filter. Also (Antimorphism), (6-1) and (12-2) yield

$$DD^*D = D^{**}D^*D^{**} \subseteq (D^*DD^*)^* \subseteq D^{*<*} \subseteq D^{**<} = D^< = D,$$

so D is also a coset.

When N is diagonal, any filter $D \subseteq S$ satisfies $D = D^{<<} = D^{**}$ by (6-5), and is thus a coset by the above argument. Moreover, for any $c, d \in D$, we have $a, b', b \in S$ with $D \ni a <_{b'} b < c, d$. Taking $c', d' \in D$ with $b <_{c'} c$ and $b <_{d'} d$, we see that $cc'bb'd = bb'd$ and $c'bb'd = c'dd'bb'd \in N$ by diagonality, because $c'dd'b, d'b, d'bb'd \in N$. Thus, $d \sim_D bb'd <_{c'} c$, by (9-7), that is, $c \in (d^{\sim_D})^<$, showing that $D \subseteq (d^{\sim_D})^< \subseteq D^< \subseteq D$. □

Definition 12.5. In a groupoid G, we call $I \subseteq G$ an *ideal* if

$$g \in I \quad \text{or} \quad h \in I \quad \Rightarrow \quad gh \in I, \tag{Ideal}$$

for all $(g, h) \in G^2$.

In particular, if $I \subseteq G$ is an ideal then $g \in I$ implies $gg^{-1} \in I$ and hence $g^{-1} = g^{-1}gg^{-1} \in I$, that is, $I = I^{-1} = II$. It follows that ideals are precisely the replete/isomorphism-closed full subgroupoids. Moreover, if G has an étale topology, the subspace topology on any ideal is again étale; see [1]. In particular, $\mathcal{D}(S)$ is an étale subgroupoid of $C(S)$, by the following result.

Proposition 12.6. *$\mathcal{D}(S)$ is an ideal of $C(S)$ (and hence an étale groupoid).*

Proof. If $C \in C(S)$, $D \in \mathcal{D}(S)$ and $(C, D) \in C^2$ then cD is directed, by (12-1), and hence so is $(CD)^* = (cD)^*$ and $(CD)^{**}$, by Proposition 12.2 and (7-2). Likewise, $(D, C) \in C^2$ implies that $(DC)^{**}$ is directed and hence $\mathcal{D}(S)$ is an ideal. □

Next, we note that units in $\mathcal{D}(S)$ are just units in $C(S)$ generated by N.

Proposition 12.7. *Let $U \in C(S)$ be a unit coset.*

(1) *$(U \cap N)^<$ is a directed unit coset.*
(2) *$(U \cap N)^< = U$ when U is directed.*
(3) *$(U \cap N)^< = U^{Z<}$ when N is diagonal.*

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

PROOF. (1) First we claim that $U \cap N$ is directed. Indeed, if $m, n \in U \cap N$ then we have $U \ni t <_{m'} m$ and $U \ni u <_{n'} n$. Noting that $t = (tm')m \in NN \subseteq N$ and $u = (un')n \in NN \subseteq N$, it follows that $tu \in NN \subseteq N$. As U is a unit coset, $tu \in UU = UU^* \subseteq U$. Thus, $U \cap N \ni tu < m, n$, by (N-Invariance). This proves the claim, and hence $(U \cap N)^<$ and $(U \cap N)^{<**}$ are also directed, by Proposition 12.2.

Moreover, by Proposition 8.2,

$$\emptyset \neq U \cap N \subseteq (U \cap N)^< \subseteq (U \cap N)^{<<<} \subseteq (U \cap N)^{<**}.$$

Also, for any $u \in (U \cap N)^{<**} \subseteq U^{<**} = U$ and $n \in U \cap N \subseteq (U \cap N)^{<**}$, directedness yields $t \in (U \cap N)^{<**} \subseteq U$ with $t < u, n$. Then $t \in N$ too, as $n \in N$, that is, $t \in U \cap N$ and hence $u \in (U \cap N)^<$. This proves that $(U \cap N)^< = (U \cap N)^{<**}$ and hence $(U \cap N)^<$ is a unit coset, by Propositions 8.2 and 12.3.

(2) In particular, if $U = (U \cap N)^<$ then U is directed. Conversely, if U is directed then, as above, for any $u \in U$ and $n \in U \cap N$, we have $t \in U$ with $t < u, n$ and hence $t \in N$, as $n \in N$. Thus, $u \in (U \cap N)^<$, showing that $U \subseteq (U \cap N)^< \subseteq U^< = U$.

