ON A RESTRICTED CLASS OF BLOCK DESIGN GAMES

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1. Introduction. Block design games have been developed by Richardson **(12)** and by Hoffman and Richardson **(8)**, who proved a number of theorems concerning such games by studying the number of elements in a blocking coalition. Hoffman and Richardson listed as unsolved (except for PG(2, 3)) the following problem: What is the minimum number of elements in a blocking coalition of a block design game?

This note considers blocking coalitions in those games that are dual to block designs having $\lambda = 1$ and $r - k > 0$. For such games certain blocking coalitions are shown to be related to sets of mutually disjoint blocks in the design to which the game is dual. In particular, for Steiner triple systems the largest odd-numbered set of mutually disjoint triples is shown to yield a minimum blocking coalition in the dual. A lower bound for the number of elements in the largest set of mutually disjoint triples is found, which results in a smaller upper bound for a minimum blocking coalition than that heretofore known for the duals of Steiner triple systems.

Some extensions of the results of this paper show that some of the games that are dual to designs having $\lambda = 1$, $r - k > 0$, and $k > 3$ have easily obtained minimum blocking coalitions and have no equitable main simple solutions.

2. Notation. In general, we use v, b, k, r, λ as parameters of the balanced incomplete block design (written BIBD) consisting of v elements arranged in b blocks of k elements each with each element occurring r times in the design and any pair of distinct elements appearing together in the same block exactly λ times. Following Hoffman and Richardson **(8)**, we use v^*, b^*, etc. as parameters of the block design and unstarred parameters in the dual. $|B|$ will denote the number of elements in a blocking coalition. Elements belonging to the same block will be called *collinear*. The usual set-theoretic notation will be used.

3. Preliminary results. The parameters of a BIBD satisfy

$$vr = bk, \tag{1}$$

$$r(k-1) = \lambda(v-1). \tag{2}$$

If we postulate the existence of at least two distinct blocks so that $v > k$, then

$$r \geqslant k. \tag{3}$$

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This is equivalent to Fisher's inequality:

$$b \geqslant v; \tag{4}$$

cf. **(4, 9, or 13)**. The designs having $r = k$ are called symmetric designs. Finite projective planes of order n are symmetric, balanced incomplete block designs with $k = n + 1$ and $\lambda = 1$. Finite Euclidean planes are balanced incomplete block designs with $\lambda = 1$, $k = n$, $r = n + 1$ so that $r - k = 1$. It has long been known in connection with the search for finite models of a Bolyai–Lobachevsky plane that designs are impossible for $\lambda = 1$ and $r - k = 2$. A general necessary relation between r and k is given by the following: A BIBD with $\lambda = 1$, and $k = p^m q$, where p is a prime, $(p, q) = 1$, satisfies $r \equiv 0$, $1 \pmod{p^m}$. This follows from (1) and (2). The necessary conditions for the existence of a BIBD are sometimes stated in the form **(7)**:

$$\lambda(v - 1) \equiv 0 \pmod{(k - 1)}, \tag{5}$$

$$\lambda v(v - 1) \equiv 0 \pmod{k(k - 1)}. \tag{6}$$

If k is a power of a prime, then a BIBD with $\lambda = 1$ has one of the following sets of parameters:

$$\text{Case I:}\quad v = (k - 1)kt + 1, \quad b = t(k - 1)kt + 1), \quad r = kt. \tag{7}$$

$$\text{Case II:}\quad v = (k - 1)kt + k, \quad b = ((k - 1)t + 1)(kt + 1), \quad r = kt + 1. \tag{8}$$

If k is composite, v, b, k, r, with $\lambda = 1$, can sometimes be found so as to satisfy (1) and (2) with $r \equiv 0$, $1 \pmod{p^m}$ and $r \not\equiv 0$, $1 \pmod{k}$; but such designs have not been constructed **(9**, p. 127**)**. In general (1) and (2) and consequently (5) and (6) are not sufficient for the existence of a BIBD.

BIBD with $\lambda = 1$ and $k = 3$ are known as Steiner triple systems (written STS). Owing to the work of Reiss **(11)** and Moore **(10)**, it is possible to construct an STS for every $t \geqslant 1$ in Cases I and II above. Hanani **(7)** has shown that for $\lambda = 1$ and $k = 4$, (5) and (6) are sufficient conditions. Hanani has also shown that (5) and (6) are sufficient for $\lambda = 1$ and $k = 5$ except possibly for the case $v = 141$.