(3) By (9-1), $U^Z \subseteq \mathsf{s}(U) \cap Z \subseteq U \cap N$ and hence $U^{Z<} \subseteq (U \cap N)^<$. For the reverse inclusion, assume that N is diagonal and take $u \in (U \cap N)^<$ so we have $t, u' \in S$ with $U \cap N \ni t <_{u'} u$. Then we can further take $s, t' \in S$ with $U \ni s <_{t'} t$. As $t't, t, tu' \in N$ and N is diagonal, $t'tu' \in N$ and hence $U^Z \ni t't <_{u'} u$, showing that $(U \cap N)^< \subseteq U^{Z<}$. □

While $\mathcal{D}(S)$ might have a nice basis, it need not be Hausdorff or even T_1 (when $C \subsetneqq D$, for $C, D \in \mathcal{D}(S)$, any neighbourhood of C is a neighbourhood of D). For better separation properties, we can consider ultrafilters

$$\mathcal{U}(S) = \{U \subseteq S : U \text{ is a maximal proper filter}\},$$

at least when Z has a zero, that is, $0s = s0 = 0 \in Z$, for all $s \in S$.

THEOREM 12.8. *If Z has a zero then $\mathcal{U}(S)$ is a locally Hausdorff ideal.*

PROOF. First, note that $0 <_0 s$, for all $s \in S$, as $0 \in Z \subseteq N$. So a filter $F \subseteq S$ is proper precisely when $0 \notin F$.

Next, we show that every $U \in \mathcal{U}(S)$ is a coset. By (6-2), $U = U^{<<} \subseteq U^{**}$. As $0 \notin U$, it follows that $0 \notin U^*$, $0 \notin U^{**}$ and hence $0 \notin U^{**<}$. By Proposition 12.2, U^* and U^{**} are directed and hence $U^{**<}$ is a proper filter containing U^{**} and hence U. As U is an ultrafilter, $U = U^{**} = U^{**<}$ and hence U is a coset, by Proposition 12.3.

Now suppose that C is another coset with $(C, U) \in \mathcal{C}^2$ and take any $c, d \in C$ and $c' \in C^*$ with $cc'd = d$. We already saw above that $(CU)^{**} = (cU)^{**}$ is a filter. Extending to an ultrafilter $V \supseteq (cU)^{**}$, we see that $(c'V)^{**}$ is another filter containing U, so $U = (c'V)^{**}$ by maximality. Then $(cU)^{**} = (cc'V)^{**} = V$, that is, $(CU)^{**} = (cU)^{**}$ was already maximal. Likewise, $(U, C) \in \mathcal{C}^2$ implies that $(UC)^{**}$ is an ultrafilter, showing that the ultrafilters form an ideal in $\mathcal{C}(S)$.

It follows that $\mathcal{U}(S)$ is an étale subgroupoid of $\mathcal{C}(S)$ by [1, Proposition 2.7]. In particular, $\mathcal{U}(S)$ is locally homeomorphic to its unit space, so to prove that $\mathcal{U}(S)$ is locally Hausdorff it suffices to show that its unit space is Hausdorff. To see this, take any distinct unit ultrafilters $U, V \in \mathcal{U}(S)$. By (Multiplicativity), $(UV)^<$ is again a filter. By Proposition 8.2, we have $m \in U \cap N$ and $n \in V \cap N$. By (N-Invariance), $U \subseteq (Un)^< \subseteq (UV)^<$ and $V \subseteq (mV)^< \subseteq (UV)^<$. As U and V are distinct and maximal, we must have $(UV)^< = S$. In particular, we must have $u \in U$ and $v \in V$ with $uv < 0$ and hence $uv = 0$. Thus, $\mathcal{U}_u \cap \mathcal{U}_v = C_u \cap C_v \cap \mathcal{U}(S) = \emptyset$. Indeed, if we had $W \in \mathcal{U}_u \cap \mathcal{U}_v$ then $0 = uv \in WW = WW^* \subseteq \mathsf{r}(W) = W$, as W has to be a unit coset, again by Proposition 8.2. But this would mean $W = 0^< = S$ and, in particular, $W \notin \mathcal{U}(S)$. This shows that the unit space of $\mathcal{U}(S)$ is Hausdorff and hence the entirety of $\mathcal{U}(S)$ is locally Hausdorff. □

REMARK 12.9. We have more to say about $\mathcal{U}(S)$ in future work. For the moment we simply remark that ultrafilters are more natural to consider than arbitrary filters when dealing with groupoid C*-algebras and more general 'bumpy semigroups'; see [1, 4].