The dual of a BIBD with $\lambda = 1$ and parameters v^*, b^*, k^*, r^* is a partially balanced incomplete block design with $v = b^*$, $b = v^*$, $k = r^*$, $r = k^*$, and $\lambda i = 0, 1$. If we take the v elements as players and the b blocks as the minimal winning coalitions, we have a block design game as shown in **(8)**. In non-game theoretic terms we can define a blocking coalition as a set of elements that intersects every block but contains no complete block. In order to show that a block design game has no equitable main simple solution, it is sufficient to show that there exists a blocking coalition, B, such that $|B| \leqslant k$ **(8)**. Hoffman and Richardson **(8)** show that the dual of every non-trivial STS has $|B| = k$. The non-trivial STS are those that have $r^* - k^* > 0$. In the following we seek $|B| < k$ and some information concerning minimum blocking coalitions.

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4. Some properties of Steiner triple systems.

LEMMA 1. *In a non-trivial* STS, D^*, *any block belongs to a set of at least three blocks, no pair of which have an element in common.*

Proof. Form the incidence matrix of D^*, as follows. Call the elements $v_1, v_2, \ldots, v_{v^*}$ and the blocks $b_1, b_2, \ldots, b_{b^*}$. Let row i show which blocks contain v_i by entering 1 in the ith row and jth column if b_j contains v_i and 0 if b_j does not contain v_i. Let the elements of b_1 be labelled v_1, v_2, v_3. Since any element is contained in r^* blocks, let $b_1, b_2, \ldots, b_{r^*}$ contain v_1. Similarly, let $b_1, b_{r^*+1}, b_{r^*+2}, \ldots, b_{2r^*-1}$ contain v_2 and let $b_1, b_{2^*}, b_{2^*+1}, \ldots, b_{3r^*-2}$ contain v_3. These b_j, $j = 1, 2, \ldots, 3r^* - 2$, are all distinct since $\lambda = 1$. We note now that for Case I,

$$3r^* - 2 = 9t - 2 < 6t^2 + t = b \qquad \text{for all } t \geqslant 2.$$

For Case II,

$$3r^* - 2 = 9t - 1 < (3t + 1)(2t + 1) = b \qquad \text{for all } t \geqslant 1.$$

It follows that for all non-trivial STS there is a block not accounted for above, say b_{3r^*-1}, which contains three new elements, say v_4, v_5, v_6. Since $\lambda = 1$, the 4th row of the incidence matrix will contain 1 in one and only one of columns $2, 3, \ldots, r^*$ and 1 in one and only one of columns $r^* + 1, r^* + 2, \ldots, 2r^* - 1$ and 1 in one and only one of columns $2r^*, \ldots, 3r^* - 2$. The same result holds for the 5th row and the 6th row. In other words, in the incidence matrix, each of rows 4, 5, 6 is incident with four previously noted blocks and $r^* - 4$ others. Hence to accommodate all the blocks that contain v_4 or v_5 or v_6, we add $3(r^* - 4)$ to the $3r^* - 1$ blocks given above for a total of $6r^* - 13$ blocks.

In Case I, $6r^* - 13 = 18t - 13 < 6t^2 + t = b$ for $t \geqslant 2$. In Case II, $6r^* - 13 = 18t - 7 < 6t^2 + 5t + 1 = b$ for $t \geqslant 1$. These conditions upon t take care of all STS having $r^* - k^* > 0$. Therefore in any non-trivial STS the number of blocks containing elements of two disjoint blocks is less than the total number of blocks. It follows that D^* contains for any pair of disjoint blocks a third block forming with the pair a set of three mutually disjoint blocks. The condition $r^* - k^* > 0$ ensures the existence of at least one block that is disjoint to any given block.

LEMMA 2. *In a non-trivial Steiner triple system* D^*, *q mutually disjoint blocks, $q \geqslant 3$, have elements in common with at most* $3q(r^* - q)$ *additional blocks.*

Proof. Let p represent the number of blocks of D^* that contain elements of, and are distinct from, the q mutually disjoint blocks. These blocks will be called p-blocks and q-blocks respectively. We form a submatrix of the adjacency matrix for the blocks of D^* as follows. Let any pair of blocks having exactly one element in common be called *adjacent*. Form a $q \times p$ submatrix by entering 1 in the ith row, jth column if the ith q-block and the jth p-block are adjacent. Enter 0 in the ith row, jth column if the ith q-block and the jth p-block are not adjacent. Let this matrix be designated by A.

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If the jth p-block meets u of the q mutually disjoint blocks, $u = 1, 2, 3$, then the jth column of A will contain u ones. Let p_u designate the number of columns containing u ones and we shall refer to such columns as p_u-columns. Then

$$p = p_1 + p_2 + p_3. \tag{9}$$

We note that the q mutually disjoint blocks contain $3q$ distinct elements each of which is replicated r^* times in D^*, which means $r^* - 1$ times in the set of p-blocks. Therefore, the total number of ones in A is

$$3q(r^* - 1) = 3p_3 + 2p_2 + p_1. \tag{10}$$

By eliminating p_1 in (9) and (10) and solving for p, we obtain

$$p = 3q(r^* - 1) - (2p_3 + p_2). \tag{11}$$

We seek a minimum value for $(2p_3 + p_2)$, given that $p_2, p_3 \geqslant 0$, and a further condition that we now derive. From A we obtain a relation involving p_2 and p_3 by considering the 2×2 submatrices consisting entirely of ones. Any pair of the q mutually disjoint blocks will generate in A nine 2×1 submatrices consisting entirely of ones, and any (unordered) pair of these nine submatrices will form a 2×2 submatrix of the required form; conversely, it is clear that any 2×2 submatrix consisting entirely of ones occurs in this way. Consequently the number of such matrices is

$$qC_2 \cdot 9C_2 = 18q(q - 1). \tag{12}$$

We now count in two ways the number of incidences of ones with 2×2 submatrices consisting entirely of ones. First of all, the total number of such incidences is four times the number of 2×2 submatrices consisting entirely of ones. Secondly, we note that if a 1 appears in a p-column, it is in eight of the specified 2×2 submatrices; and if a 1 is in a p_3 column it is in sixteen of the specified 2×2 submatrices. By equating these two values for the number of incidences of ones with the specified 2×2 submatrices, we obtain:

$$4 \cdot 18q(q - 1) = (2p_2)8 + (3p_3)16 \tag{13}$$

or

$$2p_2 + 3(2p_3) = 9q(q - 1). \tag{14}$$

In (14) we have $2 < 3$, $p_2 \geqslant 0$, $2p_3 \geqslant 0$, and $9q(q - 1)$ a positive constant. Therefore, $p_2 + 2p_3$ will be a minimum when $p_2 = 0$. But $p_2 = 0$ implies from (14) that

$$2p_3 = 3q(q - 1). \tag{15}$$

We may now conclude that

$$\min(2p_3 + p_2) = 3q(q - 1). \tag{16}$$

By substituting $3q(q - 1)$ for $\min(2p_3 + p_2)$, we obtain, from (11),

$$p \leqslant 3q(r^* - q), \tag{17}$$

as required.

LEMMA 3. *Any block of a non-trivial Steiner triple system belongs to a set of t mutually disjoint blocks and any block of a non-trivial Steiner triple system with $r^* = 3t + 1$ belongs to a set of $t + 1$ mutually disjoint blocks.*

Proof. From Lemma 2, q mutually disjoint blocks have elements in common with at most $3q(r^* - q)$ additional blocks. Therefore, a set of q mutually disjoint blocks can be extended to a set of $q + 1$ mutually disjoint blocks whenever

$$q + 3q(r^* - q) < b^*. \tag{18}$$

We now show that this inequality holds for any $q < t$. In particular, when $r^* = 3t + 1$, the inequality holds for any $q \leqslant t$. By substituting $k = 3$ in (7) and (8), we have

$$\text{Case I:} \quad v^* = 6t + 1, \quad b^* = 6t^2 + t, \quad r^* = 3t, \tag{19}$$

and

$$\text{Case II:} \quad v^* = 6t + 3, \quad b^* = 6t^2 + 5t + 1, \quad r^* = 3t + 1. \tag{20}$$

We now seek q such that

$$\text{Case I:} \quad q + 3q(3t - q) < 6t^2 + t \tag{21}$$

and

$$\text{Case II:} \quad q + 3q(3t + 1 - q) < 6t^2 + 5t + 1. \tag{22}$$

The inequality (21) above may be rewritten in the form of a quadratic inequality:

$$3q^2 - q(9t + 1) + (6t^2 + t) > 0, \qquad t > 0, \tag{23}$$

which has for its solution set all real numbers q such that

$$q < t \quad \text{or} \quad q > 2t + \tfrac{1}{3}. \tag{24}$$

We note that v^*/k^* represents an upper bound for the number of mutually disjoint blocks in a BIBD so that $q > 2t + \frac{1}{3}$ is impossible.