12.1. Zakrzewski universality. By Proposition 12.6, the identity embedding of $\mathcal{D}(S)$ in $C(S)$ is an étale functor. For Zakrzewski universality, we need a Zakrzewski morphism in the opposite direction.

DEFINITION 12.10. For any $C \in C(S)$, we define

$$D \lhd C \quad \Leftrightarrow \quad D \text{ is a maximal directed subset of } C.$$

So $\mathrm{dom}(\lhd) \subseteq C(S)$ and $\mathrm{ran}(\lhd) \subseteq \mathcal{P}(S)$. In fact, $\mathrm{ran}(\lhd) \subseteq \mathcal{D}(S)$.

PROPOSITION 12.11. *If $D \lhd C$ then D is a coset and hence $D \in \mathcal{D}(S)$. If $\mathcal{D}(S) \ni D \subseteq C \in C(S)$ then*

$$\mathsf{r}(C) \cap N = \mathsf{r}(D) \cap N \quad \Leftrightarrow \quad D \lhd C \quad \Leftrightarrow \quad \mathsf{s}(C) \cap N = \mathsf{s}(D) \cap N. \tag{12-3}$$

Moreover, if $d \in C \in C(S)$ then we have a unique $D \in \mathcal{D}(S)$ with $d \in D \lhd C$.

PROOF. If $D \lhd C$ then $C \in C(S)$ and, in particular, $C = C^{**}$. As D is directed, so too are D^* and D^{**}, by Proposition 12.2. Then (6-2) yields

$$D \subseteq D^< \subseteq D^{<<} \subseteq D^{**} \subseteq C^{**} \subseteq C$$

so $D = D^{**}$, by maximality. Thus, D is a coset by Proposition 12.3.

Assume that $d \in C \in C(S)$. Taking $c, d' \in C$ with $c <_{d'} d$, we see that $c = cd'd$ and $cd' \in \mathsf{r}(C) \cap N \subseteq (\mathsf{r}(C) \cap N)^<$ (as $\mathsf{r}(C) \cap N$ is directed; see the proof of Proposition 12.7). Thus, $(\mathsf{r}(C) \cap N)^< | d$ so, by Proposition 7.7, we have a coset

$$D = ((\mathsf{r}(C) \cap N)^< d)^< \subseteq (\mathsf{r}(C)C)^< = C$$

with $\mathsf{r}(D) = \mathsf{r}((\mathsf{r}(C) \cap N)^<) = (\mathsf{r}(C) \cap N)^<$, by Proposition 12.7. Moreover,

$$\mathsf{r}(C) \cap N = \mathsf{r}(D) \cap N$$

because $\mathsf{r}(C) \cap N \subseteq (\mathsf{r}(C) \cap N)^< \cap N \subseteq \mathsf{r}(C) \cap N$. Also $(\mathsf{r}(C) \cap N)^< d \ni cd'd <_{d'} d$ and hence $d \in D \in \mathcal{D}(S)$, by Proposition 12.7 and (12-1).

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

We claim that D is actually the largest element of $\mathcal{D}(S)$ with $d \in D \subseteq C$. To see this, take any other $A \in \mathcal{D}(S)$ with $d \in A \subseteq C$. Then $\mathsf{r}(A)$ is directed, by Proposition 12.6, and hence $\mathsf{r}(A) = (\mathsf{r}(A) \cap N)^< \subseteq (\mathsf{r}(C) \cap N)^< = \mathsf{r}(D)$, by Proposition 12.7. Then (7-2) yields

$$A = (\mathsf{r}(A)A)^< = (\mathsf{r}(A)d)^< \subseteq (\mathsf{r}(D)d)^< = (\mathsf{r}(D)D)^< = D,$$

proving the claim. Moreover, note that if we had $\mathsf{r}(A) \cap N = \mathsf{r}(C) \cap N$ then we would have $\mathsf{r}(A) = (\mathsf{r}(A) \cap N)^< = (\mathsf{r}(C) \cap N)^< = \mathsf{r}(D)$ which would yield $A = D$ above. It follows that D is the unique element of $\mathcal{D}(S)$ with $d \in D \lhd C$ and also the unique element of $\mathcal{D}(S)$ with $d \in D \subseteq C$ and $\mathsf{r}(C) \cap N = \mathsf{r}(D) \cap N$, which yields the first $\Leftrightarrow$ in (12-3). The second follows by a dual argument. □

Note that the second statement above is saying that any coset C can be partitioned uniquely into maximal directed sub(co)sets. In fact, uniqueness holds even among more general families of directed cosets covering C.