Similarly, the inequality (22) can be rewritten as

$$3q^2 - q(9t + 4) + (6t^2 + 5t + 1) > 0, \qquad t > 0, \tag{25}$$

which has for its solution set all real numbers q such that

$$q < t + \tfrac{1}{3} \quad \text{or} \quad q > 2t + 1. \tag{26}$$

Again $q > 2t + 1$ is impossible.

We have now shown that given a set of q mutually disjoint blocks, $q \geqslant 3$; we may add a block so as to form a set of $q + 1$ mutually disjoint blocks whenever, for Case I, $q < t$ and, for Case II, $q \leqslant t$. From Lemma 1, $q = 3$ for all non-trivial STS. Lemma 3 now follows by induction.

5. Some game-theoretic results. The following theorem is fundamental.

THEOREM 1. *Let D be the dual of a balanced incomplete block design D^* with $\lambda^* = 1$, $r^* - k^* > 0$. If the set of blocks of D^* has a non-empty subset, Q, containing q mutually disjoint blocks with the property that $\mathfrak{C}\{v_q\}$, the complement of $\{v_q\}$, which is the set union of the elements contained in the blocks of Q, can be partitioned into subsets of $k^* - 1$ collinear elements, then the q mutually disjoint blocks together with the blocks determined in the partitioning of $\mathfrak{C}\{v_q\}$ form a blocking coalition in D.*

Proof. Denote the set of blocks determined in the partitioning of $\mathfrak{C}\{v_q\}$ by C and let B_q be the coalition described in the conclusion of the theorem so that $B_q = Q \cup C$. We note, first of all, that the hypothesis $r^* - k^* > 0$ ensures the existence in D^* of at least two mutually disjoint blocks and the hypothesis concerning the partitioning of $\mathfrak{C}\{v_q\}$ requires $k^*q \equiv v^* \pmod{(k-1)}$.

B_q blocks since any element of D^* is in $\{v_q\}$ or in $\mathfrak{C}\{v_q\}$. We now show that B_q does not contain every block on a single element by showing that any element, x, of D^* is contained in at least one block not in B_q. We note that any block of B_q contains either at least $k^* - 1$ elements of $\mathfrak{C}\{v_q\}$ or exactly k^* elements of $\{v_q\}$ (but obviously not both since a block contains exactly k^* elements).

Suppose that $x \in \{v_q\}$; then $x \in b_x$, some block of Q. Let y by an element of D^* such that $y \notin b_x$ and $y \in \{v_q\}$. Such a y exists since by hypothesis Q is not empty, and therefore $r^* - k^* > 0$ together with $k^*q \equiv v^* \pmod{(k^* - 1)}$ requires $q > 2$. Then the block determined by x and y is not in Q since the blocks of Q form a partition of $\{v_q\}$, and the block determined by x and y is not in C since it contains at most $k^* - 2$ elements of $\mathfrak{C}\{v_q\}$. Hence for any $x \in \{v_q\}$ there is a block containing x and not in B_q.

Suppose that $x \in \mathfrak{C}\{v_q\}$. Then x is in exactly one of the $k^* - 1$ subsets in the partition of $\mathfrak{C}\{v_q\}$. The block of C containing the $(k^* - 1)$-subset that contains x has at most one element of $\{v_q\}$. Moreover, if there is a block of C that contains x and also contains an element of $\{v_q\}$, there is only one such block because of the partitioning of $\mathfrak{C}\{v_q\}$. There is then an element, w, of $\{v_q\}$ that is not in the same block of C as x. The elements, w and x then determine a block of D^* that is not in Q and not in C, hence not in B_q.

Since any element of D^* is either in $\{v_q\}$ or in $\mathfrak{C}\{v_q\}$, we can conclude that B_q cannot contain every block on a single element of D^* but every element of D^* is in some block of B_q and therefore the blocks of B_q when taken as elements in the dual of D^* form a blocking coalition in D.

COROLLARY 1. *If in the dual D of a* BIBD, *D^*, with $r^* \equiv 0, 1 \pmod{k^*}$ and $\lambda = 1$, there exists a blocking coalition of the type described in the theorem with $q = (k^* - 1)n + 1$, then this blocking coalition has $k - n$ members.*

Proof. $|B_q|$ is the sum of the number of blocks in Q and the number of blocks in C, that is,

$$|B_q| = q + (v^* - k^*q)/(k^* - 1). \tag{27}$$

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By substituting for v^* the expressions given in (7) and (8), we obtain in both cases $|B_q| = k - n$.

COROLLARY 2. *Any game that is dual to a* BIBD *with* $r^* \equiv 0, 1 \pmod{k^*}$ *and* $\lambda^* = 1$ *and has a blocking coalition of the type described in the theorem with* $q = k^*$ *has no equitable main simple solution.*

Proof. It is sufficient to prove the existence of a blocking coalition such that $|B| \leqslant k$ **(8)**. Corollary 2, therefore, follows from Corollary 1.

The preceding theorem has particular application to those games that are dual to Steiner triple systems.

COROLLARY 3. *Any player in a game that is dual to a non-trivial Steiner triple system belongs to a blocking coalition of* $k - 1$ *members.*

Proof. From Lemma 1 we have $q = 3$ and a partition of $\mathfrak{C}\{v_q\}$ into pairs is always possible when q is odd. Corollary 3 then follows from Corollary 1.

COROLLARY 4. *Any player in a game that is dual to a non-trivial Steiner triple system* D^* *belongs to a blocking coalition of* $k - n$ *members, where* n *is any positive integer such that for* t *odd* $n \leqslant (t-1)/2$, *and for* t *even* $n \leqslant t/2$ *when* $r^* = 3t$ *or* $n \leqslant (t-2)/2$ *when* $r^* = 3t + 1$.

Proof. We take a $(2n + 1)$-subset of the t, or $t + 1$, mutually disjoint blocks which from Lemma 3 include the given block and form the appropriate blocking coalition by partitioning $\mathfrak{C}\{v_q\}$ into pairs. Since $q = 2n + 1$, we have $|B| = k - n$ from Corollary 1. Since for Case I, $q < t$ and for Case II, $q \leqslant t$, n may assume the values stated in the corollary accordingly as t is odd or even.

The last two corollaries yield also the result that any game that is dual to a non-trivial Steiner triple system has no equitable main simple solution. However, we do not state this explicitly here since this result has been proved by Hoffman and Richardson **(8)**, who give a construction for a blocking coalition of k members.

COROLLARY 5. *If a* BIBD *with* $\lambda = 1$ *contains a set of* v^*/k^* *mutually disjoint blocks, the dual* D *has a minimum blocking coalition of* v^*/k^* *members.*

Proof. The proof of the theorem follows for $\mathfrak{C}\{v_q\} = \emptyset$. $|B| = v^*/k^*$ is obviously a minimum.

The duals of most of the block designs with $r^* - k^* > 0$, $\lambda^* = 1$, $r^* \equiv 1 \pmod{k^*}$ that have been given by direct construction have v^*/k^* mutually disjoint blocks and therefore have minimum blocking coalitions of v^*/k^* elements. Some of these are indicated in a later section of this paper. We make explicit the case of the finite Euclidean geometries in the following corollary.

COROLLARY 6. *If* D^* *is the system of lines in the finite Euclidean space* EG(m, p^n) *of* m *dimensions over the Galois field* GF(p^n) *for* $m \geqslant 2$ *and* $p^n \geqslant 2$,

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the dual D has a minimum blocking coalition of $p^{n(m-1)}$ members and has no equitable main simple solution.

Proof. A set of $p^{n(m-1)}$ mutually parallel lines in D^* provides the required minimum blocking coalition in D and $p^{n(m-1)} < r^* = k$; hence D has no equitable main simple solution.

We note that the duals of the finite projective spaces are not so readily handled since only the odd-dimensional cases satisfy $r^* \equiv 1 \pmod{k^*}$. We can, however, dispose of the problem of whether the dual of the system of lines in a finite projective space has an equitable main simple solution by enlarging the construction devised by Hoffman and Richardson **(8)** to obtain a blocking coalition of k members in the dual of a Steiner triple system.