PROPOSITION 12.12. *For any $C \in \mathcal{C}(S)$,*

$$C^{\rhd} = \{D \in \mathcal{D}(S) : D \lhd C\}$$

is the unique subfamily of $\mathcal{D}(S)$ with union C and a common source.

PROOF. Take a family $\mathscr{D} \subseteq \mathcal{D}(S)$ with $C = \bigcup \mathscr{D}$. We claim that

$$\mathsf{s}(C) \cap N = \bigcup_{D \in \mathscr{D}} \mathsf{s}(D) \cap N. \tag{12-4}$$

To see this, take any $n \in \mathsf{s}(C) \cap N$. By Proposition 12.11, we have $B \lhd C$ with $\mathsf{s}(B) \cap N = \mathsf{s}(C) \cap N$ so $n \in \mathsf{s}(B) = (B^*B)^<$, that is, we have $b \in B$ and $b' \in B^*$ such that $b'b < n$. Taking $e, f \in B$ with $e <_{b'} f$, directedness yields $a \in B$ with $a < b, e$ and hence $a <_{b'} f$. As $C = \bigcup \mathscr{D}$, we have $D \in \mathscr{D}$ containing a, so $b \in D$ and $b' \in D^*$, which means that $n \in (D^*D)^< \cap N = \mathsf{s}(D) \cap N$, proving the claim.

If the directed cosets in $\mathscr{D}$ also have a common source then the only way (12-4) could hold is if $\mathsf{s}(C) \cap N = \mathsf{s}(D) \cap N$, for all $D \in \mathscr{D}$. Then Proposition 12.11 again yields $\mathscr{D} = C^{\rhd}$, as required. □

If N is diagonal, we can also determine when $D \lhd C$ from C^Z and D^Z.

PROPOSITION 12.13. *For any $C \in \mathcal{C}(S)$ and $D \in \mathcal{D}(S)$,*

$$D \lhd C \quad \Rightarrow \quad C^Z = D^Z. \tag{12-5}$$

The converse also holds when $D \subseteq C$ and N is diagonal.

PROOF. Note that $C^Z = (\mathsf{s}(C) \cap N)^Z$ because, by (9-1),

$$C^Z = (C^Z)^Z \subseteq (\mathsf{s}(C) \cap N)^Z \subseteq \mathsf{s}(C)^Z = C^Z.$$

Thus, by (12-3), $D \lhd C$ implies $\mathsf{s}(C) \cap N = \mathsf{s}(D) \cap N$ and hence $C^Z = D^Z$.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

Conversely, if N is diagonal, $D \subseteq C$ and $C^Z = D^Z$ then Proposition 12.7 yields

$$\mathsf{s}(C) \cap N \subseteq (\mathsf{s}(C) \cap N)^< = C^{Z<} = D^{Z<} = (\mathsf{s}(D) \cap N)^< = \mathsf{s}(D).$$

Thus, $\mathsf{s}(C) \cap N = \mathsf{s}(D) \cap N$ and hence $D \lhd C$, by (12-3). □

Next, we show that $\lhd$ is a Zakrzewski morphism such that the coset representation $\mathcal{C}$ factors through the directed coset representation $\mathcal{D}$ via preimages of $\lhd$.

THEOREM 12.14. *$\lhd$ is a Zakrzewski morphism from $\mathcal{C}(S)$ to $\mathcal{D}(S)$ with*

$$\mathcal{C} = \overline{\lhd} \circ \mathcal{D}. \tag{12-6}$$

PROOF. If $D \lhd C$ then $D^* \lhd C^*$. If $D' \lhd C'$ too and $\mathsf{s}(C) = \mathsf{r}(C')$ then Proposition 12.7 and (12-3) yield $\mathsf{s}(D) = (\mathsf{s}(C) \cap N)^< = (\mathsf{r}(C') \cap N)^< = \mathsf{r}(D')$ and

$$\mathsf{r}(C \cdot C') \cap N = \mathsf{r}(C) \cap N = \mathsf{r}(D) \cap N = \mathsf{r}(D \cdot D') \cap N,$$

so $D \cdot D' \lhd C \cdot C'$. This shows that $\lhd$ is functorial.

Now suppose that $\mathsf{r}(D) \lhd U$, for some unit coset U. For any $d \in D$, we see that $\mathsf{r}(D)|d$ so $U|d$ and $D = (\mathsf{r}(D)d)^< \subseteq (Ud)^<$, as $\mathsf{r}(D) \subseteq U$. By Proposition 7.7, $(Ud)^<$ is a coset with $\mathsf{r}(Ud) = U$. Thus, $\mathsf{r}(D) \cap N = U \cap N = \mathsf{r}(Ud) \cap N$ and hence $D \lhd (Ud)^<$, by (12-3), showing that $\lhd$ is star-surjective. On the other hand, if we had another $C \in \mathcal{C}(S)$ with $D \lhd C$ and $\mathsf{r}(C) = U$ then $C = (\mathsf{r}(C)d)^< = (Ud)^<$, as $d \in D \subseteq C$, showing that $\lhd$ is also star-injective.

Lastly, note that if $\mathcal{D}_a \ni D \lhd C$ then certainly $C \in \mathcal{C}_a$, while conversely if $C \in \mathcal{C}_a$ then Proposition 12.11 yields $D \in \mathcal{D}_a$ with $D \lhd C$. This shows that $\lhd^{-1}[\mathcal{D}_a] = \mathcal{C}_a$, so $\lhd$ is also continuous and hence a Zakrzewski morphism satisfying (12-6). □

We call an étale representation $\mu : S \to \mathcal{B}(G)$ *Zakrzewski universal* if, for every étale representation $\theta : S \to \mathcal{B}(H)$, there exists a unique Zakrzewski morphism $\phi \subseteq G \times H$ such that $\theta = \overline{\phi} \circ \mu$, where $\overline{\phi} : \mathcal{B}(G) \to \mathcal{B}(H)$ is the preimage map.

THEOREM 12.15. *If Z is symmetric then the directed coset representation $\mathcal{D}$ is a Zakrzewski universal étale representation.*

PROOF. As $\mathcal{C}$ is an étale representation by Theorem 8.11, and $\mathcal{D}(S)$ is an ideal of $\mathcal{C}(S)$ by Proposition 12.6, it follows that $\mathcal{D}$ is also an étale representation.

To see that $\mathcal{D}$ is Zakrzewski universal, let us take another étale representation $\theta : S \to \mathcal{B}(G)$. Let ϕ be the étale morphism in the proof of Theorem 8.11 given by

$$\phi(g) = \{a \in S : g \in \theta(a)\}.$$

By Theorem 12.14, $\lhd$ is also a Zakrzewski morphism and hence so is the composition

$$\psi = \lhd \circ \phi.$$

Moreover, $\overline{\psi} \circ \mathcal{D} = \overline{\phi} \circ \overline{\lhd} \circ \mathcal{D} = \overline{\phi} \circ \mathcal{C} = \theta$, by (12-6) and the proof of Theorem 8.11.

The only thing left to prove is that ψ is unique. Accordingly, suppose that we had another Zakrzewski morphism $\psi' \subseteq \mathcal{D}(S) \times G$ with $\overline{\psi'} \circ \mathcal{D} = \theta$, that is,

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

$\psi'^{-1}[\mathcal{D}_a] = \theta(a)$, for all $a \in S$. Then we see that $g \in \theta(a)$ if and only if $D\ \psi'\ g$, for some $D \in \mathcal{D}_a$, and hence

$$\bigcup_{D\,\psi'\,g} D = \{a \in S : g \in \theta(a)\} = \phi(g).$$

As ψ' is a functorial relation, all those $D \in \mathcal{D}(S)$ with $D\,\psi'\,g$ must have a common source. By Proposition 12.12, it follows that

$$\{D \in \mathcal{D}(S) : D\ \psi'\ g\} = \phi(g)^{\rhd} = \{D \in \mathcal{D}(S) : D\ \psi\ g\}.$$

This shows that $\psi' = \psi$, as required. □

As in Proposition 4.8, we can consider the pullback bundle

$$\rho^{\lhd}_{\mathcal{D}} : {\lhd}^{\rho_{\mathcal{D}}}\, \widetilde{\mathcal{D}}(S) \to C(S),$$

which is continuously isomorphic to the coset bundle $\rho_C : \widetilde{C}(S) \to C(S)$.