THEOREM 2. *If D^* is the system of lines in the finite projective space* $\mathrm{PG}(m, p^n)$ *of m dimensions over the Galois field* $\mathrm{GF}(p^n)$ *with $m \geqslant 3$ and $p^n \geqslant 2$, then the dual D has a blocking coalition of* $1 + p^n + p^{2n} + \ldots + p^{(m-1)n}$ *members and hence has no equitable main simple solutions.*

Proof. Let O be any point in $\mathrm{PG}(m, p^n)$. Consider the pencil of lines on O. Since $\lambda^* = 1$, every point of $\mathrm{PG}(m, p^n)$ is on some line of the pencil. Choose $p^n + 1$ coplanar lines on O. Call the plane containing these lines P. Let A be a point in P such that $A \neq O$. Then on A there is in P a pencil of lines containing every point of P. In the pencil of lines of D^* on O replace the lines that are in P by the previously noted plane pencil on A. The resulting set of lines forms a blocking coalition of $r^* = 1 + p^n + {}^2p^n + \ldots + p^{(m-1)n}$ in the dual. Since we have $|B| = r^* = k$, D has no equitable main simple solution.

The method of construction of the blocking coalition in Theorem 1 enables us to solve the problem of determining the minimum number of elements in a blocking coalition in the dual of a Steiner triple system, at least to the extent of expressing a minimum blocking coalition in terms of the maximum number of mutually disjoint blocks in the system. For any given STS a minimum blocking coalition in the dual can easily be found by application of the following theorem.

THEOREM 3. *Let D^* denote a non-trivial Steiner triple system. In the set of blocks $\{b_i\}$ in D^* let M_q be the largest subset having the property that the members of M_q are mutually disjoint blocks of D^* and M_q contains q members where q is odd. Let $\{v_q\}$ be the subset of elements of D^* contained in the blocks of M_q. In the dual D let B_q be the blocking coalition consisting of the members of M_q and the blocks of D^* determined by a partition of $\mathfrak{C}\{v_q\}$ into pairs. Then B_q is a minimum blocking coalition in D.*

Proof. That B_q is a blocking coalition in D follows from Theorem 1. We now show that $|B_q|$ is a minimum. Suppose the contrary. Then there exists a blocking coalition B_x such that $|B_x| < |B_q|$. Let $\{b_1, b_2, \ldots, b_x\}$ be the largest mutually

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disjoint subset of B_x and let $|B_x| = z$. Then

$$B_x = \{b_1, b_2, \ldots, b_x, b_{x+1}, \ldots b_z\}. \tag{28}$$

B_x blocks, therefore,

$$\{v_{v^*}\} = \{b_1 \cup b_2 \cup \ldots \cup b_x \cup b_{x+1}, \ldots, \cup b_z\}, \tag{29}$$

where $\{v_{v^*}\}$ is the set of all the elements of D^*.

Since, in the expression for $\{v_{v^*}\}$ above, the first x blocks are mutually disjoint and since there are not $x + 1$ mutually disjoint blocks in B_x, it follows that

$$|(b_1 \cup b_2 \cup \ldots \cup b_j) \cap b_{j+1}| \geqslant 1 \qquad \text{for } j \geqslant x. \tag{30}$$

In computing $|B_x|$, therefore, any b_j, $j > x$, adds at most two new elements to the set union. Suppose, now, that x is odd. Then z is at least so large as to satisfy

$$v^* = 3x + 2(z - x) \tag{31}$$

and (31) implies that $z \geqslant (v^* - x)/2$. But by assumption $z < |B_q|$; hence

$$(v^* - x)/2 < q + (v^* - 3q)/2 = (v^* - q)/2. \tag{32}$$

By (32), $x \geqslant q$. This is impossible since q is the largest odd-numbered mutually disjoint subset of the set of blocks in D^*.

Suppose x is even. Then at least one block, say b_j, for some $j > x$, adds only one new element to the set union so that z is at least large enough to satisfy

$$v^* = 3x + 2((z - 1) - x) + 1, \tag{33}$$

which implies that $z \geqslant (v^* - (x - 1))/2$. But $z < |B_q|$; hence

$$(v^* - (x - 1))/2 < (v^* - q)/2, \tag{34}$$

which implies that $x - 1 > q$. However, any subset of a set of mutually disjoint blocks is a set of mutually disjoint blocks. Since x is even, any $(x - 1)$-subset of the x mutually disjoint blocks of B_x will be an odd-numbered set of mutually disjoint blocks in D^*. The inequality $x - 1 > q$, therefore, contradicts the assumption that M_q is the largest odd-numbered set of mutually disjoint blocks in D^*. Hence $|B_q|$ is a minimum.

6. Examples. We conclude this note by presenting a number of examples illustrating the preceding results.

Example 1. The dual of a Kirkman triple system of order $v^* = 6t + 3$ has a minimum blocking coalition of $2t + 1$ members. By definition, a Kirkman triple system of order $6t + 3$ is a Steiner triple system with the additional stipulation that the set of $b^* = (2t + 1)(3t + 1)$ triples be partitioned into $3t + 1$ components, each of which is a $(2t + 1)$-subset of triples with each element of the triple system appearing exactly once in a component (**13**. p. 101). Each component is thus a set of $2t + 1$ mutually disjoint blocks and Corollary 5 applies.

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Example 2. The "method of symmetrically repeated differences," due to Bose (**1**), for the construction of a BIBD yields, when applied to designs with $v^* = (k^* - 1)k^*t + k^*$ and $\lambda^* = 1$, $k^*t + 1$ transitive constituents (with respect to blocks) of $(k^* - 1)t + 1$ blocks each, and has an automorphism of order $(k^* - 1)t + 1$; and one of the transitive constituents consists of a set of $(k^* - 1)t + 1$ mutually disjoint blocks that constitute a minimum blocking coalition in the dual. For example, a Steiner triple system on 15 letters can be constructed by applying the transformation

$$T = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)(11\ 12\ 13\ 14\ 15)$$

to the basis blocks:

$$\begin{array}{lll} B_1 = (1, 4, 10), & B_6 = (2, 3, 10), & B_{11} = (6, 9, 15), \\ B_{16} = (7, 8, 15), & B_{21} = (11, 14, 5), & B_{26} = (12, 13, 5), \\ & B_{31} = (5, 10, 15). & \end{array}$$

The last five blocks; (5, 10, 15), (1, 6, 11), (2, 7, 12), (3, 8, 13), and (4, 9, 14), which are based on B_{31}, are a set of five mutually disjoint blocks (**9**, p. 117).

Example 3. There is another construction due to Bose (**1**) for BIBD with $(k^* - 1)k^*t + k^*$ elements and $\lambda^* = 1$, where the k^*t non-zero elements of a finite field with $k^*t + 1$ elements are used $k^* - 1$ times to obtain $(k^* - 1)t$ basis blocks to which is added a single basis block containing an element ∞ and $k^* - 1$ replications of the zero element of the field in the form

$$s(v^* - 1)/(k^* - 1),$$

where $s = 1, 2, \ldots, k^* - 1$, so that when the transformation
$T = (1\ 2 \ldots m)(m + 1 \ldots 2m)(2m + 1 \ldots 3m) \ldots ((k^* - 2)m \ldots v^* - 1)$, where $m = (v^* - 1)/(k^* - 1)$, is applied to the $(k^* - 1)t + 1$ basis blocks, each element appears once with every other element and each element occurs $k^*t + 1$ times in all. Here the basis blocks are mutually disjoint and form in the dual a minimum blocking coalition as noted in Theorem 1, Corollary 5.

For example, when $v^* = 40$, $b^* = 130$, $r^* = 13$, $k^* = 4$, $\lambda^* = 1$, the ten basis blocks are:

$$\begin{array}{ll} B_1 = (1, 12, 18, 21), & B_{14} = (4, 9, 20, 19), \\ B_{27} = (3, 10, 15, 24), & B_{40} = (14, 25, 31, 34), \\ B_{53} = (17, 22, 33, 32), & B_{66} = (16, 23, 28, 37), \\ B_{79} = (27, 38, 5, 8), & B_{92} = (30, 35, 7, 6), \\ B_{105} = (29, 36, 2, 11), & B_{118} = (\infty, 13, 26, 39), \end{array}$$

and the remaining blocks are obtained by applying

$$T = (1\ 2\ 3 \ldots 13)(14\ 15 \ldots 26)(27\ 28 \ldots 39)$$

(**4**, p. 89). The ten basis blocks, when taken as players in the dual, form a minimum blocking coalition.