Proposition 12.16. *Whenever $d \in D \lhd C$, we can define*

$$\iota([d, D], C) = [d, C].$$

The resulting function is a continuous isomorphism $\iota : {\lhd}^{\rho_{\mathcal{D}}}\, \widetilde{\mathcal{D}}(S) \to \widetilde{C}(S)$.

Proof. If $d \in D \lhd C$ then $C^Z = D^Z$ and ${}^ZC = {}^ZD$, by (12-5), so $d^{\sim_C} = d^{\sim_D}$, by (9-4). In particular, ι is well-defined. Whenever $d \in C \in C(S)$, Proposition 12.11 yields a unique $D \lhd C$ with $d \in D$. This shows that ι is a bijection, which is immediately seen to be a groupoid isomorphism. Moreover, for all $s \in S$, we see that $\iota^{-1}[\widetilde{C}_s] = \rho^{\lhd-1}_{\mathcal{D}}[C_s]$ and $\iota^{-1}[\widetilde{C}^s] = {\lhd}^{\rho_{\mathcal{D}}}\, \widetilde{\mathcal{D}}(S) \cap (\widetilde{\mathcal{D}}^s \times C_s)$, so ι is also continuous. □

We now have the following extension of Theorem 12.14. As before, let $\widetilde{C}$ and $\widetilde{\mathcal{D}}$ denote the maps $a \mapsto \widetilde{a} = \widetilde{a}_C$ and $a \mapsto \widetilde{a}_{\mathcal{D}} = \widetilde{a}|_{\mathcal{D}(S)}$, respectively.

Corollary 12.17. *$(\lhd, \iota)$ is a Zakrzewski–Pierce morphism from $\rho_{\mathcal{D}}$ to ρ_C such that*

$$\widetilde{C} = \frac{\iota}{\lhd} \circ \widetilde{\mathcal{D}},$$

where $\iota/\lhd : \mathcal{S}(\rho_{\mathcal{D}}) \to \mathcal{S}(\rho_C)$ is the semigroup morphism from Theorem 4.10.

Proof. By Theorem 12.14 and Proposition 12.16, $(\lhd, \iota)$ is a Zakrzewski–Pierce morphism. For any $C \in \mathrm{dom}(\widetilde{a}_C) = C_a$, we have a unique $D \in \mathcal{D}(S)$ with $a \in D \lhd C$, by Proposition 12.11. Then we see that

$$\widetilde{a}_C(C) = [a, C] = \iota([a, D], C) = \iota((\widetilde{a}_{\mathcal{D}} \circ \lhd)(C), C) = \frac{\iota}{\lhd}(\widetilde{a}_{\mathcal{D}})(C).$$

This shows that $\widetilde{a}_C = (\iota/\lhd)(\widetilde{a}_{\mathcal{D}})$, which in turn shows that $\widetilde{C} = (\iota/\lhd) \circ \widetilde{\mathcal{D}}$. □

Again, we call a bundle representation $\mu : S \to \mathcal{S}(\pi)$ *Zakrzewski universal* if, for every bundle representation $\theta : S \to \mathcal{S}(\pi')$, there exists a unique Zakrzewski–Pierce morphism (ϕ, τ) from π to π' such that $\theta = (\tau/\phi) \circ \mu$.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

THEOREM 12.18. *If Z is symmetric then the directed coset bundle representation $\widetilde{\mathcal{D}}$ is a Zakrzewski universal bundle representation.*

PROOF. As $\widetilde{C}$ is a bundle representation by Theorem 11.2, and $\mathcal{D}(S)$ is an ideal of $C(S)$ by Proposition 12.6, it follows that $\widetilde{\mathcal{D}}$ is also a bundle representation.

To see that $\widetilde{\mathcal{D}}$ is Zakrzewski universal, let us take another bundle representation $\theta : S \to \mathcal{S}(\pi)$. Let $\psi = \lhd \circ \phi$ be the Zakrzewski morphism in the proof of Theorem 12.15, where ϕ is the étale morphism in the proof of Theorem 8.11 given by $\phi(g) = \{a \in S : g \in \theta(a)\}$. Further, let $\tau : \phi^{\pi}\widetilde{C}(S) \to F$ be the continuous functor in the proof of Theorem 11.2 given by $\tau([a, \phi(g)], g) = \theta(a)(g)$. We then get another continuous functor $\sigma : \psi^{\rho_{\mathcal{D}}}\widetilde{\mathcal{D}}(S) \to F$ defined by

$$\sigma([a, D], g) = \tau(\iota([a, D], \phi(g)), g) = \tau([a, \phi(g)], g) = \theta(a)(g)$$

whenever $a \in D \lhd \phi(g)$. Thus, (ψ, σ) is a Zakrzewski–Pierce morphism with

$$\frac{\sigma}{\psi}(\widetilde{a}_{\mathcal{D}})(g) = \sigma((\widetilde{a}_{\mathcal{D}} \circ \psi)(g), g) = \sigma([a, D], g) = \theta(a)(g),$$

where D is the unique directed coset with $a \in D \lhd \phi(g)$. This shows that $(\sigma/\psi)(\widetilde{a}_{\mathcal{D}}) = \theta(a)$, which in turn shows that $\theta = (\sigma/\psi) \circ \widetilde{\mathcal{D}}$.

For uniqueness, suppose that we had another Zakrzewski–Pierce morphism (ψ', σ') with $\theta = (\sigma'/\psi') \circ \widetilde{\mathcal{D}}$. By Theorem 12.15, we must have $\psi' = \psi$. For any $([a, D], g) \in \psi^{\rho_{\mathcal{D}}}\widetilde{\mathcal{D}}(S)$,

$$\sigma'([a, D], g) = \frac{\sigma'}{\psi'}(\widetilde{a}_{\mathcal{D}})(g) = \theta(a)(g) = \sigma([a, D], g).$$

Thus, $\sigma' = \sigma$, proving uniqueness and hence universality. □

As in Corollary 11.3, it follows that even directed cosets and their equivalence classes determine when structured semigroups have faithful bundle representations.

COROLLARY 12.19. *If Z is symmetric then the following assertions are equivalent.*

(1) *(S, N, Z) has a faithful bundle representation.*
(2) *The directed coset bundle representation of (S, N, Z) is faithful.*
(3) *For all distinct $a, b \in S$, we have $D \in \mathcal{D}(S)$ with $\{a, b\} \cap D \neq \emptyset$ and*

$$a \nsim_D b.$$

PROOF. Exactly like the proof of Corollary 11.3, just using Theorem 12.18. □

Acknowledgement

Many thanks to the anonymous referee for reading the first draft very carefully and making numerous helpful comments and suggestions which have been incorporated into this final version.