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Example 4. It is well known that a non-symmetric BIBD with $r^* \equiv 1 \pmod{k^*}$ and $\lambda^* = 1$ can be constructed whenever an appropriate difference set can be found. For example, take $v^* = 21$, $b^* = 3.21 + 7$, $k^* = 3$, $r^* = 10$, $\lambda^* = 1$. An appropriate difference set is given by

$$S = (0, 3, 9 \mid 0, 1, 5 \mid 0, 2, 10 \mid 0, 7, 14),$$

so that the design is constructed by taking as blocks:
$(x, x + 3, x + 9)$, $(x, x + 1, x + 5)$, $(x, x + 2, x + 10)$, $(x, x + 7, x + 14)$, where $x = 0, 1, 2, \ldots, 20$ and sums are taken (mod 21). The last basis block provides a set of seven mutually disjoint blocks which form a minimum blocking coalition in the dual.

A number of non-isomorphic Steiner triple systems on 15 letters may be constructed by this method. We give four examples:

$$\begin{array}{llll} \mathrm{STS}_1: & (x, x + 1, x + 12), & (x, x + 2, x + 9), & (x, x + 5, x + 10); \\ \mathrm{STS}_2: & (x, x + 1, x + 4), & (x, x + 2, x + 8), & (x, x + 5, x + 10); \\ \mathrm{STS}_3: & (x, x + 1, x + 12), & (x, x + 2, x + 8), & (x, x + 5, x + 10); \\ \mathrm{STS}_4: & (x, x + 1, x + 4), & (x, x + 6, x + 13), & (x, x + 5, x + 10); \end{array}$$

We note that STS_2 above is PG(3, 2) (**15**, p. 203). The dual of any BIBD with $\lambda^* = 1$ and constructed by this method has a minimum blocking coalition of v^*/k^* members and hence no equitable main simple solution since

$$v^*/k^* < r^* = k.$$

Example 5. We present, finally, an example of a block design that has less than v^*/k^* mutually disjoint blocks. Let $v^* = 15$, $b^* = 35$, $k^* = 3$, $r^* = 7$, $\lambda^* = 1$, and let the first 15 letters of the alphabet represent the elements. Then the blocks of a BIBD with these parameters may be represented by

$$\begin{array}{lllll} B_1 = ABC, & & & & \\ B_2 = ADE, & B_8 = BDF, & B_{14} = CDG, & & \\ B_3 = AFG, & B_9 = BEG, & B_{15} = CHK, & & \\ B_4 = AHI, & B_{10} = BHJ, & B_{16} = CIJ, & B_{20} = DHL, & B_{24} = EFH, \\ B_5 = AJK, & B_{11} = BIK, & B_{17} = CLO & B_{21} = DIN, & B_{25} = EJM, \\ B_6 = ALM, & B_{12} = BLN, & B_{18} = CNE, & B_{22} = DKM, & B_{26} = EKO, \\ B_7 = ANO, & B_{13} = BMO & B_{19} = CFM, & B_{23} = DJO, & B_{27} = EIL, \end{array}$$

$$\begin{array}{llll} B_{28} = FOI, & B_{30} = FKN, & B_{32} = GHO, & B_{34} = GJN, \\ B_{29} = FJL, & B_{31} = HMN, & B_{33} = GIM, & B_{35} = GKL. \end{array}$$

This design does not contain a set of five mutually disjoint blocks. However, by Lemma 1 any block belongs to a set of three mutually disjoint blocks. Consider B_1. The blocks B_1, B_{20}, B_{28} form a set of three mutually disjoint blocks containing elements A, B, C, D, H, L, F, O, I. By partitioning the remaining elements into pairs, we obtain EG, JK, MN, which determine B_9, B_5, B_{31} respectively. A blocking coalition of $k - 1$ elements in the dual is given by B_1, B_{20}, B_{28}, B_9, B_5, B_{31}. This is a minimum blocking coalition since the largest odd-numbered set of mutually disjoint blocks in D^* contains three blocks.

The design used in this example was constructed by trial and error. We note, however, that all the 80 non-isomorphic triple systems on 15 letters have been enumerated by Cole, White, and Cummings **(2)** and independently by Hall and Swift **(6)**. No attempt has been made to classify either the preceding design or those of Examples 2 and 4 in the context of these two references.

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Added in proof. Recently, Hanani has constructed a BIBD with $\lambda = 1$, $k = 5$, and $v = 141$, this filling the gap indicated on p. 226. See Hanani, *A balanced incomplete block design*, Ann. Math. Statist., *36* (1965), 711ff.

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