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press

References

[1] T. Bice, 'An algebraic approach to the Weyl groupoid', *J. Algebra* **568** (2021), 193–240.
[2] T. Bice, 'Dauns–Hofmann–Kumjian–Renault duality for Fell bundles and structured C*-algebras', Preprint, 2021, arXiv:2104.12415.
[3] T. Bice, 'Noncommutative Pierce duality between Steinberg rings and ample ringoid bundles', Preprint, 2021, arXiv:2012.03006.
[4] T. Bice and L. O. Clark, 'Reconstructing étale groupoids from semigroups', *Forum Math.* **33**(6) (2021), 1423–1444.
[5] T. Bice and C. Starling, 'General non-commutative locally compact locally Hausdorff Stone duality', *Adv. Math.* **314** (2019), 40–91.
[6] T. Bice and C. Starling, 'Hausdorff tight groupoids generalised', *Semigroup Forum* **100**(2) (2020), 399–438.
[7] R. Brown, 'Fibrations of groupoids', *J. Algebra* **15** (1970), 103–132.
[8] S. A. Celani, 'Topological representation of distributive semilattices', *Sci. Math. Jpn.* **58**(1) (2003), 55–65.
[9] S. A. Celani and L. J. González, 'A categorical duality for semilattices and lattices', *Appl. Categ. Structures* **28**(5) (2020), 853–875.
[10] L. O. Clark, C. Farthing, A. Sims and M. Tomforde, 'A groupoid generalisation of Leavitt path algebras', *Semigroup Forum* **89**(3) (2014), 501–517.
[11] R. J. Deeley, I. F. Putnam and K. R. Strung, 'Constructing minimal homeomorphisms on point-like spaces and a dynamical presentation of the Jiang–Su algebra', *J. reine angew. Math.* **2018**(742) (2018), 241–261.
[12] R. Exel, 'Noncommutative Cartan subalgebras of C^*-algebras', *New York J. Math.* **17** (2011), 331–382.
[13] I. Gelfand, 'Normierte Ringe', *Rec. Math. [Mat. Sbornik] N.S.* **9**(51) (1941), 3–24.
[14] I. Gelfand and M. Naimark, 'On the imbedding of normed rings into the ring of operators in Hilbert space', *Rec. Math. [Mat. Sbornik] N.S.* **12**(54) (1943), 197–213.
[15] P. J. Higgins, *Notes on Categories and Groupoids*, Van Nostrand Rienhold Mathematical Studies, 32 (Van Nostrand Reinhold Co., London, 1971).
[16] P. Jipsen and M. Andrew Moshier, 'Topological duality and lattice expansions, I: A topological construction of canonical extensions', *Algebra Universalis (2)* **71** (2014), 109–126.
[17] A. Kumjian, 'On C^*-diagonals', *Canad. J. Math.* **38**(4) (1986), 969–1008.
[18] A. Kumjian, 'Fell bundles over groupoids', *Proc. Amer. Math. Soc.* **126**(4) (1998), 1115–1125.
[19] B. K. Kwaśniewski and R. Meyer, 'Noncommutative Cartan C^*-subalgebras', *Trans. Amer. Math. Soc.* **373**(12) (2020), 8697–8724.
[20] M. V. Lawson, *Inverse Semigroups, The Theory of Partial Symmetries* (World Scientific, River Edge, NJ, 1998).
[21] M. V. Lawson, 'Non-commutative Stone duality: inverse semigroups, topological groupoids and C*-algebras', *Internat. J. Algebra Comput.* **22**(6) (2012), 1250058.
[22] M. V. Lawson and D. H. Lenz, 'Pseudogroups and their étale groupoids', *Adv. Math.* **244**(Supplement C) (2013), 117–170.
[23] M. V. Lawson, S. W. Margolis and B. Steinberg, 'The étale groupoid of an inverse semigroup as a groupoid of filters', *J. Aust. Math. Soc.* **94**(2) (2013), 234–256.
[24] D. H. Lenz, 'On an order-based construction of a topological groupoid from an inverse semigroup', *Proc. Edinb. Math. Soc. (2)* **51**(2) (2008), 387–406.
[25] R. S. Pierce, *Modules over Commutative Regular Rings*, Memoirs of the American Mathematical Society, 70 (American Mathematical Society, Providence, RI, 1967).
[26] J. Renault, *A Groupoid Approach to C^*-Algebras*, Lecture Notes in Mathematics, 793 (Springer, Berlin, 1980).
[27] J. Renault, 'Cartan subalgebras in C^*-algebras', *Irish Math. Soc. Bull.* **61** (2008), 29–63.
[28] P. Resende, 'Étale groupoids and their quantales', *Adv. Math.* **208**(1) (2007), 147–209.

[29] A. Sims, 'Étale groupoids and their C*-algebras', Preprint, 2018, arXiv:1710.10897.
[30] P. Stachura, 'Short and biased introduction to groupoids', *J. Knot Theory Ramifications* **27**(7) (2018), Article no. 1841010.
[31] B. Steinberg, 'A groupoid approach to discrete inverse semigroup algebras', *Adv. Math.* **223**(2) (2010), 689–727.
[32] M. H. Stone, 'Topological representations of distributive lattices and Brouwerian logics', *Cat. Mat. Fys.* **67** (1) (1938), 1–25.
[33] J. Varela, 'Duality of C^*-algebras', in: *Recent Advances in the Representation Theory of Rings and C^*-Algebras by Continuous Sections (A Seminar Held at Tulane University, New Orleans, LA, 1973)*, Memoirs of the American Mathematical Society, 148 (American Mathematical Society, Providence, RI, 1974), 97–108.
[34] H. Wallman, 'Lattices and topological spaces', *Ann. of Math. (2)* **39**(1) (1938), 112–126.
[35] S. Zakrzewski, 'Quantum and classical pseudogroups. I. Union pseudogroups and their quantization', *Comm. Math. Phys.* **134**(2) (1990), 347–370.

TRISTAN BICE, The Department of Abstract Analysis of the Institute of Mathematics of the Czech Academy of Sciences,
Žitná 25, Prague
e-mail: bice@math.cas.cz

https://doi.org/10.1017/S1446788722000064 Published online by Cambridge University Press